01/30/06 Continuous random variables

Office hours: W 2-3 PM
F 4-5 PM

Random variable $X$ with continuous state space $S$
(Generally $S \subseteq \mathbb{R}^d$ or $\mathbb{C}^d$)

Specify a random variable $X$ by specifying:

1) state space $\Omega \subseteq S$

2) collection of "measurable" subsets of $S$
   - this is called $\sigma$-algebra $B$

3) a probability measure
   \[ P_X(B) = \text{Prob}(X \in B) \]
   for $B \in B$
For $S \subseteq \mathbb{R}^n$, $\mathcal{C}$ then usually

$B = \text{Borel } \sigma\text{-algebra}$

- smallest $\sigma\text{-algebra}$ that contains all open and closed cubes

- a $\sigma\text{-algebra } B$ is a collection of sets $S$ such that:
  1) $\emptyset \subseteq B$
  2) $B \subseteq B \Rightarrow \forall S \subseteq B \in B$
  3) $\left\{ \bigcup_{j=1}^{\infty} \mathcal{B}_j \right\} \subseteq B$

Also useful = Lebesgue-measurable $\sigma\text{-algebra } B$

- smallest complete $\sigma\text{-algebra}$ containing all open and closed cubes

- if $B \subseteq B$ and $\mathcal{P}(B) = 0$ then

  $A \subseteq B \Rightarrow \mathcal{P}(A) = 0$ and $A \in B$
Suppose $P_X$ is absolutely continuous with Lebesgue measure. Geometrically, small sets have small probability.

$$\Rightarrow \text{there exists a probability density function (PDF)}$$

$$p(x) \text{ s.t.,}$$

$$P(B) = \text{Prob}(X \in B) = \int_B p(x) \, dx$$

for $B \in \mathcal{B}$

\[ \begin{array}{c}
  \text{Formally,} \\
  \text{Prob}(\mid X - x \mid \leq \Delta x) \\
  = \int_{x-\Delta x}^{x+\Delta x} p(x') \, dx' \\
  \approx p(x) \Delta x + o(\Delta x) \text{ for small } \Delta x.
\end{array} \]

More generally (even in multi-d) $\text{Prob}(X \in D_x) \approx p(x) \text{ Vol}(D_x)$ for small sets $D_x$ containing $x$.\]
How can we describe continuous r.v.s more practically?

Consider first \( S = \mathbb{R} \)

Define cumulative distribution function (CDF) for r.v. \( X \)

\[
F_X(x) = \text{Prob}(X \leq x) \quad \text{for } x \in \mathbb{R}
\]

This gives complete information about \( X \):

\[
\text{Prob}(X \in [a, b]) = \text{Prob}(a \leq X < b)
\]

\[
= P_X([a, b])
\]

\[
= \text{Prob}(X \in \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} (a-2^{-j}, b-2^{-k}])
\]

Why do this?

\[
= \lim_{j \to \infty} \lim_{k \to \infty} F_X(b-2^{-k}) - F_X(a-2^{-j})
\]
\[
\text{Prob}(X \in (c, d]) = \text{Prob}(c < X \leq d) \\
\quad = \text{Prob}(X \leq d \text{ and (not } X \leq c)) \\
\quad = \text{Prob}(X \leq d) - \text{Prob}(X \leq c) \\
\quad = F_X(d) - F_X(c)
\]

What does \( CDF \) look like, for smoothly distributed \( x \):

If \( a \in \mathbb{R} \) is a sticky/absorption
So CDFs are practical but mathematically general in 1-dim.
- not so neat in multi-D
- not so intuitive due to nonlocality
HW 1 due date postponed to 02/09, 2 PM (Thursday)

Examples

1) Uniform distribution

\[ X \sim U [a, b] \]

PDF: \[ p(x) = \frac{1}{b-a} \quad \text{for} \quad a \leq x \leq b \]

= 0 \quad\text{otherwise}
Generally \[ F_X(x) = \int_{-\infty}^{x} \lambda e^{-\lambda x} \, dx \]

2) Exponentially distributed r.v. \( X \)

PDF: \[ p(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x \geq 0 \]
\[ = 0 \quad \text{for} \quad x < 0 \]

where \( \lambda > 0 \) is a parameter \((<X> = \frac{1}{\lambda})\)
3) Gaussian distribution (normal)

\[ X \sim N(\mu, \sigma^2) \]

mean \( \mu \)

standard deviation \( \sigma^2 \)

PDF: \[ p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ \langle X \rangle = \mu \]

\[ \langle (X - \mu)^2 \rangle = \sigma^2 > 0 \]

\[ F_X(x) = \text{erf} \left( \frac{x - \mu}{\sigma} \right) \]
How to calculate with continuous r.v.'s and probability density,

remark if $P_\Sigma$ is not absolutely continuous but has a nice set of points $\{a_j\}$ which have nonzero prob, $\text{Prob}(\Sigma = a_j) = p_j$, so that $P_\Sigma$ has a discrete and continuous component, then in applications, one can still use PDF's with $\delta$-facs:

$$p(x) = p_{\text{cont}}(x) + \sum p_j \delta(x-a_j)$$
How do we calculate averages involving continuous r.v.'s?

\[ \langle f(X) \rangle \]

If \( X \) has a PDF \( p(x) \)

\[ \langle f(X) \rangle = \int_{S} f(x) p(x) \, dx \]

(just generalizes sums from discrete case)

\[ \sum_{x} \]

In particular

\[ \langle \sum_{x} \rangle = \int_{S} x \cdot p(x) \, dx \]

etc.
More generally (w/o assume PDF) then 
\[ \langle f(\mathbf{X}) \rangle = \int f(x) \, dp_{\mathbf{X}}(x) \]

Lebesgue integral

\[ (\text{If } \mathbf{X} \text{ is a.c., then } \) \]
\[ dp_{\mathbf{X}}(x) = p(x) \, dx \]

One can generalize from discrete state space:

i) multi-dimensional state spaces
   - CDPS awkward
   - POPs natural

ii) relations between r.v.s
   - then joint PDFs
   - independence: \( \Phi \)

\( \Phi \) Let \( \mathbf{X} \) be a collection of continuous r.v.s with PDF \( p(x) \). (joint PDF)
If $X_1, X_2$ independent

$$p(x) = p_1(x_1) p_2(x_2)$$

PDF for $p_1$ PDF for $p_2$ for $X_1, X_2$

(iii) Conditional prob/expr

- same idea but more technical