Bessel Functions and Kepler's Equation

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1. INTRODUCTION. Toward the end of a first course in differential equations, we may often use the method of Frobenius to show that *Bessel's differential equation of integral order n*

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{1}$$

has a solution

$$y = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}k!(k+n)!} \tag{2}$$

called the *Bessel function of the first kind of order n*.

This essay is a partial account of connections between (1), (2), Bessel, and Kepler's Equation, a transcendental equation of celestial mechanics with a rich and extensive history.

2. KEPLER'S EQUATION. After years of work, Johannes Kepler announced three laws of planetary motion early in the seventeenth century.

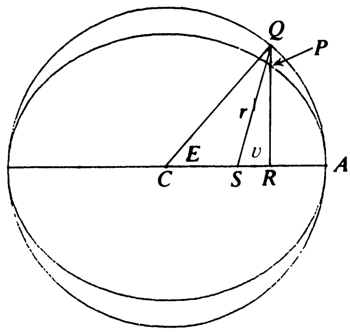
Kepler's three laws state that the planets move in elliptical orbits in a common plane with the sun at one focus, that for each planet the line connecting the sun with the planet sweeps out equal areas in equal times and that the ratio of the square of the period of revolution of each planet to the cube of the semimajor axis of its orbit is the same for all planets.

Kepler stated the first two laws in 1609 in the *Astronomia Nova* and the third in 1619 in *The Harmony of the World*. As we know now, these laws are only approximations, but for the six planets known at the time and to the limits of observation then they were essentially exact.

Kepler's Equation is a consequence of the first two laws only.

Suppose a planet moves in the counterclockwise direction in an elliptical orbit with the sun at one focus which has eccentricity e, $0 < e < 1$, has semimajor axis a, and is traveled once in time T. In the figure, A denotes perihelion, C center of the orbit, and S the position of the sun. If, having passed through A, the planet after elapsed time t is at position P, we wish to express the polar coordinates of P, (r, v), relative to S at time t.

The quantity v = angle PSA is called the *true anomaly* of the planet at time t. The circle centered at C with radius a is called the eccentric circle. If we draw the line P perpendicular to radius CA and mark R, its intersection with CA, and Q, its intersection with the eccentric circle, the quantity E = angle QCA is called the *eccentric anomaly* of the planet at time t.



The relation between r and v is

$$r = \frac{a(1-e^2)}{1+e\cos v}. \tag{3}$$

With trigonometry and algebra, we may derive

$$r = a(1 - e\cos E) \quad \text{and} \quad \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{4}$$

Thus (r, v) may be obtained from E.

Kepler's Equation relates E to t by means of a quantity $M = 2\pi t/T$ called the *mean anomaly* of the planet at time t.

The relation between E and M (and so t) comes through Kepler's second law.

$$\text{Area } PSA = (t/T)(\text{Area enclosed in the orbit}) = (1/2)Ma^2\sqrt{1-e^2}$$

and

$$\text{Area } PSA = \text{Area } PSR + \text{Area } PRA = \frac{1}{2}a^2\sqrt{1-e^2}\,(E - e\sin E).$$

The result is Kepler's Equation (KE):

$$M = E - e\sin E.$$

If we know t and M, and if we can solve (KE) for E, then we can find (r, v) from (4). More details and background appear in [**10**].

3. LAGRANGE'S SOLUTION OF (KE). From the time of Kepler, many efforts had been made to solve (KE), at least approximately. In 1770, J. L. Lagrange [**7**] showed that under suitable conditions an equation of the form

$$w = a + t\phi(w) \tag{5}$$

would have a solution for w and for any function f it would be true that

$$f(w) = f(a) + \sum_{n=1}^{\infty}\frac{t^n}{n!}\frac{d^{n-1}}{da^{n-1}}\{f'(a)[\phi(a)]^n\}. \tag{6}$$

In the special case: $f(z) = z, \phi(z) = \sin z, a = M, t = e$ and $w = E$, (5) becomes (KE) and (6) reads

$$E = M + \sum_{n=1}^{\infty}\frac{e^n}{n!}\frac{d^{n-1}}{dM^{n-1}}(\sin^n M) = M + \sum_{n=1}^{\infty}a_n(M)e^n. \tag{7}$$

Lagrange applied his result to (KE) in **[8]**, and later there were many efforts to find explicit formulas for the coefficients $\{a_n(M)\}$ in (7).

4. BESSEL AND (KE). F. W. Bessel, an eminent German astronomer of the early nineteenth century, was well-acquainted with Lagrange's solution of (KE). Where Lagrange used repeated differentiation, Bessel used integration and described a pretty solution of (KE) in the form

$$E = M + \sum_{n=1}^{\infty} b_n(e) \sin nM, \tag{8}$$

that is, as a Fourier sine series.

Here is what he did. If $E = g(M)$ is the solution of (KE), g has $M = 0$ and $M = \pi$ as fixed points, and if

$$g(M) - M = \sum_{n=1}^{\infty} b_n(e) \sin nM \tag{9}$$

on the interval $0 \le M \le \pi$, then

$$b_n(e) = \frac{2}{\pi} \int_0^{\pi} [g(M) - M] \sin nM \, dM = \frac{2}{n\pi} \int_0^{\pi} \cos nM \, dg(M).$$

Since $M = E - e \sin E = g(M) - e \sin(g(M))$,

$$b_n(e) = \frac{2}{n} \left\{ \frac{1}{\pi} \int_0^{\pi} \cos(nE - ne \sin E) \, dE \right\}.$$

In modern notation Bessel's definition of $J_n(x)$ was

$$J_n(x) = \frac{1}{\pi} \int_0^{\infty} \cos(nE - x \sin E) \, dE \tag{10}$$

and Bessel's solution of (KE) was

$$E = M + \sum_{n=1}^{\infty} \left\{ \frac{2}{n} J_n(ne) \right\} \sin nM. \tag{11}$$

It is not at all obvious that (10) and (2) describe the same function. In a landmark paper of 1824 **[2]** Bessel showed that the functions (10) indeed do satisfy the differential equation (1) and at $x = 0$ have the same value and derivative value as (2). In this paper Bessel also derived many of the standard identities for $J_n(x)$, but there is no explicit mention of (KE). The solution of (KE) leading to (11) was actually done in an 1818 letter to W. Olbers **[1]** and Bessel expressed his surprise that nobody had heretofore discovered it.

5. OTHER ANTECEDENTS TO BESSEL FUNCTIONS. Although it is simpler to say that Bessel invented Bessel functions in order to solve (KE), he didn't invent them and he didn't consider (KE) the most important of the problems of celestial mechanics which lead him to them.

Lagrange, **[8]**, also tried to write his solution of (KE) in the form (8) and determined in series form the coefficients $b_1(e), b_2(e), b_3(e)$. While Lagrange seems to be the first to have encountered Bessel functions in form (2) in the context of (KE), functions in form (2) had been encountered in special cases as early as 1703 and in considerable generality by Euler around 1764, [**13**, p. 356].

Even in the area of celestial mechanics Bessel's definition (10) finds some close antecedent in the work of S. D. Poisson [**9**], [**12**, p. 6]. And almost simultaneously with Bessel, F. Carlini [**3**], [**4**] derived a series expression for the true anomaly, v, in the form

$$v = M + \sum_{n=1}^{\infty} G_n(e) \sin nM$$

by starting with Lagrange's theorem. Carlini's expressions for $G_n(e)$ were not in the form of integrals, but in modern notation they satisfy the identities, [**11**, p. 59],

$$G_n(e) = \frac{2}{n} J_n(ne) + \sum_{m=1}^{\infty} \alpha^m \left[J_{n-m}(ne) + J_{n+m}(ne) \right], \qquad e = 2\alpha/(1+\alpha^2).$$

Carlini's work attracted very little attention until C. G. J. Jacobi in 1850 translated it into German, correcting parts of it and adding extensive commentary, [**5**]. In 1893, M. W. Kapteyn, [**6**], motivated by the literature of (KE) and celestial mechanics, studied the possibility of representing functions in the form

$$f(x) = \sum_{n=0}^{\infty} c_n(f) J_n(nx),$$

which are now called *Kapteyn series*, and of which the first is (11) solving (KE).

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