Homework 3 due Friday, November 15 at 5 PM.

Another key result from mathematical analysis regarding convergence is that:

If \( C \) is a compact (finite and bounded) contour and if the functions \( f_n(z) \) converge uniformly to \( f(z) \) on \( C \), then

\[
\lim_{n \to \infty} \int_C f_n(z) \, dz = \int_C f(z) \, dz
\]

The point of statements like this is we want to be able to infer from the convergence of sequences or series that operations on these sequences and series of functions also can be done to their limits (integration, differentiation, multiplication.)

Recall that differentiation is the most subtle operation for which to prove convergence. Many series converge whose derivatives don't converge, i.e.:

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \cos n^2 x
\]

Formal derivative:

\[
\sum_{n=1}^{\infty} \frac{-\sin n^2 x}{n} \quad \text{does not converge}
\]

Let's apply the root test and Weierstrass M-test to make some general statements about power series, which are of central interest in complex analysis.

\[
\sum_{n=0}^{\infty} a_n z^n
\]

This power series converges absolutely and uniformly over any region \( \{z : |z| \leq R_1\} \)

where \( R = \left( \lim_{n \to \infty} \sup |a_n|^{1/n} \right)^{-1} \) and \( R_1 < R \). This value of \( R \) is called the radius of convergence of the power series.
The series definitely diverges outside the radius of convergence, by the root test. But on the circle with radius equal to the radius of convergence, the series may converge absolutely, converge conditionally, or diverge -- have to check. (See Dettmann Sec. 4.3.)

For the Weierstrass M-test for uniform convergence, take: \( M_n = |a_n|R_1^n \)

**Corollary:** the series \( \sum_{n=0}^{\infty} b_n z^{-n} \) converges absolutely and uniformly over any region \( \{z \in \mathbb{C}: |z| \geq \rho_1 \} \) with \( \rho_1 > \rho \) where \( \rho = (\limsup_{n \to \infty} |b_n|^n)^{\frac{1}{n}} \).

Proof: Simply notice this is a power series in terms of the variable \( \xi = \frac{1}{z} \) and apply the above result for power series.

Now let's apply these general ideas from analysis to the Cauchy's Integral formula. Suppose that we have a function \( f(z) \) which is analytic in some annular domain of the form:

\( D = \{z \in \mathbb{C}: \rho < |z - z_0| < R\}. \) Then \( f(z) \) can be represented as a **Laurent series** (generalization of a Taylor series):

\[
 f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n
\]

And this series converges absolutely and uniformly to \( f(z) \) over any compact subset of \( D \), in particular, over closed annuli of the form

\( \{z \in \mathbb{C}: p_1 \leq |z - z_0| \leq R_1 \} \) with \( p < p_1 < R_1 < R \).

Moreover, the Laurent coefficients \( a_n \) can be computed as follows:
Moreover, the Laurent coefficients $a_n$ can be computed as follows:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Where $C$ is any simple closed contour encircling $z_0$ once in a counterclockwise (positive) orientation, lying entirely in $D$.

As a corollary to the proof, one can show that if $f(z)$ is actually analytic inside a disc $D = \{ z \in \mathbb{C} : |z - z_0| < R \}$, then $f(z)$ has a Taylor series representation:

$$f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n$$

which converges absolutely and uniformly to $f(z)$ over any compact subset of $D$, in particular, any closed disc of the form $\{ z \in \mathbb{C} : |z - z_0| \leq R_1 \}$ with $R_1 < R$. The Taylor series coefficients $a_n$ can be computed via similar contour integrals as for Laurent series.

Proof of Laurent series representation:

We'll use Cauchy's integral formula with the contour $C = C_1 \cup C_2$ with:

$$C_1 = \{ z \in \mathbb{C} : |z| = R \}$$

counterclockwise
Let's look at these integrals separately. Consider the integral over $C_1$. We will expand \( \frac{1}{\zeta - z} \) as a geometric series:

\[
\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} r^n
\]

Converges absolutely for $|r| < 1$

The geometric series expansion converges absolutely for $\left| \frac{z - z_0}{\zeta - z_0} \right| < 1$. But in the $C_1$ integral, $|\zeta - z_0| = R_1$ and $|z - z_0| < R_1$ so this condition holds.
And in fact, this series representation converges, for a fixed \( z \),
uniformly over \( \zeta \in C_1 \) by the Weierstrass M-test with \( M_n = \frac{|z - z_0|^n}{R_1^n} \),
which gives a dominating geometric series since \( |z - z_0| < R_1 \).

Therefore, plugging in the above geometric series into Cauchy's integral formulas, we obtain:

\[
\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{\zeta^n} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^n} \, d\zeta
\]

Notice that the Laurent coefficients are derived for a specific simple closed contour \( C_1 \) but by the principle of deformation of contours, these coefficients could just as well be computed from any simple closed contour in \( D \) encircling \( z_0 \) once.

By similar manipulations on the contour integral over \( C_2 \), we get the negative powers.

We expand the geometric series the other way (so that it converges).

\[
\frac{1}{z - z_0} \sum_{n=1}^{\infty} \left( \frac{z_0 - z}{z - z_0} \right)^n = \frac{1}{z - z_0} \left( \frac{1}{1 - \frac{z_0 - z}{z - z_0}} \right)
\]

Converges uniformly and absolutely over \( \zeta \in C_2 \) for a fixed \( z \) with
Plug this in, interchange the infinite sum with the contour integral by uniform convergence again, and we get:

\[
\sum_{\mathcal{C}_2} \frac{f(s) \, ds}{(s - z)} = \sum_{\mathcal{C}_2} f(s) \left( -\frac{1}{z - z_0} \sum_{n = 0}^{\infty} (\frac{s - z_0}{z - z_0})^n \right) \, ds
\]

\[
= \sum_{n = 0}^{\infty} -\sum_{\mathcal{C}_2} f(s) \left( \frac{(s - z_0)^n}{(z - z_0)^{n+1}} \right) \, ds
\]

But recall that \(\mathcal{C}_2\) is taken with negative orientation (clockwise). So if we reverse the direction of integration to make it positive (counterclockwise), the negative sign will go away, and by principle of deformation of contour, the integration can be done over any simple closed contour in the domain \(D\) encircling \(z_0\) once.

The part of the Laurent series involving negative powers of \(z - z_0\) is called the principal part of the Laurent series: \(\sum_{n = -\infty}^{-1} a_n (z - z_0)^n\)

And we'll see that it plays a crucial role in calculations (because it describes the structure of the singularities, which have some hope of contributing to contour integrals.)

A few important comments about Laurent series:
The Laurent series expansion is unique over a specified domain $D$. That is, if we can express

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$$

for some complex constants $b_n$, then these constants $b_n$ must be the same as the $a_n$ presented above. Proof is to integrate both sides of the new Laurent series expansion against $(z - z_0)^m$ and integrate along contours as above.

However, one can have different Laurent series representations over different regions.

This can't happen with Taylor series, but can happen with Laurent series because one can find nonoverlapping annuli, usually separated by singularities, over which the function is analytic.

Not only can Laurent series be obtained from a given function, but a Laurent series which is convergent over some annulus can be shown to define an analytic function over that annulus.