MATH-6640 **Spring 2005**

COMPLEX VARIABLES AND INTEGRAL TRANSFORMS WITH APPLICATIONS

Assignment 4

1. Use the properties of the Laplace transform, and the short table of transforms presented in class, to find the following:

   (a) Transform of
   $$\mathrm{Si}(t) = \int_0^t \frac{\sin u}{u}\, du,$$
   where $\mathrm{Si}(t)$ is the Sine Integral function which occurs in the study of optics.

   (b) The Laplace inverse of
   $$\frac{1}{(s+\omega_1)^2 + \omega_2^2}.$$

   (c) The Laplace inverse of
   $$\frac{e^{-5s}}{(s-3)^3}.$$

   (d) The Laplace inverse of
   $$\frac{s^2}{s^2+1},$$
   if it exists. (Why should there be a question?) Don't be hasty.

2. Consider the differential equation
   $$\frac{d^3y}{dt^3} + \omega^3 y = f(t), \quad \omega > 0, \quad y(0) = y'(0) = y''(0) = 0.$$

   (a) Show that the Laplace transform $Y(s)$ of the solution satisfies
   $$Y(s) = \frac{F(s)}{s^3+\omega^3},$$
   where $F(s)$ is the transform of $f(t)$.

   (b) Deduce that the inverse transform of
   $$Z(s) = \frac{1}{s^3+\omega^3}$$
   is given by
   $$z(t) = \frac{e^{-\omega t}}{3\omega^2} - \frac{2}{3\omega^2}\exp(\omega t/2)\cos\left(\frac{\sqrt{3}}{2}\omega t - \frac{\pi}{3}\right).$$
   How will the above result allow you to find a representation for $y(t)$?

3. Show that the inverse Laplace transform of the function
   $$F(s) = \frac{1}{\sqrt{s^2+\omega^2}}$$
   is given by
   $$f(t) = \frac{1}{\pi}\int_{-\omega}^{\omega} \frac{e^{itr}}{\sqrt{\omega^2 - r^2}}\, dr = \frac{2}{\pi}\int_0^1 \frac{\cos(\omega\rho t)}{\sqrt{1-\rho^2}}\, d\rho.$$
   Hint: Deform the Bromwich contour around the branch points $s = \pm i\omega$, then show that the large contour at infinity and the small contours encircling the branch points provide vanishingly small contributions. The contributions from both sides of the cut add to give the desired result.

4. Show that the inverse Laplace transform of the function

$$F(s) = \frac{\ln s}{s^2 + \omega^2}$$

is given by

$$\frac{\pi}{2\omega} \cos \omega t + \frac{\ln \omega}{\omega} \sin \omega t - \int_0^\infty \frac{e^{-rt}}{r^2 + \omega^2}\, dr.$$

Hint: Choose the branch cut along the negative real axis. Show that contributions from the contour at infinity and from the segment encircling the branch point are vanishingly small.