From last time, we stated the Cauchy-Goursat theorem, namely that if we are given a closed contour \( C \), with \( f \) analytic on the interior of \( C \) and \( f \) continuous on the closure of the interior (interior plus the boundary) then

\[
\oint_C f \, dz = 0
\]

Some important consequences (direct analogues of multivariable calculus concepts):

Definition: A simply connected domain \( D \) is a domain (open region) such that any closed contour \( C \) lying entirely within \( D \) has all interior points belonging to \( D \).

Given a simply connected domain \( D \) and a function \( f \) which is analytic on \( D \), we have the following corollaries of the Cauchy-Goursat theorem:

1. \( f \) has a single-valued antiderivative on \( D \)
2. \( \oint_C f(z) \, dz = 0 \) for any closed contour \( C \) lying within \( D \)
3. Given any two points \( z(a) \) and \( z(b) \) lying within the domain \( D \), the contour integral \( \int_C f(z) \, dz \) has the same value for any contour \( C \) lying within \( D \) that starts at \( z(a) \) and ends at \( z(b) \).

Proof by pictures:
Let’s prepare for discussing Cauchy’s Integral Formula by first considering more generally the nature of complex functions defined by indefinite integrals.

Two basic important types of such indefinite integrals:

Type A) Antiderivatives \( F(z) = \int_{z_0}^{z} f(\zeta) d \zeta \) which from the above discussion is well-defined and single-valued provided that we are working on a simply connected domain \( D \) over which \( f \) is analytic. One can moreover check by direct definition of contour integrals and derivatives that \( \frac{dF}{dz} = f \) on \( D \), so that in particular the antiderivative \( F \) is analytic on \( D \).

Examples: Error function:

\[ \Phi(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-s^2} ds \]

\( \Phi(z) \) is analytic on \( \mathbb{C} \) except for \( z = 0 \).

- This function often arises in solving partial differential equations with twice as many space derivatives as time derivatives.

Exponential integral: \( Ei(z) = \int_{-\infty}^{z} \frac{e^{\xi}}{\xi} d \xi \)

In principle, one should discuss what it means to integrate from -infinity, but one can either interpret this informally or on the Riemann sphere.

To define a single-valued form for the exponential integral, or any antiderivative of a function that is analytic over some region with some singularities, one needs to specify a simply connected subdomain of where the integrand is analytic for the paths of integration to lie. Here we do this by (following convention) cutting the positive real axis. By always interpreting the integral defining \( Ei(z) \) to avoid crossing the positive real axis, we obtain a well-defined single-valued analytic function definition for \( Ei(z) \) on the (simply connected) domain.
The positive real axis would in fact serve as a branch cut for a multivalued version of $Ei(z)$ (infinite-sheeted like log because same kind of singularity at the origin). And then the single-valued function we defined would just be a branch (the principal branch) of the multi-valued function.

Why is the exponential integral interesting? First of all note that because

$$
\cos f = \frac{e^{is} + e^{-is}}{2}, \quad \sin f = \frac{e^{is} - e^{-is}}{2i}.
$$

$E_i(z)$ can be related to

$$
C_i(z) = \int_{-\infty}^{z} \cos f \, df, \quad S_i(z) = \int_{0}^{z} \sin f \, df.
$$

Also, consider integrals of the form:

$$
\mathcal{A} \int_{-\infty}^{z} \frac{P(f)}{Q(f)} e^{f} \, df
$$

where $P, Q$ are polynomials. Such an integral can be evaluated by decomposing the rational function $P/Q$ into partial fractions, and this will reduce the integral to a sum of terms of the form:

$$
\int_{-\infty}^{z} (f-a)^n e^{f} \, df, \quad n \in \mathbb{Z}
$$

For $n > 0$, integrate by parts repeatedly to relate these integrals to $n=0$, which just gives an exponential.

For $n < 0$: integrate by parts (but now integrating $(z-a)^n$ repeatedly and differentiating the exponential) until you relate the given integral to an integral of the form with $n=-1$. Then by changing variables $z \to z+a$ we arrive at the exponential integral. Therefore, any
antiderivative of the form * can be expressed as a sum of exponentials and exponential integrals, each multiplied by polynomials.

Type B: Functions defined by integration with respect to a parameter

\[ F(z) = \int_C g(z, \zeta) d\zeta \]

for a fixed contour \( C \)

Before presenting the basic theory for such integrals, we list some important examples:

- **Cauchy integrals:**
  \[ F(z) = \int_C \frac{e^z}{Q(z)} \, d\zeta \]
  Which arises as the solution of an ordinary differential equation of the form
  \[ Q \left( \frac{d}{dx} \right) f = 0 \]
  after a Laplace transform.

- **Stieltjes integrals:**
  \[ F(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{1 + z x} \]
  Arise in calculations in turbulence, material science, quantum field theory, ....

- **Gamma function:**
  \[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \]
  Generalizes the factorial \( \Gamma(n) = (n - 1)! \) for \( n \in \mathbb{N} \)

- **Legendre polynomials:**
  \[ P_n(z) = \frac{1}{n! 2^n} \frac{d^n}{dz^n} (z^2 - 1)^n \]
  \[ = \frac{1}{2^{n+1} n!} \int_C \frac{(z^2 - 1)^n}{(z - \zeta)^{n+1}} \, d\zeta \]
Where $C$ is any positively oriented simple contour enclosing $z$.

This is an example of representing a real-valued special function in terms of a contour integral. This is often useful because integral representations are easier to apply analytical techniques to in order to obtain asymptotics and bounds on the behavior of the function. See books on special functions for examples.

**Proposition:** If $g(z, \zeta)$ is analytic w.r.t. $z$ for $z \in D$ and $\partial g(z, \zeta)/\partial z$ is continuous for $\zeta \in C$ (a compact contour) and $z \in D$, then

$$F(z) \equiv \int_C g(z, \zeta) d\zeta$$

is an analytic function on the domain $D$.

**Remark:** Here the domain $D$ and the contour $C$ need not have any relation to each other, nor need $D$ be simply connected.

Note the technical assumption on $g(z, \zeta)$
Is strong for the $z$ argument (analytic) but not so strong for the $\zeta$ (just continuous after differentiated by $z$).

**Proof:** Let's consider the difference quotient whose limit must exist for $F$ to be analytic.

$$\lim_{h \to 0} \frac{F(z + h) - F(z)}{h} = \int_C \frac{g(z + h, \zeta) - g(z, \zeta)}{h} d\zeta$$

$$= \int_C \frac{g(z + h, \zeta) - g(z, \zeta)}{h} d\zeta$$

Where $z \in D$

Because $g$ is analytic (and therefore differentiable) at $z$, we can write:

$$g(z + h, \zeta) - g(z, \zeta) = \frac{\partial g(z, \zeta)}{\partial z}$$
Pointwise convergence!

But we also know that both $g$ and $\frac{\partial g}{\partial z}$ are continuous functions, so since $\eta$ is just a linear combination of these functions, it is also continuous (w.r.t. $\zeta$, $z$). But a fundamental result of analysis is that if a sequence of continuous functions converge pointwise on a compact set then the convergence is actually uniform over that compact set.

This allows us to conclude that

$$\lim_{h \to 0} m(z, \delta h) = 0$$

uniformly for $s \in C$

pointwise for $z \in D$

But integrals of uniformly convergent functions on compact sets converge to the integral of the limit:

$$\lim_{h \to 0} \int_C \frac{g(z+h,s) - g(z,s)}{h} \, ds = 0$$

This gives us the statement in the proposition:
Some of the examples we presented above did not involve compact contours. The proposition doesn't apply directly to those cases. One needs to instead show that the integrals over noncompact contours can be approximated by integrals over compact contours by introducing a decay estimate of some type for the integrand.

\[
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} (z \in C)
\]

\[
\frac{df}{dz} = \int_C \frac{d\Phi(z, \Phi) ds}{dz}
\]

For \( z \in C \), so is analytic.