How can we determine whether a communication class in a countable state Markov chain is:

- transient
- null recurrent
- positive recurrent

Topological analysis is not enough anymore!

Any communication class with one-way connections to another communication class must be transient, because there is a finite probability to follow a path that leads to another communication class from which there is no return.

Any closed communication class with finitely many states is positive recurrent.

But can't decide the transience/recurrent property of a closed communication class with infinitely many states from topology alone. Need some analysis. It will suffice to assume that we are looking at an irreducible Markov chain since we can treat a closed communication class as an irreducible Markov chain.

We will develop a systematic procedure for deciding whether an
Proposition 1: An irreducible Markov chain has a stationary distribution if and only if it is positive recurrent.

Proof: We showed, in our proof of existence for FSDT MCs that positive recurrence is enough to guarantee the existence of a stationary distribution.

In the other direction, why does the existence of a stationary distribution imply that MC must be positive recurrent?

Given a stationary distribution \( \pi \), if we initialize the MC with this stationary distribution then \( P(X_n = j) = \pi_j \) for all \( n \).

But for both null recurrent and transient MCs, one can show \( \lim_{n \to \infty} P(X_n = j) = 0 \). These two statements contradict each other. So a MC with a stationary distribution must be positive recurrent.

Proposition 2: If an irreducible Markov chain does not have an invariant measure, then it must be transient.

Proof: Our proof for the existence of stationary distribution for irreducible FSDT MCs in fact, under examination, shows that any recurrent MC has an invariant measure.

This is not a decisive test however; if one has a MC with an invariant measure that does not have a stationary distribution, could be null recurrent or could be transient.

Proposition 3: Decisive test for transience for an irreducible MC

1) Choose any reference state \( i_s \in S \). Let \( Q \) be the matrix obtained by deleting the \( i_s \) row and column from the probability transition matrix \( P \). If the only bounded, nonnegative (column vector) solution \( \hat{x} \) to the equation \( \hat{x} = Q \cdot \hat{x} \) is \( \hat{x} = 0 \), then the MC is recurrent. Otherwise it is transient.

2) Similarly, with the same definition of \( Q \), if the equation \( \hat{x} = Q \cdot \hat{x} \) has any unbounded solution \( (|\hat{x}|_\infty = \sup_{j \in S, j \neq i_s} |x_j| = \infty) \), then the MC is recurrent. (This is not an if and only if statement).

We'll prove the first statement; the second statement is derived in Karlin and Taylor Sec. 3.4.
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Proof of first statement in Proposition 3:

Choose the reference state \( i^* \) and define:

\[
\beta_j = P(T_{i^*}(1) = \infty | X_0 = j) \text{ for any } j \neq i^*.
\]

This is the probability that starting from state \( j \), that state \( i^* \) is never visited.

Recurrence is equivalent to the statement:

\[
\beta_j = 0 \text{ for all } j \in S \setminus i^*.
\]

Now we set up an equation for the \( \beta_j = \{\beta_j\}_{j \neq i^*} \).

Define \( \hat{U} = I - \beta \).

\[
\hat{U}_j = P(X_n = i^* \text{ for some } n \mid X_0 = j).
\]

To compute an equation for these quantities, we will do a first step analysis very similarly to the absorption probability calculations, and their connection to answering questions about irreducible MCs (which of two states gets hit first).

We employ the same strategy: turn the target state \( i^* \) into an absorbing state; that doesn’t change the answer to the question.

This modified Markov chain, with the canonical decomposition, will have the probability transition matrix:

\[
\hat{P} = \begin{pmatrix}
\begin{pmatrix}
1 \\
E
\end{pmatrix} & 0 \\
R & Q
\end{pmatrix}
\]

All states \( j \neq i^* \) in the modified MC must be transient because they have a finite probability to hit the absorbing state \( i^* \). Now we will set up the formula for the absorption probability in state \( i^* \) from any other
state $j$ in this modified MC. Notice that, contrary to the finite state case, it is not guaranteed to be absorbed by the single absorbing state.

The same first step analysis as before gives the recurrence relation:

$$ \tilde{\beta} = \tilde{\beta} + \tilde{R} $$

(The $U$ and $R$ are normally matrices in absorption probability calculations, but here they're just column vectors because we only have one absorbing state in the modified MC.)

$$ \tilde{\beta} = \tilde{I} - \beta $$

$$ \tilde{I} - \beta = Q \tilde{I} - \beta $$

$$ Q \tilde{\beta} = \beta + Q \tilde{I} - \tilde{I} + \tilde{R} $$

We'll show this is $0$.

$$ (Q \tilde{I}) \tilde{\beta} - 1 + \tilde{R} = \sum_{i \in S} P_{ji} \tilde{I} - 1 + \tilde{P}_{ji} $$

$$ = \sum_{i \in S} P_{ji} \tilde{I} - 2 = \tilde{I} - 2 = 0 $$

So now we've shown that $Q \cdot \beta = \beta$. But this equation can have multiple solutions, but $\beta$ as defined is not just any solution, but some particular solution of this equation that can have multiple solutions. So
which of the solutions to this equation is the right one?

Lemma: $\tilde{\beta}$ is the maximal solution to the equation $Q\tilde{x} = \tilde{x}$ satisfying the following property:

- $0 \leq x_j \leq 1$ for all $j \in S \setminus i_s$

This means that any other solution $\tilde{y}$ to the equation $Q\tilde{x} = \tilde{x}$ must satisfy $y_j \leq \beta_j$ for all $j \in S \setminus i_s$.

Proof of Lemma:

First we write $\beta_j = \lim_{n \to \infty} P(X_n \neq i_s | X_0 = j)$ under the modified Markov chain dynamics.

$$\beta_j = \lim_{n \to \infty} \mathbb{E}_j \left( \prod_{t=0}^{n-1} \left( \frac{1}{\sum_{k \neq i_s} \pi_k} \right) \right)$$

$$= \lim_{n \to \infty} \mathbb{E}_j \left( \prod_{t=0}^{n-1} \left( \lambda^t \right) \right)$$

$$= \lim_{n \to \infty} \left( \lambda^n \right)$$

Now let $\tilde{y}$ be any solution to $Q\tilde{y} = \tilde{y}$ satisfying $0 \leq \tilde{y} \leq 1$.

Then:

$$\tilde{y} = \lambda \tilde{y} \leq \lambda \frac{1}{\lambda}$$

$$\tilde{y} = \lambda \tilde{y} \leq \lambda (\lambda \frac{1}{\lambda}) = \lambda^2 \frac{1}{\lambda}$$

Continuing by induction,

$$\tilde{y} \leq \lambda^n \frac{1}{\lambda^n} \quad \text{for} \quad n = 1, 2, \ldots$$
Before returning to the classification problem, note that this lemma gives a useful result for how to calculate the probability that, in a countable state irreducible Markov chain, one will ever visit a state \( i \) from another state \( j \).

That's fine, but what does this have to do with the proposition we're trying to prove.

So if the only bounded nonnegative solution to \( Qx = x \) is \( 0 \), then the class of solutions to the problem in the lemma is just the single solution \( \hat{0} \) which has maximal solution \( \hat{\beta} = \hat{0} \), which is equivalent to recurrence of original MC.

What about in the other direction? If the condition of the proposition 3 is negated, that means there exists some nonzero, bounded, nonnegative solution to \( Qx = x \). By rescaling the solution by its maximum value, we obtain a solution \( y = c \hat{x} \) that satisfies \( Qy = y \) and \( 0 \leq y \leq 1 \), that is, a nonzero solution satisfying the conditions of the lemma. Because \( \hat{\beta} \) is the maximal solution, \( \hat{\beta} \neq 0 \). And therefore the original MC must be transient.

Putting together the above considerations and propositions, we have a systematic way for classifying the transience/recurrence properties of MCs.

1. Decompose the Markov chain into communication classes, and decide the properties of whatever communication classes can be determined from topology alone.
   - Any non-closed communication class must be transient.
   - Any closed communication class with finitely many states must be positive recurrent.
2. This only leaves closed communication classes with infinitely many states. Treat each such class as an irreducible MC in its own right for the purpose of the following analysis. Two parallel tracks which can be followed in either order:
   a. Look for an invariant measure (\( \nu P = \nu \) with \( \nu \geq 0 \))
      i. If you can find an invariant measure that can be normalized (\( \sum_{j \in S} v_j < \infty \)) into a stationary distribution, the MC must be positive recurrent.
      ii. If you can show that no invariant measure exists, then the MC must be transient.
      iii. If you find an invariant measure that can't be normalized, then you only know the
class is transient or null recurrent. Have to go to test b.

b. **Decisive test for transience.** Choose any convenient reference state \( i_* \) and form the matrix \( Q \) by deleting the corresponding row and column from the probability transition matrix. Look for solutions to the equation \( \dot{x} = Q \dot{x} \).

i. If you find a **nonzero, nonnegative, bounded solution**, then the MC must be transient.

ii. If you find an **unbounded solution**, then the MC must be recurrent.

iii. If you can prove that the only nonnegative, bounded solution is \( 0 \), then the MC must be recurrent.

Next time, we will show how this strategy can be employed to classify a general birth-death MC.