Homework 3 extended until Friday, November 22 at 5 PM.

Continuing the theory for how to evaluate integrals of the form:

\[ \int_{-\infty}^{\infty} e^{ix} \frac{p(x)}{q(x)} \, dx \]

(This is typical; to relate a real-valued integral over an infinite interval to a contour integral, write the real-valued integral as a limit of a finite integral.)

So by residue calculus, if we extend the integrand into the complex plane, and call the closed contour drawn above \( C_R \):

\[ \lim_{R \to \infty} \int_{C_R} e^{iz} \frac{p(z)}{q(z)} \, dz = 2\pi i \sum \text{Res} \left( z=\gamma \right) e^{iz} \frac{p(z)}{q(z)} \]

\[ \lim_{R \to \infty} \int_{C_R} e^{iz} \frac{p(z)}{q(z)} \, dz = \begin{cases} 0 & \text{if } \gamma \text{ inside } \text{contour} \in \text{UHP} \text{ computable} \\ \end{cases} \]
On the other hand if $\xi$ is real and negative, then...

Through similar reasoning:

$$\int_{-\infty}^{\infty} e^{i\xi x} \frac{p(x)}{q(x)} \, dx = -2\pi i \left[ \sum_{\text{Zeros of } q} \frac{e^{i\xi z}}{q(z)} \right]_{\text{LHP}}$$

Apply to our example:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \lim_{R \to \infty} \text{PV} \int_{-R}^{R} e^{ix} \, dx$$
More simply:

\[
\oint \frac{e^{iz}}{z} \, dz = 0
\]

\[
\lim_{R \to \infty} \left( \int_{-R}^{-\varepsilon} e^{ix} \, dx + \int_{\varepsilon}^{R} e^{ix} \, dx \right)
\]

\[
= \lim_{\varepsilon \to 0} \left( \int_{-\varepsilon}^{\varepsilon} e^{iz} \, dz \right)
\]

\[
= -\pi \text{Im} \left( \text{Res} \left( \frac{e^{iz}}{z} \right) \right)
\]

\[
\text{Res} \left( \frac{e^{iz}}{z} \right) = \frac{e^{-\varepsilon}}{\varepsilon}
\]

More simply:
Integrals of rational functions of trig functions over a period or half-period.

To convert to contour integral, change variables; express in terms of $z = e^{i\theta}$.

Then the given integral will become an integral over the closed contour of the unit circle traversed counterclockwise. The integrand will just be a rational function.
function. Compute the contour integral by the sum of the residues inside (or minus the sum outside).

For our example:

\[ z = e^{i\theta} \]

\[ \cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) = \frac{1}{2} (z + \frac{1}{z}) \]

\[ (\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) = \frac{1}{2i} (z - \frac{1}{z}) \]

\[ dz = ie^{i\theta} d\theta = i \, z \, d\theta \]

\[ \int_0^{2\pi} \frac{d\theta}{a + \cos \theta} = \int_C \frac{1}{iz} \, dz \]

\[ C = \{ z = e^{i\theta}, 0 \leq \theta \leq 2\pi \} \]

\[ \{ z : |z| = 1 \}, \text{ (counter clockwise) } \]

We've done parameteric integration in reverse!

Integrals involving fractional power laws and rational functions

\[ \int_0^{\infty} x^\alpha R(x) \, dx \]

\[ \alpha' > -1 \]

Example: \[ \int_0^{\infty} \frac{dx}{\sqrt{x} (x+1)} \]
The basic idea for approaching these integrals is to integrate back and forth on opposite sides of a branch cut of the multivalued function that emerges from extending \( x^\alpha \) into the complex plane, and then close with a contour at infinity. Note that one should choose the branch cut to make the problem work well, not necessarily in the standard location.

\[
\int_0^\infty \frac{dx}{\sqrt{x} (x+1)} = \lim_{R \to \infty} \int_0^R \frac{dx}{\sqrt{x} (x+1)}
\]

Extending the integrand into the complex plane; \( \frac{1}{\sqrt{x}} \) will have a multivalued extension; let's choose a single-valued branch. We will choose a branch so that the branch cut is actually along the desired integration segment.

So we will take the branch of \( z^{-\frac{1}{2}} \) to be:

\[
z^{-\frac{1}{2}} = r^{-\frac{1}{2}} e^{i \theta / 2} \quad \text{with} \quad 0 \leq \theta < 2\pi
\]

where \( z = re^{i \theta} \)

This agrees with \( x^{-\frac{1}{2}} \) when \( \theta = 0 \), along the desired contour of integration.
Notice that on the contour \( J_R \) the integrand is precisely the same as our original contour, except that the multivalued function \( z^{-1/2} \) will take a different value. On the original contour, \( \theta = 0 \) and \( z^{1/2} = x^{1/2} \) but on
Putting the branch cut where we put it means that the integrals over the original contour and over $J_R$ won’t cancel, even though they approach the same contour going in different directions, because they’re on opposite sides of the branch cut, and so the multivalued complex function takes different values.

So, from the above we have:

$$\lim_{R \to \infty} \left[ \int_{0}^{R} \frac{e^{-\frac{t}{2}}}{1+\frac{1}{2}} \, dt + \int_{R}^{0} \frac{e^{-\frac{t}{2}}}{1+\frac{1}{2}} \, dt \right] = 2\pi$$

$$\lim_{R \to \infty} \left[ \int_{0}^{R} \frac{x^{-\frac{1}{2}}}{1+x} \, dx + \int_{R}^{0} \frac{x^{-\frac{1}{2}}}{1+x} \, dx \right] = 2\pi$$

$$\lim_{R \to \infty} \int_{0}^{R} \frac{dx}{\sqrt{x}(1+x)} = \pi$$

The point of drawing the branch cut along the contour of integration was that it allowed us to draw large closing contour at infinity without having to cross the branch cut, and it allowed the integration along the contour coming back to not simply cancel out the original integral, but be related to it.
Second example of this:

To extend the given real integral into the complex plane, we will define a branch of \((1 - z^2)^{-\frac{1}{2}}\) so that the branch cut is along the integration contour.

\[
\left(1 - z^2\right)^{-\frac{1}{2}} = \begin{cases} 
(-1)^{-\frac{1}{2}} \left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}} & \text{for } 2 - 12 = r_1 e^{i\theta_1} , \quad -\pi < \theta_1 \leq \pi \\
(-1)^{-\frac{1}{2}} \left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}} & \text{for } 2 + 12 = r_2 e^{i\theta_2} , \quad -\pi < \theta_2 \leq \pi
\end{cases}
\]
Can't do residue calculus inside the contour because nasty branch cut is there. But can do residue calculus outside the contour:

\[ \int_{\gamma} \frac{dz}{(z-a)^{1/2}} = -2\pi i \left( \text{Res}_{z=a} \left( \frac{1-z}{a-z} \right)^{-1/2} + \text{Res}_{z=\infty} \left( \frac{1-z}{a-z} \right)^{-1/2} \right) \]

Corrected after lecture

\[ = -2\pi i \left( (1-a^2)^{-1/2} + 0 \right) \]

\[ \left( \text{Res}_{z=a} \left( \frac{1-z}{a-z} \right)^{-1/2} \right) = \text{Res}_{z=0} \frac{1}{z^2} \left( \frac{1-z}{a-z} \right)^{-1/2} \]

\[ = \text{Res}_{z=0} \frac{1}{z^2} \left( \frac{1-z}{a-z} \right)^{-1/2} \]

\[ \text{Res}_{z=0} \frac{1}{z^2} \left( \frac{1-z}{a-z} \right)^{-1/2} \]

\[ = \text{Res}_{z=0} \frac{(z^2-1)^{-1/2}}{a} \]

\[ = 0 \quad \text{(removable singularity)} \]

Let's show that the little loops avoiding the singularity aren't important:
Let's show that the little loops avoiding the singularity aren't important:

\[ \lim_{\epsilon \to 0} \left| \int_{A_\epsilon} \frac{dz}{(z-a)^{1/2}} \right| \leq \lim_{\epsilon \to 0} M_\epsilon \leq 0 \]

\[ M_\epsilon = \sup_{z \in A_\epsilon} \left| \frac{(1-z^2)^{-1/2}}{(z-a)} \right| = \sup_{z \in A_\epsilon} \frac{r_1^{-1/2} r_2^{-1/2}}{|z-a|} \]

\[ r_1 = |z-1| \]
\[ r_2 = |z+1| \]

On \( A_\epsilon \):
\[ |z-1| = 3 \Rightarrow r_1^{-1/2} = \frac{1}{\sqrt{3}} \]
\[ |z+1| \geq 1-3 = \Rightarrow r_2^{-1/2} \leq \frac{1}{\sqrt{-3}} \]

\[ \left| \frac{1}{z-a} \right| \geq \frac{1}{a-1} \]

\[ M_\epsilon \leq \frac{\frac{1}{\sqrt{3}}}{\frac{1}{a-1}} \]

\[ \lim_{\epsilon \to 0} \left( \int_{A_\epsilon} \frac{dz}{(z-a)^{1/2}} \right) \leq \lim_{\epsilon \to 0} M_\epsilon \leq \lim_{\epsilon \to 0} 2 \pi \left( \frac{\frac{1}{\sqrt{3}}}{\frac{1}{a-1}} \right) \]

\[ = 0 \]

Similarly:
\[ \lim_{\epsilon \to 0} \left( \int_{A_\epsilon} \frac{dz}{(z-a)^{1/2}} \right) > 0 \]

\[ \lim_{\epsilon \to 0} \left( \int_{J_\epsilon} \frac{dz}{(z-a)^{1/2}} + \int_{K_\epsilon} \frac{dz}{(z-a)^{1/2}} \right) + 2 \pi i (1-a^2)^{-1/2} \]
\[ \int_{m}^{\infty} \int_{0}^{\infty} \theta_{1} \theta_{2} = 0 \quad \text{so} \quad (1 - z^2)^{-\frac{1}{2}} = (\cdot^2) \quad \text{and} \quad z = \sqrt{-1} \]

\[ \int_{m}^{\infty} \int_{0}^{\infty} \frac{d\theta_{1} d\theta_{2}}{(a - z) \sqrt{1 - z^2}} \]

\[ \int_{m}^{\infty} \int_{0}^{\infty} \frac{d\theta_{1} d\theta_{2}}{(a - z) \sqrt{1 - z^2}} \]

\[ 2 \int_{-1}^{1} \frac{dx}{(x - a) \sqrt{1 - x^2}} = \pi \sqrt{1 - a^2} \]

\[ \int_{-1}^{1} \frac{dx}{(x - a) \sqrt{1 - x^2}} = \pi \sqrt{1 - a^2} \]