Introduction

Consider a vibrating membrane. We ask: what do the nodal lines tell us about the membrane?

This study is motivated by several experiments. One experiment begins with a homogeneous vibrating beam. The beam is excited at the second or higher natural frequency. The beam divides itself into parts separated by nodes. The nodal positions and the frequency are measured. Then mass is added to the beam in a small area. The nodal positions move toward the added mass and the corresponding natural frequency is lowered.

The second experiment is a vibrating plate which is excited at a natural frequency. Sand is sprinkled on the plate. It bounces away from points that vibrate and accumulates along nodal lines. The nodal lines divide the plate into domains which are either long and thin with large diameter or small, isolated regions. If we add mass to the plate, the natural frequencies are lowered. The nodal lines and nodal domains change; long thin domains break into isolated regions and some isolated regions join with other domains.

In this paper we consider a related two-dimensional problem. We have a homogeneous, rectangular membrane. We assume that there is a force acting on the membrane and that the force depends linearly on the displacement. We will show that the amplitude of the force, \( q \), is uniquely determined by a subset of the nodal lines, up to an additive constant. Letting \( \bar{q} \) be the average of \( q \), an algorithm for calculating \( q - \bar{q} \) is given. The algorithm determines \( q - \bar{q} \) at a dense set of points from the differences of pairs of eigenvalues. These eigenvalues are calculated for domains defined by the nodal lines.

Our results in this paper extend the one dimensional results of McLaughlin and Hald. [Mc], [HMc1], [HMc2]. There inverse nodal problems are defined; and it is shown that the one dimensional potential for the Sturm-Liouville problem with Dirichlet or mixed boundary conditions is uniquely determined, up to an additive constant, by a dense set of nodes. A crucial property in the analysis of the Sturm-Liouville problem is that the distance between consecutive eigenvalues becomes larger and larger. Since this property holds, perturbation theory can be applied. Asymptotic forms for sufficiently large eigenvalues, the corresponding eigenfunctions and nodal positions can be obtained. The uniqueness theorem follows.

In order to develop our two-dimensional results we must have a clear understanding of how the eigenvalues, i.e., the squares of the natural frequencies, and the corresponding
mode shapes of the membrane change as \( q \) changes. For the vibrating membranes, even for the rectangular membranes considered here, and even for rectangular membranes for which all the eigenvalues are distinct, arbitrarily large eigenvalues can be arbitrarily close together. This presents considerable difficulty as the arbitrarily small differences become small divisors in an asymptotic expansion, see Kato [K]. To overcome this difficulty, we begin with the \( q = 0 \) case and establish a dense, well defined set of rectangular membranes for which almost all of the eigenvalues are well separated. The membranes are chosen so that the square of the ratio of the sides is not well approximated by rational numbers. The condition we give is nearly the same as that given by Moser in [M]. After selecting the 'good' rectangles, we give three conditions that the eigenvalues must satisfy. We show that almost all eigenvalues are at least a given distance from their nearest neighbor and a large distance from a selected set of neighbors. We show that almost all eigenfunctions have roughly the same number of oscillations in the \( x \) and \( y \) directions. Then for the selected rectangles, we consider only those 'good' eigenvalue, eigenfunction pairs which have these properties.

Using the \( q = 0 \) case as our base problem, perturbation theory results are obtained for the \( q \neq 0 \) case. We establish leading order terms for the 'good' eigenvalues and eigenfunctions. Explicit estimates are given for the remainder terms for the eigenvalues and explicit \( L^\infty \) bounds are given for the remainder terms for the eigenfunctions. Our work here is influenced by perturbation results for the two and three dimensional periodic problem by Feldman, Knörer, and Trubowitz, [FKT], and Friedlander, [F2]. An essential difference between our work and the results of those authors is that [FKT] and [F2] contain order estimates for the \( L^2 \) bounds for the remainder terms for the eigenfunctions and order estimates for the remainder terms for the eigenvalues. For us, explicit bounds are necessary; and \( L^\infty \) bounds for the eigenfunctions are required so that we can determine the approximate location of the nodal lines in the \( q \neq 0 \) case.

Our main results establish the solution of the inverse nodal problem. Here we encounter the second major difficulty in our analysis. The nodal domains for eigenfunctions corresponding to 'good' eigenvalues for the \( q = 0 \) problem have small diameter. The nodal domains for the corresponding eigenfunctions for the \( q \neq 0 \) problem usually do not. We show that we can cut the large nodal domains into smaller approximate nodal domains; the cut is made along lines where the eigenfunction is small. The position of the cut is independent of \( q \). Having established this, we use the Rayleigh-Ritz formulation of the eigenvalue problem to establish the formula for approximating \( q - f \) at a dense set of points. Our perturbation theory results are used to establish bounds for the error in making the approximation. The uniqueness theorem then follows.