MATH-6640 **Spring 2005**

COMPLEX VARIABLES AND INTEGRAL TRANSFORMS WITH APPLICATIONS

Assignment 4 Solutions

1. Use the properties of the Laplace transform, and the short table of transforms presented in class, to find the following:

   (a) Transform of

   $$\mathrm{Si}(t) = \int_0^t \frac{\sin u}{u}\, du,$$

   where $\mathrm{Si}(t)$ is the Sine Integral function which occurs in the study of optics.

   Use

   $$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\, du$$

   to get

   $$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{1+u^2}\, du = \frac{\pi}{2} - \tan^{-1} s.$$

   Then, use

   $$\mathcal{L}\int_0^t f(u)\, du = \frac{F(s)}{s}$$

   to get

   $$\mathcal{L}\int_0^t \frac{\sin u}{u}\, du = \frac{1}{s}\left\{\frac{\pi}{2} - \tan^{-1} s\right\}.$$

   (b) The Laplace inverse of

   $$\frac{1}{(s+\omega_1)^2+\omega_2^2}.$$

   Recall that

   $$\mathcal{L}^{-1}\frac{1}{s^2+\omega_2^2} = \sin \omega_2 t.$$

   Then,

   $$\mathcal{L}^{-1}\frac{1}{(s+\omega_1)^2+\omega_2^2} = e^{-\omega_1 t}\sin \omega_2 t.$$

   (c) The Laplace inverse of

   $$\frac{e^{-5s}}{(s-3)^3}.$$

   Recall that

   $$\mathcal{L}^{-1}\frac{1}{s^3} = \frac{1}{2}t^2.$$

   Therefore,

   $$\mathcal{L}^{-1}\frac{1}{(s-3)^3} = \frac{1}{2}e^{3t}t^2 = f(t), \text{ say.}$$

   Then,

   $$\mathcal{L}^{-1}\frac{e^{-5s}}{(s-3)^3} = f(t-5)H(t-5) = \frac{1}{2}e^{3(t-5)}(t-5)^2H(t-5).$$

(d) The Laplace inverse of

$$\frac{s^2}{s^2+1}.$$

if it exists. (Why should there be a question?) Don't be hasty.

Since

$$\frac{s^2}{s^2+1} = 1 - \frac{1}{s^2+1},$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \delta(t) - \sin t.$$

2. Consider the differential equation

$$\frac{d^3y}{dt^3} + \omega^3 y = f(t), \quad \omega > 0, \quad y(0) = y'(0) = y''(0) = 0.$$

(a) Show that the Laplace transform $Y(s)$ of the solution satisfies

$$Y(s) = \frac{F(s)}{s^3+\omega^3},$$

where $F(s)$ is the transform of $f(t)$.

Laplace transformation of the ODE leads to

$$s^3Y(s) + \omega^3 Y(s) = F(s)$$

whence

$$Y(s) = \frac{F(s)}{s^3+\omega^3}.$$

(b) Deduce that the inverse transform of

$$Z(s) = \frac{1}{s^3+\omega^3}$$

is given by

$$z(t) = \frac{e^{-\omega t}}{3\omega^2} - \frac{2}{3\omega^2}\exp(\omega t/2)\cos\left(\frac{\sqrt{3}}{2}\omega t - \frac{\pi}{3}\right).$$

How will the above result allow you to find a representation for $y(t)$?

By using partial fractions we may write

$$\begin{aligned}\frac{1}{s^3+\omega^3} &= \frac{1}{3\omega^2}\frac{1}{s+\omega}\\ &+ \frac{1}{\sqrt{3}\omega^2}\frac{\frac{\sqrt{3}\omega}{2}}{(s-\omega/2)^2+3\omega^2/4}\\ &- \frac{1}{3\omega^2}\frac{s-\omega/2}{(s-\omega/2)^2+3\omega^2/4}\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3+\omega^3}\right\} &= \frac{1}{3\omega^2}e^{-\omega t} + e^{\omega t/2}\left[\frac{1}{\sqrt{3}\omega^2}\sin\frac{\sqrt{3}\omega t}{2} - \frac{1}{3\omega^2}\cos\frac{\sqrt{3}\omega t}{2}\right]\\ &= \frac{1}{3\omega^2}e^{-\omega t} - \frac{2}{3\omega^2}e^{\omega t/2}\cos\left(\frac{\sqrt{3}\omega t}{2} - \frac{\pi}{3}\right).\end{aligned}$$

We can now compute y by using convolution.

3. Show that the inverse Laplace transform of the function

$$F(s)=\frac{1}{\sqrt{s^2+\omega^2}}$$

is given by

$$f(t)=\frac{1}{\pi}\int_{-\omega}^{\omega}\frac{e^{itr}}{\sqrt{\omega^2-r^2}}\,dr=\frac{2}{\pi}\int_0^1\frac{\cos(\omega\rho t)}{\sqrt{1-\rho^2}}\,d\rho.$$

Hint: Deform the Bromwich contour around the branch points $s=\pm i\omega$, then show that the large contour at infinity and the small contours encircling the branch points are vanishingly small. The contributions on both sides of the cut add to give the desired result.

The branch points are $-\omega i$ and ωi. We take the corresponding cuts as $(-\infty i,-\omega i)$ and $(-\infty i,\omega i)$. The branches are chosen as follows:

Let $s-i\omega=r_1e^{i\theta_1}$, $s+i\omega=r_2e^{i\theta_2}$, with $-\pi/2<\theta_1,\theta_2<3\pi/2$. The two cuts, where coincident, cancel, leaving only the segment of the cut joining the two branch points. The Bromwich contour can now be deformed into a closed contour, traversed counterclockwise, that surrounds the cut and the two branch points (argue why). Show that the small circles encircling the branch points make vanishingly small contributions. That leaves just the two straight contours, L_1 going from $-i\omega$ to $i\omega$ to the right of the cut and L_2 from $i\omega$ to $-i\omega$ to the left of the cut.

On L_1, let $s=ir$. Then, $s-i\omega=(\omega-r)e^{-i\pi/2}$, while $s+i\omega=(\omega+r)e^{i\pi/2}$, where r varies from $-\omega$ to ω. We obtain $\sqrt{s^2+\omega^2}=\sqrt{\omega^2-r^2}$.

Similarly on L_2, let $s=ir$. Then, $s-i\omega=(\omega-r)e^{3i\pi/2}$ while $s+i\omega=(\omega+r)e^{i\pi/2}$, where r varies from ω to $-\omega$. We now obtain $\sqrt{s^2+\omega^2}=\sqrt{(\omega^2-r^2)e^{2\pi i}}=-\sqrt{\omega^2-r^2}$.

Therefore,

$$\frac{1}{2\pi i}\int_{L_1}\frac{e^{st}}{\sqrt{s^2+\omega^2}}\,ds=\frac{1}{2\pi i}\int_{-\omega}^{\omega}\frac{e^{irt}}{\sqrt{\omega^2-r^2}}i\,dr$$

and

$$\frac{1}{2\pi i}\int_{L_2}\frac{e^{st}}{\sqrt{s^2+\omega^2}}\,ds=-\frac{1}{2\pi i}\int_{-\omega}^{\omega}\frac{e^{irt}}{(-\sqrt{\omega^2-r^2})}i\,dr=\frac{1}{2\pi i}\int_{-\omega}^{\omega}\frac{e^{irt}}{\sqrt{\omega^2-r^2}}i\,dr.$$

On adding the two integrals, we get

$$\mathcal{L}^{-1}\frac{1}{\sqrt{s^2+\omega^2}}=\frac{1}{\pi}\int_{-\omega}^{\omega}\frac{e^{irt}}{\sqrt{\omega^2-r^2}}\,dr=\frac{2}{\pi}\int_0^{\omega}\frac{\cos rt}{\sqrt{\omega^2-r^2}}\,dr=\frac{2}{\pi}\int_0^1\frac{\cos(\omega\rho t)}{\sqrt{1-\rho^2}}\,d\rho.$$

4. Show that the inverse Laplace transform of the function

$$F(s)=\frac{\ln s}{s^2+\omega^2}$$

is given by

$$\frac{\pi}{2\omega}\cos\omega t+\frac{\ln\omega}{\omega}\sin\omega t-\int_0^{\infty}\frac{e^{-rt}}{r^2+\omega^2}\,dr.$$

Hint: Choose the branch cut along the negative real axis. Show that contributions from the contour at infinity and from the segment encircling the branch point are vanishingly small.

With the branch cut along the negative real axis, we choose the principal branch for the logarithm, i.e., for $s=re^{i\theta}$, $\ln s=\ln r+i\theta$, $-\pi<\theta<\pi$.

There are simple poles at $\pm i\omega$. As the contour is deformed into the keyhole contour around the branch cut, we must pick up the residues at the two poles.

The residue at $s=i\omega$ is

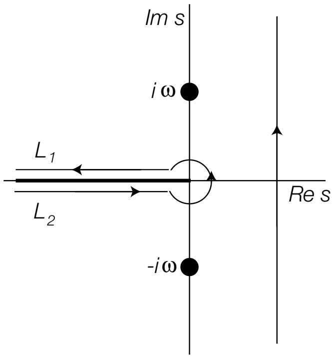
$$R_+=\frac{1}{2\pi i}\lim_{s\to i\omega}(s-i\omega)\frac{e^{st}\ln s}{s^2+\omega^2}=\frac{1}{2\pi i}\lim_{s\to i\omega}\frac{e^{st}\ln s}{s+i\omega}=\frac{1}{2\pi i}\frac{e^{i\omega t}[\ln\omega+i\pi/2]}{2i\omega}.$$

Similarly the residue at $s = -i\omega$ is

$$R_- = \frac{1}{2\pi i}\lim_{s\to -i\omega}(s+i\omega)\frac{e^{st}\ln s}{s^2+\omega^2} = \frac{1}{2\pi i}\lim_{s\to -i\omega}\frac{e^{st}\ln s}{s-i\omega} = \frac{1}{2\pi i}\frac{e^{-i\omega t}[\ln\omega - i\pi/2]}{(-2i\omega)}.$$

We then get

$$2\pi i(R_+ + R_-) = \frac{\pi}{2\omega}\cos\omega t + \frac{\ln\omega}{\omega}\sin\omega t.$$



On the keyhole contour, the contribution from the small circle around the branch point is zero (indicate why). On the straight segment L_1 of the contour above the cut,

$s = re^{i\pi} = -r$, $\ln s = \ln r + i\pi$, and r goes from 0 to ∞.

On the straight segment L_2 below the cut,

$s = re^{-i\pi} = -r$, $\ln s = \ln r - i\pi$, and r goes from ∞ to 0.

The sum of the integrals from the two segments is

$$\begin{aligned}\int_{L_1} + \int_{L_2} \frac{1}{2\pi i}\,\frac{e^{st}\ln s}{s^2+\omega^2}\,ds &= \frac{1}{2\pi i}\int_0^\infty \frac{e^{-rt}[\ln r + i\pi]}{\omega^2+r^2}\,(-dr)\\ &- \frac{1}{2\pi i}\int_0^\infty \frac{e^{-rt}[\ln r - i\pi]}{\omega^2+r^2}\,(-dr)\\ &= \frac{1}{2\pi i}\int_0^\infty \frac{e^{-rt}[-2i\pi]}{\omega^2+r^2}\,dr\\ &= -\int_0^\infty \frac{e^{-rt}}{\omega^2+r^2}\,dr\end{aligned}$$

The addition of all the terms leads to

$$\frac{\pi}{2\omega}\cos\omega t + \frac{\ln\omega}{\omega}\sin\omega t - \int_0^\infty \frac{e^{-rt}}{\omega^2+r^2}\,dr.$$