01/23/06 Fundamentals of Probability Theory

Homework 1 posted due February 6 (Monday)

Probability space: Ω

Discrete: coin flips

terrorist status (yes/no) in some population

Can you see me now?

Continuous: location of particle ($\mathbb{R}^3$)

Abstract: Financial markets over the next year

Status of New Orleans after an Atlantic hurricane

Elements $\omega \in \Omega$ describe complete information for one possible outcome

An "event" A is a set of possible outcomes $A \subseteq \Omega$

$$\text{Prob}(A) = \sum_{\omega \in A} \text{Prob}(\omega)$$ for discrete state spaces

The $\text{Prob}(\omega)$ must be specified by model.

"Prob" behaves like a positive measure.

$$\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B)$$

$\text{Prob}(A \cup B) = \text{Prob}(A \text{ or } B)$

if the events A and B are mutually exclusive $(A \cap B = \emptyset)$

$$\text{Prob}(A) \geq 0$$

$$\text{Prob}(\emptyset) = 0$$

$$\text{Prob}(\Omega) = 1$$

$$\text{Prob}(\Omega \setminus A) = \text{Prob}(\Omega) - \text{Prob}(A) = 1 - \text{Prob}(A)$$

$\text{Prob}(\Omega \setminus A) = \text{Prob}(\text{not } A)$

Technically the probability measure "Prob" may only be defined for <u>some collection</u> of subsets of Ω.

– important for continuous and abstract prob spaces.

Independence: ~~Events A and B have no cause~~

Two events A, B are independent if the outcome of A happening or not happening has no connection to whether B happens.

$$\mathrm{Prob}(A \text{ and } B) = \mathrm{Prob}(A)\,\mathrm{Prob}(B)$$

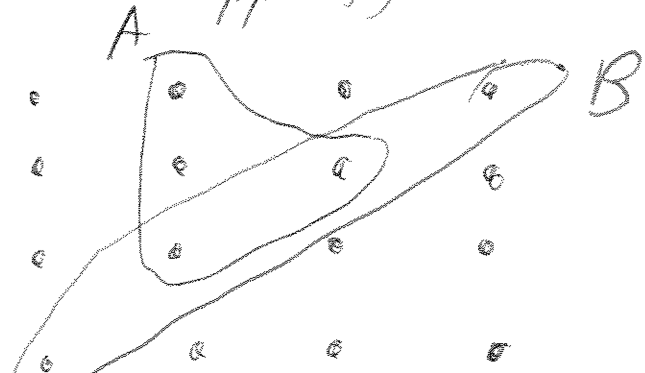
$$\mathrm{Prob}(A \text{ and } B) = \mathrm{Prob}(A \cap B)$$

More general notion of connection of two events

Conditional probability:

$$\mathrm{Prob}(A \mid B) = \frac{\mathrm{Prob}(A \text{ and } B)}{\mathrm{Prob}(B)} \quad \left(\text{true in general}\right)$$

$\mathrm{Prob}(A \mid B) = \mathrm{Prob}(A$ happens given that B happens$)$



If uniform probability

$$\text{Prob}(A) = 1/4$$

$$\text{Prob}(B) = 1/4$$

$$\text{Prob}(A \text{ and } B) = \frac{1}{8}$$

$$\text{Prob}(A \mid B) = \frac{1/8}{1/4} = 1/2$$

Statistical independence

$$\iff \text{Prob}(A \mid B) = \text{Prob}(A)$$

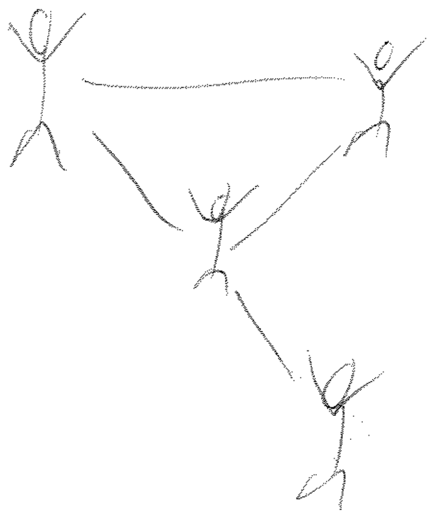
if A, B independent.

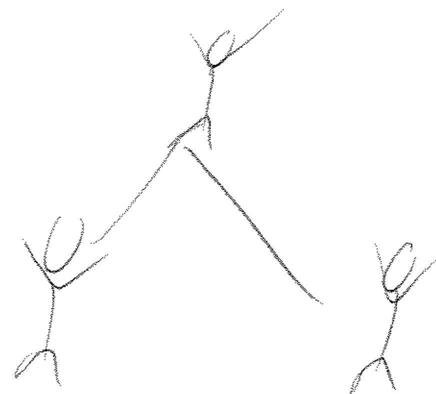
~~Probability~~

Random variables are "measurable" functions on probability space.

The range of a random variable is called the <u>state space</u>.

Measurable means, for random variables X with discrete state space, that $\text{Prob}(X = x)$ is well defined for any value $x \in S$ (state space)





More formally:

$$X : \Omega \to S$$

$$\mathrm{Prob}(X = x) = \sum_{\substack{w \in \Omega : \\ X(w) = x}} \mathrm{Prob}(w)$$

To be a little more precise we make the probability measures on the various spaces more precise:

For $A \subseteq \Omega$ $\qquad \mathrm{Prob}(A) \equiv P(A)$

For ~~$B \in S$~~ $\qquad \mathrm{Prob}(B) = P_X(B)$

$B \subseteq S$

$$P_X(B) = \sum_{\substack{w \in \Omega : \\ X(w) \in B}} P(w)$$

$P(\{w\})$

P_X is a measure induced on S by pushing forward the measure P on Ω through the function $X : \Omega \to S$

P is a measure defined on some "measurable" collection of subsets of Ω.

For discrete state spaces, a practical way to work is to define:

$$p_x = P_{\underline{X}}(\{x\}) = \text{Prob}(\underline{X} = x)$$

$$P_{\underline{X}}(B) = \sum_{x \in B} p_x$$

This allows one to work with the set of positive reals $\{p_x\}_{x \in S}$ rather than measures

Practical info about random variable $\underline{X}$ (discrete state space still)

Probability distribution: $\{p_x\}_{x \in S}$
- complete statistical info about $\underline{X}$

Useful condensations of info about $\mathbb{X}$:

Mean (average or expected value):

$$\mu_{\mathbb{X}} = \langle \mathbb{X} \rangle = \mathbb{E}\,\mathbb{X}$$

$$= \sum_{x \in S} x \, p_x$$

More generally for any function f defined on S

$$\langle f(\mathbb{X}) \rangle = \sum_{x \in S} f(x) \, p_x$$

Average/expected value of $f(\mathbb{X})$

Variance:

$$\sigma_{\mathbb{X}}^2 = \langle (\mathbb{X} - \mu_{\mathbb{X}})^2 \rangle$$

reflects how much $\mathbb{X}$ tends to deviate from mean value.

Simple rules for manipulating

$\langle \cdot \rangle$: behave like sums + integrals

~~For example, can formally view~~ :

This is a linear operation:

$$\langle a f(\mathbb{X}) + b g(\mathbb{X}) \rangle$$

$$= a \langle f(\mathbb{X}) \rangle + b \langle g(\mathbb{X}) \rangle$$

for $a, b \in \mathbb{C}$ (deterministic)

Parameters which are not random behave like parameters independent of integration variable.

- can factor out of integral.

$$\int y x^2 \, dx = y \int x^2 \, dx$$

Example calculation:

$$\sigma_{\mathbb{X}}^2 = \langle (\mathbb{X} - \mu_{\mathbb{X}})^2 \rangle$$

$$= \langle \mathbb{X}^2 - 2\mu_{\mathbb{X}} \mathbb{X} + \mu_{\mathbb{X}}^2 \rangle$$

$$= \langle \mathbb{X}^2 \rangle - 2\mu_{\mathbb{X}} \langle \mathbb{X} \rangle + \mu_{\mathbb{X}}^2 \langle 1 \rangle$$

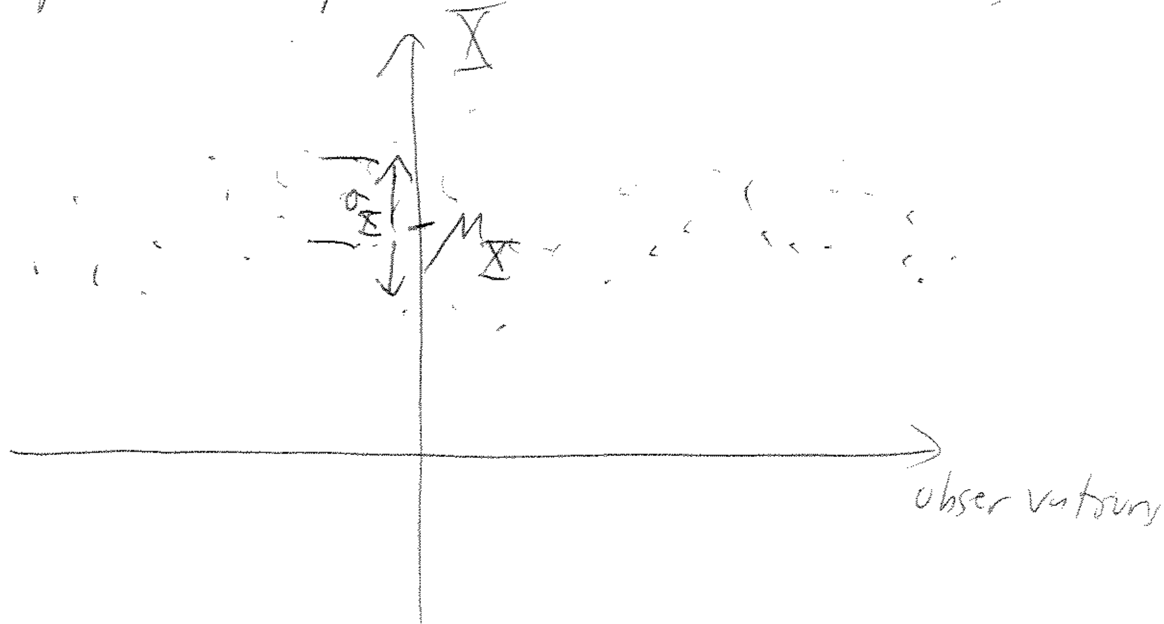
because $\mu_{\mathbb{X}}$ is not random.

$$= \langle X^2 \rangle - 2\mu_X^2 + \mu_X^2$$

$$\sigma_X^2 = \langle X^2 \rangle - \mu_X^2$$

Standard deviation: $\sigma_X = \sqrt{\sigma_X^2}$

- typical departure from mean

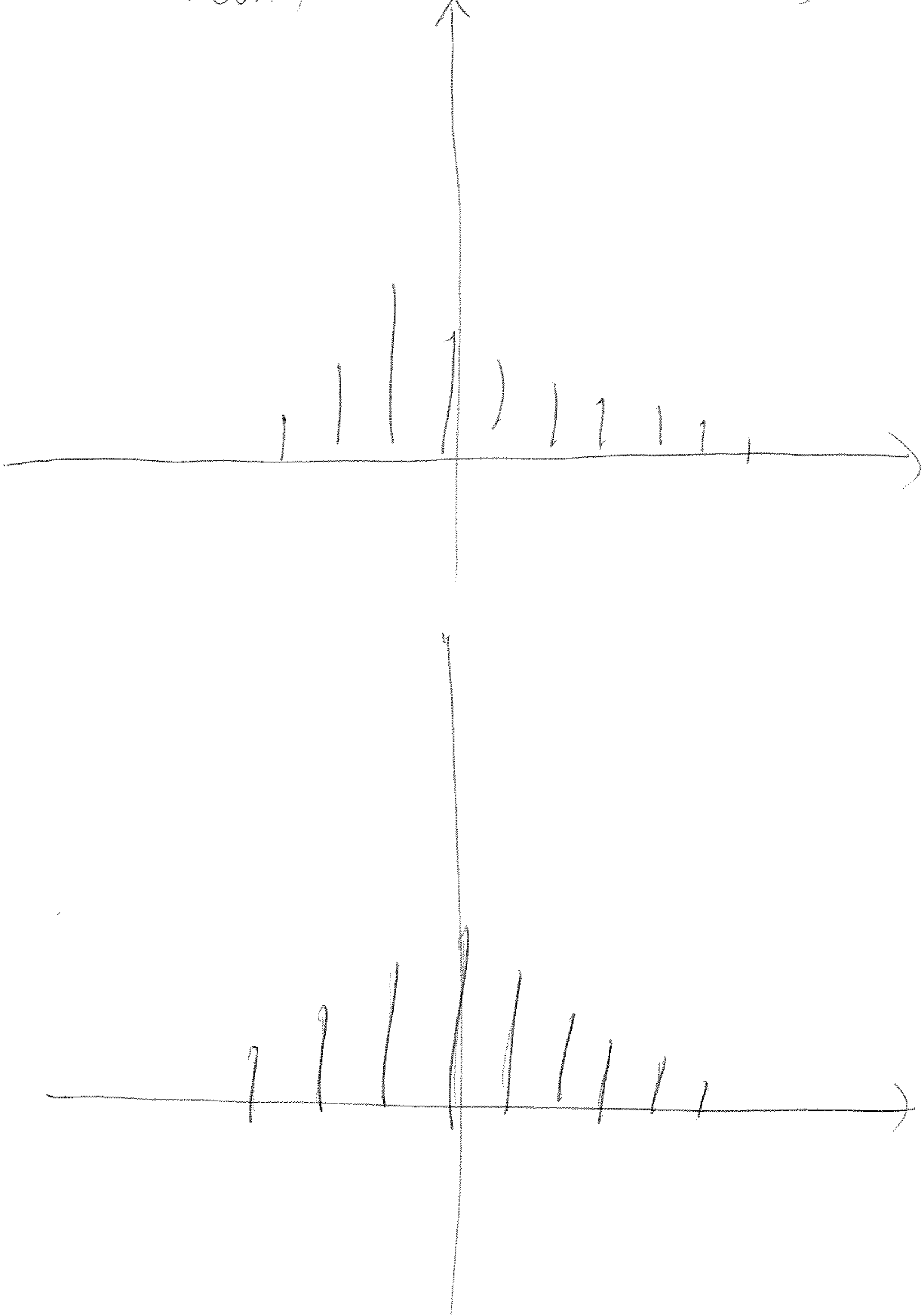


Further information about random variables from higher moments

$$\langle X^n \rangle = \sum_{x \in S} x^n p_x \quad \text{for } n = 1, 2, 3, \ldots$$

Actually "cumulants" are better way to organize information
- modified moments

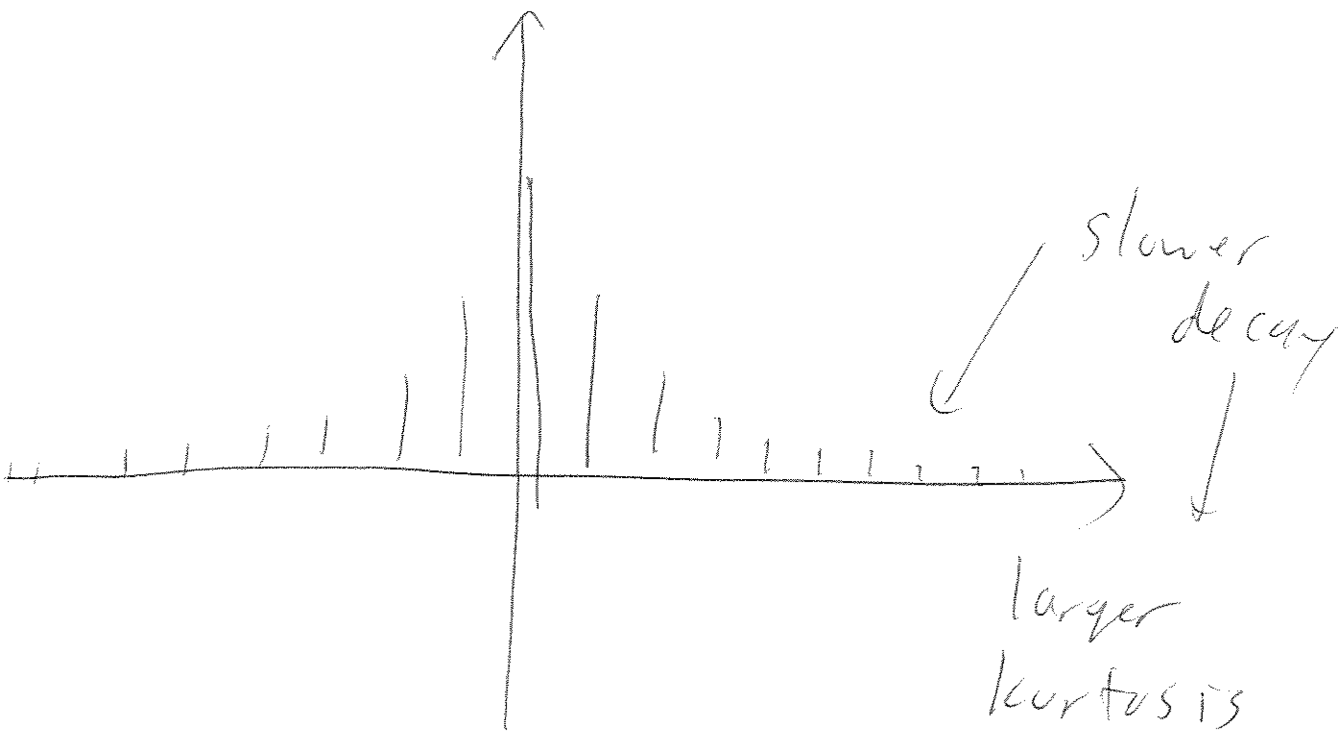
Third moment/cumulant: skewness

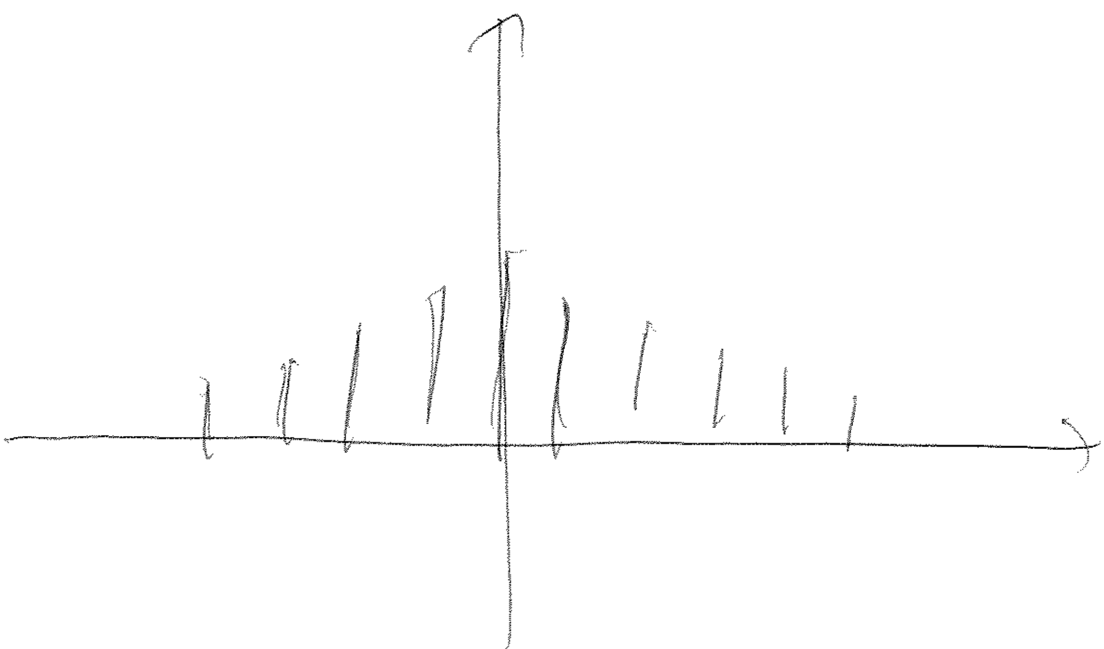


Fourth moment/cumulant: kurtosis

flatness factor

- rapidness of decay in tails





Examples of random variables with discrete state space:

1) Binomial distribution

$$S = \{0, 1, 2, \ldots, N\}$$

$$\text{Prob}(\underline{X} = j) = \binom{N}{j} p^j (1-p)^{N-j}$$

where p is a prescribed value $0 < p < 1$.

$$\binom{N}{j} = {}_N C_j = \frac{N!}{(N-j)!\,j!}$$

Interpretation: Bernoulli trials (coin flips)

- N independent trials
- p = probability of positive outcome in any given trial.

$\underline{X}$ = # positive outcomes in N trials.

2) Uniform distribution

$$S = \{0, 1, \ldots, N\}$$

$$\text{Prob}(X = j) = \frac{1}{N+1} \quad \text{for } 0 \le j \le N.$$

Equal likelihood for each outcome.

Office hours

Wed 3-4 PM

Fri 4-5 PM