Homework 2 posted soon, due Tuesday, March 17.

Reading: K, "Wiener process"

I will now drop the explicit reference to realizations when referring to random processes, so we will write the Wiener process more simply as $W(t)$ rather than $W(t, \omega)$.

First we'll define and describe the mathematical properties of the Wiener process and then explain why it is the appropriate continuum limit of our discrete-time simplest model for Brownian motion.

The one-dimensional Wiener process (mathematical Brownian motion) $W(t)$ is uniquely defined by the following properties:

- $W(t)$ is continuous in (almost) every realization,
- $W(0) = 0$
- The increments of $W(t)$ are independent on disjoint intervals, meaning that if $t_1 < t_2 \leq t_3 < t_4$ then $W(t_2) - W(t_1)$ is independent of $W(t_4) - W(t_3)$
- Each increment $W(t_2) - W(t_1)$ is a Gaussian random variable with mean 0 and variance $|t_2 - t_1|$.

Note: It is increments, not the values of $W(t)$ that are independent over disjoint intervals!

One can actually show, using the Levy-Khinchine theorem, that essentially any continuous random process with independent increments has to be related to the Wiener process by translation and rescaling. The central limit theorem (suitably generalized) tells you that continuity plus independent increment property can only give you Gaussian result, unless you break the technical conditions for the CLT, meaning that high moments don't exist (Levy process). If one removes the continuity assumption, there are many useful stochastic processes (called jump processes, including continuous-time Markov chains) that have the independent increment property.
Wiener process (continuous noise): small changes happen all the time, smoothly

Jump process: most of the time, nothing happens, but there are times at which a punctuated $O(1)$ change happens.

In some of the mathematical texts, you will read about the Kolmogorov extension theorem which guarantees that the description we gave uniquely defines a continuous stochastic process. (The continuity actually is forced (up to pathological choices) by the other statements.)

Now we'll turn to the practical issues with working with the Wiener process:

- whenever possible, refer to increments of the Wiener process because their statistics are well-defined, and they’re independent over disjoint intervals. When dealing with the Wiener process at multiple times, try to break up the calculation into increments over disjoint time intervals. Note that $W(0) = 0$ is known, so often useful to take $0$ as a starting point for an increment.

Example calculation:

$$\text{Cov}(W(t_1), W(t_2)) = \min(t_1, t_2) \quad \text{whenever } 0 < t_1, t_2.$$ 

How do we show this?

Let’s first calculate:

$$\text{Cov}(W(t_1), W(t_2)) = \mathbb{E}(W(t_1)W(t_2)) - \mathbb{E}(W(t_1))\mathbb{E}(W(t_2))$$

$$\mathbb{E} W(t_1) = \mathbb{E} \left( \int_0^{t_1} \omega_s \, ds \bigg| \mathcal{F}_0 \right)$$

$$= \mathbb{E} \left( W(t_1) - W(0) + W(0) \right)$$

increment over $[0, t_1]$

$$\mathbb{E} W(t) = \mathbb{E} \left( W(t) - W(0) + W(0) \right)$$

$$= \mathbb{E} \left( W(t) - W(0) \right) + \mathbb{E} W(0)$$
\[
\text{Mean: } 0 \\
\text{Variance: } t_1
\]

\[
= 0 + 0
\]

\[
\mathbb{E}(W(t_1)) = 0
\]

\[
\mathbb{E}(W(t_2)) = 0
\]

\[
\mathbb{E}(W(t_1)W(t_2))
\]

**Special Case: \( t_1 = t_2 \)**

\[
\mathbb{E}(W(t_1)W(t_1)) = \mathbb{E}(W(t_1)^2)
\]

\[
\text{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2
\]

\[
= \text{Var}(W(t_1)) + (\mathbb{E}W(t_1))^2
\]

\[
= \text{Var}(W(t_1) - W(0)) + (\mathbb{E}(W(t_1) - W(0)))^2
\]

*because \( W(0) = 0 \).*

\[
= t_1 + 0^2
\]

\[
\mathbb{E}(W(t_1)W(t_1)) = t_1 \quad \text{if} \quad t_1 = t_2
\]

**Now suppose \( t_1 < t_2 \)**

\[
\mathbb{E}(W(t_1)W(t_2)) = \\
\mathbb{E}\left(W(t_1)\left(W(t_2) - W(t_1) + W(t_1)\right)\right)
\]
\[
\begin{align*}
\text{when working with the Wiener process computationally, then the key idea again is to think about expressing the calculation in terms of increments over nonoverlapping intervals. Simply increment from 0 to the first time of interest, then from the first time of interest to the second time of interest, etc. Every time one increments from } t_i \text{ to } t_{i+1}, \text{ one has to simulate } W(t_{i+1}) - W(t_i) \text{ which is a Gaussian random variable with mean zero and variance } |t_{i+1} - t_i|, \text{ and is independent of previously generated increments.}
\end{align*}
\]
Each component $W_j(t)$ is an independent standard Wiener process.

We will now show how this Wiener process (mathematical Brownian motion) serves as a continuous-time limit of our discrete-time simplest model for Brownian motion.

We wrote down the continuous-time model as an SDE:

$$dX(t) = \sqrt{c} \, dW(t)$$

Note that the differential of mathematical Brownian motion is subtle, because even though $W(t)$ is a nice, continuous, Gaussian random process, its time derivative is nasty:

The Wiener process is continuous but not differentiable in an ordinary sense (its derivative can be interpreted in the sense of random generalized functions or random distributions as "mathematical white noise"). More precisely, the Wiener process just barely fails to be Holder-continuous with exponent $1/2$.

So the practical point is that dealing with $dW(t)$ is subtle but if you can work directly with the Wiener process w/o differential, $W(t)$, then calculations are much more straightforward.
So for our simplest SDE:

$$dX(t) = \sqrt{c} \, dW(t)$$

we can dispense with worrying about the meaning of $dW(t)$ by simply integrating both sides:

$$X(t) - X(0) = \sqrt{c} \left( W(t) - W(0) \right)$$

$$X(t) = X(0) + \sqrt{c} W(t)$$

Let's imagine that we observe this continuous-time process at discrete times separated by $\Delta t$: $X^{(n)} = X(n\Delta t)$

Then we could write: $X^{(n+1)} = X((n+1)\Delta t) = X(n\Delta t) + \sqrt{c} \left( W((n+1)\Delta t) - W(n\Delta t) \right)$

This would agree with our discrete-time stochastic model for Brownian motion:

$$X^{(n+1)} = X^{(n)} + Z^{(n)} \quad \text{where}$$

$$Z^{(n)} \equiv \sqrt{c} \left( W((n+1)\Delta t) - W(n\Delta t) \right)$$

These random kicks, so defined, have the following properties following from the properties of the Wiener process:

- $\{Z^{(n)}\}$ independent by independent increment property
- Gaussian
- mean 0
- $\text{Cov} \left( Z^{(n)}, Z^{(n)} \right) = c \Delta t$ because:

$$\text{Cov} \left( Z^{(n)}, Z^{(n)} \right) = \text{Cov} \left( \sqrt{c} \left( W((n+1)\Delta t) - W(n\Delta t) \right), \sqrt{c} \left( W((n+1)\Delta t) - W(n\Delta t) \right) \right)$$

$$= \sqrt{c} \sqrt{c} \text{Cov} \left( W((n+1)\Delta t) - W(n\Delta t), W((n+1)\Delta t) - W(n\Delta t) \right)$$

$$= \sqrt{c} \sqrt{c} \text{Cov} \left( W((n+1)\Delta t) - W(n\Delta t), \frac{W((n+1)\Delta t) - W(n\Delta t)}{\sqrt{c}} \right)$$

$$= c \Delta t$$

(note that covariance matrix of any vector of iid random variables must be multiple of identity matrix)

Therefore the discretization of the continuous-time mathematical
Brownian motion agrees with our discrete-time model.

Now we have a simplest continuous-time model for Brownian motion:

\[ dX(t) = \sqrt{\epsilon} dW(t) \]

or:

\[ X(t) = X(0) + \sqrt{\epsilon} W(t) \]

This model is physically valid when observed with resolution \( \Delta t \) large compared to the momentum relaxation time; will have to improved if one wants a higher resolution model.

This model has been developed and represented in a Lagrangian (trajectory-based) framework.

- stochastic differential equation description

We will now look at a very useful alternative perspective, which is Eulerian (probability-based) framework.

- Fokker-Planck equation/forward–Kolmogorov equation: deterministic partial differential equations

To prepare to set up this framework, we need a few facts from probability theory.

**Law of Total Probability**

If \( \{C_j\}_{j=1}^m \) is a partition of sample space \( \Omega \), then:

\[
P(A) = \sum_{j=1}^{m} P(A|C_j)P(C_j)
\]

Partition means that:

- \( C_j \cap C_{j'} = \emptyset \) for \( j \neq j' \)
- \( \bigcup_{j=1}^{m} C_j = \Omega \)

In particular, if the event \( A = \{Y \in B\} \), this takes the form:

\[
P(Y \in B) = \sum_{j=1}^{m} P(Y \in B|C_j)P(C_j)
\]