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Unique identifiability of elastic parameters from time-dependent interior displacement measurement

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Abstract
We consider the question: what can be determined about the stiffness distribution in biological tissue from indirect measurements? This leads us to consider an inverse problem for the identification of coefficients in the second-order hyperbolic system that models the propagation of elastic waves. The measured data for our inverse problem are the time-dependent interior vector displacements. In the isotropic case, we establish sufficient conditions for the unique identifiability of wave speeds and the simultaneous identifiability of both density and the Lamé parameters. In the anisotropic case, counterexamples are presented to exhibit the nonuniqueness and to show the structure of the set of shear tensors corresponding to the same given data.

1. Introduction

Elastography is a proposed imaging technique for human tissue. The goal is to extend the doctor's palpation exam (see [10, 30]), where fingers press against the skin to detect regions that are stiffer than normal tissue. To accomplish the goal three experiments have been proposed:

- static experiment: the tissue is compressed;
- dynamic sinusoidal excitation: a time harmonic excitation made on the boundary creates a time harmonic wave in the tissue;
- transient elastography: a short time-dependent pulse on the boundary creates a propagating wave in the tissue.

In each of these cases the *interior* displacement is measured on a fine grid of points using ultrasound [6, 9, 22, 29, 32] or magnetic resonance imaging [5, 19, 21]. The elastography problem then is to construct high resolution images of tissue stiffness characteristics from the measured displacement. This high resolution/high contrast image is expected for two

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reasons: (1) one is that *interior* measurements are used instead of boundary measurements; and (2) the shear wave speed can be substantially more than double in abnormal stiff tissue [12, 19, 23, 30, 31]. Some elegant reconstruction algorithms for transient elastography have also been proposed, among which we refer to [17, 27]. The former is based on the asymptotic expansion of geometric optics and the latter is based on using propagating fronts to recover wave speeds.

The purpose of this paper is to focus on unique identifiability for the transient elastography experiment. In this experiment the time- and space-dependent propagating wave has amplitudes of the order of microns [6, 29, 32]. Since stiffness is an elastic property and the wave amplitudes are small, the displacement satisfies the linear equations of elasticity. In this initial paper we also assume that the medium is isotropic. In this case then the relevant elastic properties are the density, ρ, the Lamé parameters, λ and μ, or the compression and shear waves speeds, $\sqrt{(\lambda+2\mu)/\rho}$ and $\sqrt{\mu/\rho}$, respectively. Furthermore in soft tissue the compression wave speed is approximately $1500\ \mathrm{m\ s^{-1}}$ while the shear wave speed in normal tissue is 1–3 $\mathrm{m\ s^{-1}}$. This large difference means that the compression wave has a very long wavelength with a much shorter shear wavelength in low frequency excitation experiments [6, 29]. This difference is used to argue that experiments can be designed where the shear wave displacement in the axial direction (normal to the tissue boundary) can be isolated and that an approximate mathematical model for this displacement is a scalar wave equation with stiffness coefficient, μ, density, ρ, and wave speed, $\sqrt{\mu/\rho}$. For this reason we consider both the scalar wave equation model and the linear equations of elasticity. In all cases we will assume that the medium begins at rest and that a displacement or a traction force on the boundary of the tissue initiates a wave that propagates into the tissue. A fundamental idea used in our analysis is that the wave has a propagating front.

Our uniqueness results are for the inverse problem: find elastic parameters from a single *interior* time-dependent scalar or vector displacement measurement. We will establish a series of uniqueness results for the elastic parameters in the region where the wave has propagated, that is in the region where the solution is nonzero for some time during the measurement period. More specifically two of our culminating results are:

(1) that there is at most one pair (ρ, μ) corresponding to a time-dependent solution of the scalar wave equation when μ is either given on the boundary or is determined from the boundary traction force;
(2) that there is at most one pair (ρ, μ) corresponding to a time-dependent vector solution of the linear equations of elasticity when λ/ρ is given throughout the tissue and λ is either given on the boundary or determined by the boundary traction force.

Note again that for all of our results we assume the homogeneous initial condition. In addition, we give examples to show that a single interior displacement measurement is not enough to establish a uniqueness theorem for the parameters in a general anisotropic medium.

To put our results in perspective, we describe some related results, referring to [3, 4, 7, 13–15, 25, 26, 28]: Imanuvilov and Yamamoto investigate, for the scalar wave equation, the identification problem of lower-order coefficients (usually referred to as potentials) [13] and, more recently, [14], they establish uniqueness and stability for the identification of μ using a Carleman estimate assuming $\rho \equiv 1$. But they need an *a priori* assumption for μ and a special type of initial condition which could be difficult to control in the transient elastography experiments. Note that the homogeneous initial condition that is assumed in this paper is rather natural from the experimental viewpoint, since the medium is at rest until the boundary excitation begins. Rachele establishes the unique identifiability of wave speeds [25] and the density [26] from the knowledge of the Dirichlet-to-Neumann map on the boundary for the linear equations of elasticity under the assumption that there are no caustics. Richter [28] shows the unique identifiability of μ in a steady state single elliptic equation $\nabla\mu \cdot \nabla u + \mu\Delta u = q$ for a known q and with the measured data u given. However, in his approach he requires *a priori* assumptions of the knowledge of μ on the inflow portion of the boundary and $\inf_{x\in\Omega} \max\{|\nabla u(x)|, \Delta u(x)\} > 0$ that is generally not true in the problem we investigate. See also Knowles [15] for an extension of Richter's assumptions and arguments for the determination of parameters in the aquifer identification problem and Cox and Gockenbach [7] for the simultaneous reconstruction of μ and λ. In [7], the authors extend Richter's argument to the two-dimensional static elastic system. Note also Barbone and Bamber [3] and Barbone and Gokhale [4] consider uniqueness and nonuniqueness issues in the *incompressible* elastography problem for the static and dynamic sinusoidal excitation cases.

Our paper is organized as follows. In section 2, the mathematical models of the forward problem are discussed. In section 3, using the propagating front we establish the *shrink and spread argument*, which says that the solution starting out as zero in a region and satisfying both the finite propagation speed (hyperbolic property) and the unique continuation (elliptic property) at each time slice in that region must be identically zero for all time. This property is a basic ingredient in our uniqueness proofs. In section 4, we establish the unique identifiability of wave speeds in isotropic media for both scalar and vector displacement cases. In section 5, the simultaneous identifiability is investigated both for the Dirichlet and the Neumann cases. For the Dirichlet case, an *a priori* specification of a certain elastic parameter on the boundary is required. In addition, in this section we present counterexamples that exhibit this boundary specification cannot be removed. In section 6, anisotropic media are considered. We present counterexamples that show the unique identification of any elastic parameters (even wave speeds) is impossible in general anisotropic media. Also the nonuniqueness structure is clarified.

Many issues remain for transient elastography. For example:

1. so far, one, or at most two, components of the three-dimensional elastic displacement can be measured;
2. soft tissue is nearly incompressible; and
3. real tissue can be anisotropic and our counterexamples suggest multiple measurements are needed.

Uniqueness issues for all these cases should be addressed and we will investigate these problems in future papers.

2. Mathematical model

Our forward problem is described by the following hyperbolic initial-boundary value problems which model the wave propagation in an elastic body. Throughout this paper, let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) be an open connected $\mathcal{C}^2$ domain and $T > 0$ be fixed.

2.1. Scalar shear displacement case

Assume that the density $\rho \in \mathcal{C}^0(\bar{\Omega})$ and the shear modulus $\mu \in \mathcal{C}^1(\bar{\Omega})$ satisfy $\rho(x), \mu(x) \geqslant \alpha_0 > 0$. Assume also that a scalar shear displacement u in an isotropic medium is governed by the following initial-boundary value problem:

$$\nabla \cdot (\mu(x)\nabla u(x,t)) = \rho(x)u_{tt}(x,t) \qquad \text{in } \Omega \times (0,T) \tag{2.1}$$

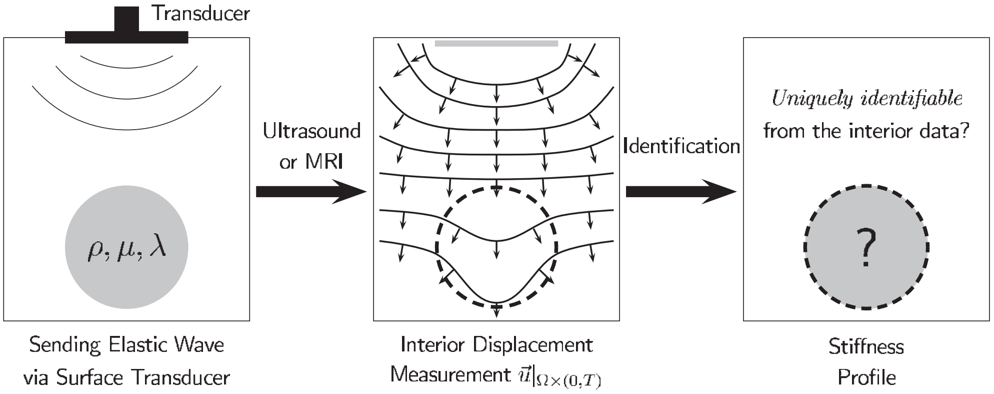


Figure 1. An illustrative diagram representing the inverse problem for the identification of the stiffness profile related to the elastic parameters ρ, μ and λ from the measurement of the time-dependent interior vector displacement $\vec{u}|_{\Omega\times(0,T)}$.

where the medium is initially at rest satisfying the homogeneous initial condition $u(x,0) = u_t(x,0) = 0$ on Ω and one of the following boundary conditions:

$$\begin{aligned} &u(x,t) = f(x,t) \qquad \text{on } \partial\Omega\times(0,T), \\ &\mu(x)\nabla u(x,t)\cdot\nu(x) = g(x,t) \qquad \text{on } \partial\Omega\times(0,T), \end{aligned} \tag{2.2}$$

where ν is the outward normal to $\partial\Omega$ and x is a point in $\mathbb{R}^n$ ($n = 2, 3$). It is well known (see [18, 20]) that there exists a unique solution $u \in H^2(\Omega\times(0,T))$ if the Dirichlet boundary condition $f \in H^{5/2}(\partial\Omega\times(0,T))$ satisfies some compatibility conditions such as $f(\cdot,0) = f_t(\cdot,0) = 0$. For the Neumann case, $g \in H^{3/2}(\partial\Omega\times(0,T))$ is required.

In an anisotropic medium, that will be considered in section 6, the shear modulus μ in (2.1) and (2.2) must be substituted by the symmetric positive-definite shear tensor $M \in [\mathcal{C}^1(\bar{\Omega})]^{n\times n}$.

2.2. *Vector displacement case in isotropic media*

Assume that the density $\rho \in \mathcal{C}^1(\bar{\Omega})$ and the Lamé parameters $\mu, \lambda \in \mathcal{C}^2(\bar{\Omega})$ satisfy $\rho(x), \mu(x), \lambda(x) \geqslant \alpha_0 > 0$. Then the vector elastic displacement $\vec{u}$ in an isotropic medium is governed by the following initial-boundary value problem:

$$\nabla(\lambda\nabla\cdot\vec{u}) + \nabla\cdot(\mu(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})) = \rho\vec{u}_{tt} \qquad \text{in } \Omega\times(0,T) \tag{2.3}$$

where again we assume the medium is initially at rest satisfying the homogeneous initial condition $\vec{u}(x,0) = \vec{u}_t(x,0) = 0$ on Ω. We assume also that one of the following boundary conditions:

$$\begin{aligned} &\vec{u}(x,t) = \vec{f}(x,t) \qquad \text{on } \partial\Omega\times(0,T), \\ &[(\lambda(x)\nabla\cdot\vec{u})I + \mu(x)(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})]\nu(x) = \vec{g}(x,t) \qquad \text{on } \partial\Omega\times(0,T), \end{aligned}$$

where ν is the outward normal to $\partial\Omega$, $(\cdot)^{\mathrm{T}}$ denotes the transpose of matrices and I is the identity matrix, is satisfied. Here $\nabla\cdot$ represents the divergence of vectors and the matrices according to the context, which will be rigorously defined in section 3.

Our inverse problem is to identify the elastic parameters ρ, μ, λ or some combination of them, such as the shear wave speed $c_{\mathrm{s}} := \sqrt{\mu/\rho}$ and/or the compression wave speed $c_{\mathrm{p}} := \sqrt{(\lambda + 2\mu)/\rho}$ from a single time-dependent interior displacement $u|_{\Omega\times(0,T)}$ in the scalar case or $\vec{u}|_{\Omega\times(0,T)}$ in the vector case (see figure 1 for an illustration).

3. Shrink and spread argument

In this section, we develop the *shrink and spread argument* which is a main tool for the proof of the unique identifiability in our inverse problem. Roughly speaking, this argument says that, in a given subregion, the solution that: (1) satisfies both hyperbolic and elliptic equations; and (2) is zero in that subregion for some time $t = t_0$, must vanish in that region for all $t \geqslant t_0$. We begin with some basic definitions.

Definition 3.1. *Let* A, B *be* 3×3 *matrices with* A_j *and* B_j *as the* jth *row vectors of* A *and* B*, respectively. Then the inner product, norm, cross product, divergence and curl for matrices are defined by* $A \cdot B = \mathrm{tr}(A^{\mathrm{T}}B)$*,* $|A|^2 = A \cdot A$ *and*

$$A \times B = \begin{pmatrix} A_2 \cdot B_3 - A_3 \cdot B_2 \\ A_3 \cdot B_1 - A_1 \cdot B_3 \\ A_1 \cdot B_2 - A_2 \cdot B_1 \end{pmatrix}, \qquad \nabla \cdot A = \begin{pmatrix} \nabla \cdot A_1 \\ \nabla \cdot A_2 \\ \nabla \cdot A_3 \end{pmatrix}, \qquad \nabla \times A = \begin{pmatrix} \nabla \times A_1 \\ \nabla \times A_2 \\ \nabla \times A_3 \end{pmatrix},$$

where $(\cdot)^{\mathrm{T}}$ *and* $\mathrm{tr}(\cdot)$ *denote the transpose and the trace of the matrices, respectively.*

For completeness and easy referral we give the following identities. For any function ψ, vector $\vec{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ and 3×3 matrix A which are all sufficiently smooth, we have the following:

$$\begin{aligned} &\nabla \cdot (A\vec{u}) = \nabla \cdot (A^{\mathrm{T}}) \cdot \vec{u} + A^{\mathrm{T}} \cdot \nabla\vec{u}, \\ &\nabla \times (A\vec{u}) = (\nabla \times (A^{\mathrm{T}}))^{\mathrm{T}}\vec{u} + \nabla\vec{u}^{\mathrm{T}} \times A, \\ &\nabla \times (\psi\vec{u}) = \nabla\psi \times \vec{u} + \psi\nabla \times \vec{u}, \\ &\nabla \times (\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}) = (\nabla(\nabla \times \vec{u}))^{\mathrm{T}}, \\ &\nabla \times \nabla \times \vec{u} = -\Delta\vec{u} + \nabla(\nabla \cdot \vec{u}), \end{aligned} \tag{3.1}$$

where $\nabla\vec{u}$ denotes the Jacobian matrix of $\vec{u}$ and $\Delta\vec{u} = \nabla \cdot \nabla\vec{u}$ is the Laplacian of $\vec{u}$.

One of the most important properties of a solution of a hyperbolic equation is that it has a finite propagation speed. An important property of a solution of an elliptic equation is the unique continuation principle. Both of these ideas will be important in our uniqueness proofs. Here we give a rigorous definition of these two notions.

Definition 3.2. *Let* $B_\epsilon(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < \epsilon\} \subset \Omega$ *be an open ball in* Ω.

(a) $\vec{U} = (U_1, \ldots, U_m) \in [H^2_{\mathrm{loc}}(\Omega \times (0, T))]^m$ *is said to have a finite propagation speed in* $B_\epsilon(x_0) \times (0, T)$ *with the maximum speed* $c > 0$ *if for any* $t_0 \in [0, T)$, $\vec{U}(\cdot, t_0) = \vec{U}_t(\cdot, t_0) = 0$ *in* $B_\epsilon(x_0)$ *implies that* $\vec{U} = 0$ *a.e. in a space–time cone* $\bigcup_{0<s<\epsilon/c} C_{\mathrm{s}}$*, where* $C_{\mathrm{s}} = C_{\mathrm{s}}(x_0, t_0, \epsilon, c) := B_{\epsilon - cs}(x_0) \times \{t = t_0 + s\}$.

(b) $\vec{U} = (U_1, \ldots, U_m) \in [H^1_{\mathrm{loc}}(\Omega)]^m$ *is said to have a unique continuation principle in* Ω *if* $\vec{U} = 0$ *in an open subset of* Ω *implies that* $\vec{U} = 0$ *in* Ω.

In section 3.3, an important ingredient in the proofs will be that these two properties hold simultaneously.

3.1. Finite propagation speed

Since the displacement u or $\vec{u}$ is a solution of the hyperbolic equation (2.1) or (2.3), the propagation speed must be finite, as shown in the following two theorems.

Theorem 3.3. *Assume that* $\rho \in \mathcal{C}^0(\bar{\Omega})$ *and* $\mu, \lambda \in \mathcal{C}^1(\bar{\Omega})$ *satisfy* $\rho(x), \mu(x), \lambda(x) \geqslant \alpha_0 > 0$*. Let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a solution of the hyperbolic system*

$$\nabla(\lambda\nabla \cdot \vec{u}) + \nabla \cdot (\mu(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})) = \rho\vec{u}_{tt} \qquad \text{in } \Omega \times (0, T). \tag{3.2}$$

Then for any open ball $B_\epsilon(x_0) \subset \Omega$, $\vec{u}$ *has a finite propagation speed in* $B_\epsilon(x_0) \times (0, T)$ *with the maximum speed* $c = \sup_{x \in B_\epsilon(x_0)} \sqrt{(\lambda(x) + 2\mu(x))/\rho(x)}$.

Proof. Fix any $t_0 \in [0, T)$ and assume that

$$\vec{u}(\cdot, t_0) = \vec{u}_t(\cdot, t_0) = 0 \qquad \text{in } B_\epsilon(x_0). \tag{3.3}$$

We must show that $\vec{u} = 0$ a.e. in $\bigcup_{0<s<\epsilon/c} C_{\mathrm{s}}$, where $C_{\mathrm{s}} = B_{\epsilon - cs}(x_0) \times \{t = t_0 + s\}$. Let

$$e(s) := \frac{1}{2} \int_{C_{\mathrm{s}}} \left\{ \rho |\vec{u}_t|^2 + \lambda |\nabla \cdot \vec{u}|^2 + \frac{\mu}{2} |\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}|^2 \right\} \mathrm{d}x$$

represent the elastic energy contained in C_{s}. We will show that $e(s) = 0$ for all $s \in (0, \epsilon/c)$.

For a fixed $s \in (0, \epsilon/c)$ define $\Lambda(s) := \bigcup_{0<\tau<s} C_\tau$. Taking the inner product of (3.2) with $\vec{u}_t$ and integrating in $\Lambda(s)$, using the first identity in (3.1) we get

$$\begin{aligned} 0 &= \int_{\Lambda(s)} \vec{u}_t \cdot \{\rho \vec{u}_{tt} - \nabla \cdot ((\lambda \nabla \cdot \vec{u}) I + \mu(\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}))\} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\Lambda(s)} \left\{ \left(\frac{\rho}{2} |\vec{u}_t|^2 \right)_t + \Sigma \cdot \nabla \vec{u}_t - \nabla \cdot (\Sigma \vec{u}_t) \right\} \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

where $\Sigma = (\lambda \nabla \cdot \vec{u}) I + \mu(\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}})$. Since $\mu(\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}})$ is symmetric, we have

$$\begin{aligned} \Sigma \cdot \nabla \vec{u}_t &= (\lambda \nabla \cdot \vec{u}) I \cdot \nabla \vec{u}_t + \frac{\mu}{2} (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}) \cdot (\nabla \vec{u}_t + \nabla \vec{u}_t^{\mathrm{T}}) \\ &= \left(\frac{\lambda}{2} |\nabla \cdot \vec{u}|^2 + \frac{\mu}{4} |\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}|^2 \right)_t. \end{aligned}$$

Thus we have

$$0 = \frac{1}{2} \int_{\Lambda(s)} \left\{ \left(\rho |\vec{u}_t|^2 + \lambda |\nabla \cdot \vec{u}|^2 + \frac{\mu}{2} |\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}|^2 \right)_t - 2 \nabla \cdot (\Sigma \vec{u}_t) \right\} \mathrm{d}x \, \mathrm{d}t. \tag{3.4}$$

Applying the space–time divergence theorem, we have

$$0 = \frac{1}{2} \int_{\partial \Lambda(s)} \left\{ \left(\rho |\vec{u}_t|^2 + \lambda |\nabla \cdot \vec{u}|^2 + \frac{\mu}{2} |\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}|^2 \right) \nu_t - 2 (\Sigma \vec{u}_t) \cdot \nu_x \right\} \mathrm{d}S_{x,t}, \tag{3.5}$$

where $\partial \Lambda(s) = C_{\mathrm{s}} \cup C_0 \cup (\cup_{0<\tau<s} \partial C_\tau)$ is the space–time boundary of $\Lambda(s)$, $\mathrm{d}S_{x,t}$ is the space–time boundary element and (ν_x, ν_t) is the space–time outward normal to $\partial \Lambda(s)$. Since (ν_x, ν_t) is explicitly given by

$$(\nu_x, \nu_t) = \begin{cases} (0, 1) & \text{on } C_{\mathrm{s}}, \\ (0, -1) & \text{on } C_0, \\ \dfrac{1}{\sqrt{1 + c^2}} \left(\dfrac{x - x_0}{|x - x_0|}, c \right) & \text{on } L := \bigcup_{0<\tau<s} \partial C_\tau, \end{cases} \tag{3.6}$$

equation (3.5) can be rewritten as

$$e(s) - e(0) = \frac{-c}{2\sqrt{1 + c^2}} \int_L \left\{ \rho |\vec{u}_t|^2 + \lambda |\nabla \cdot \vec{u}|^2 + \frac{\mu}{2} |\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}|^2 - \frac{2}{c} \Sigma \vec{u}_t \cdot \frac{x - x_0}{|x - x_0|} \right\} \mathrm{d}S_{x,t}. \tag{3.7}$$

We know that $e(s) \geqslant 0$ and $e(0) = 0$. Our intermediate goal is then to show that the right-hand side of (3.7) is not positive, implying that $e(s) \equiv 0$. To do this we first use the

Cauchy–Schwarz inequality and the fact that $c \geqslant \sqrt{(\lambda + 2\mu)/\rho}$. Then the integrand of the right-hand side of (3.7) is greater than or equal to

$$\begin{aligned}
&\rho|\vec{u}_t|^2 + \lambda|\nabla\cdot\vec{u}|^2 + \frac{\mu}{2}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2 - \frac{2}{c}\lambda|\nabla\cdot\vec{u}||\vec{u}_t| - \frac{2\mu}{c}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}||\vec{u}_t| \\
&\quad = \lambda\left(\frac{\rho}{\lambda + 2\mu}|\vec{u}_t|^2 + |\nabla\cdot\vec{u}|^2 - \frac{2}{c}|\nabla\cdot\vec{u}||\vec{u}_t|\right) \\
&\qquad + 2\mu\left(\frac{\rho}{\lambda + 2\mu}|\vec{u}_t|^2 + \frac{1}{4}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2 - \frac{1}{c}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}||\vec{u}_t|\right) \\
&\quad \geqslant \lambda\left(\sqrt{\frac{\rho}{\lambda + 2\mu}}|\vec{u}_t| - |\nabla\cdot\vec{u}|\right)^2 + 2\mu\left(\sqrt{\frac{\rho}{\lambda + 2\mu}}|\vec{u}_t| - \frac{1}{2}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|\right)^2 \geqslant 0.
\end{aligned}$$

Thus we have $0 \leqslant e(s) \leqslant e(0) = 0$, implying $e(s) = 0$ for all $s \in (0, \epsilon/c)$.

Finally we use a standard argument to show that $\vec{u} = 0$ a.e. With the lower bound α_0 for ρ, μ, and λ, we get the following L^2-estimate for $\vec{u}_t$ in the cone $\Lambda(\epsilon/c) = \bigcup_{0<s<\epsilon/c} C_{\mathrm{s}}$:

$$\|\vec{u}_t\|^2_{L^2(\Lambda(\epsilon/c))} \leqslant \int_{\Lambda(\epsilon/c)} \frac{\rho}{\alpha_0}|\vec{u}_t|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{2}{\alpha_0}\int_0^{\epsilon/c} e(s)\,\mathrm{d}s = 0.$$

The homogeneous initial condition (3.3) then implies $\vec{u}(x, t) = 0$ a.e. in $\Lambda(\epsilon/c)$, which completes the proof. □

For the scalar shear displacement u, we also have a finite propagation speed. Since the proof is parallel to theorem 3.3, we give only the outline of the proof.

Theorem 3.4. *Assume that* $\rho \in C^0(\bar{\Omega})$ *and* $\mu \in C^1(\bar{\Omega})$ *satisfy* $\mu(x), \rho(x) \geqslant \alpha_0 > 0$. *Let* $u \in H^2(\Omega \times (0, T))$ *be a solution to the hyperbolic equation*

$$\nabla\cdot(\mu(x)\nabla u(x, t)) = \rho(x)u_{tt}(x, t) \qquad \text{in } \Omega \times (0, T). \tag{3.8}$$

Then for any open ball $B_\epsilon(x_0) \subset \Omega$, u *has a finite propagation speed in* $B_\epsilon(x_0) \times (0, T)$ *with the maximum speed* $c = \sup_{x\in B_\epsilon(x_0)} \sqrt{\mu(x)/\rho(x)}$.

Proof. Fix any $t_0 \in [0, T)$ and assume that $u(\cdot, t_0) = u_t(\cdot, t_0) = 0$ in $B_\epsilon(x_0)$. Then we must show that $u = 0$ a.e. in $\bigcup_{0<s<\epsilon/c} C_{\mathrm{s}}$ where $C_{\mathrm{s}} = B_{\epsilon-cs}(x_0) \times \{t = t_0 + s\}$. Let

$$e(s) := \tfrac{1}{2}\int_{C_{\mathrm{s}}} \{\rho|u_t|^2 + \mu|\nabla u|^2\}\,\mathrm{d}x.$$

Analogously as in the proof of theorem 3.3, taking the inner product of (3.8) with u_t and integrating in $\Lambda(s)$, we get

$$0 = \tfrac{1}{2}\int_{\Lambda(s)} \{(\rho|u_t|^2 + \mu|\nabla u|^2)_t - 2\nabla\cdot(\mu u_t\nabla u)\}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.9}$$

To show that $e(s) = 0$ and then that $u = 0$ a.e. we again apply the space–time divergence theorem. Using the explicit form (3.6) of the outward normal, we have

$$e(s) - e(0) = \frac{-c}{2\sqrt{1 + c^2}}\int_{\bigcup_{0<\tau<s}\partial C_\tau} \left\{\rho|u_t|^2 + \mu|\nabla u|^2 - \frac{2\mu}{c}u_t\nabla u\cdot\frac{x - x_0}{|x - x_0|}\right\}\mathrm{d}S_{x,t}. \tag{3.10}$$

Since $\sqrt{\mu}/c \leqslant \sqrt{\rho}$, the integrand of (3.10) is greater than or equal to

$$\rho|u_t|^2 + \mu|\nabla u|^2 - 2\sqrt{\rho\mu}|u_t||\nabla u| = (\sqrt{\rho}|u_t| - \sqrt{\mu}|\nabla u|)^2 \geqslant 0.$$

Thus we have $0 \leqslant e(s) \leqslant e(0) = 0$ for all $s \in (0, \epsilon/c)$. Hence it follows immediately as in the proof of theorem 3.3 that $u(x, t) = 0$ a.e. in $\Lambda(\epsilon/c)$. □

3.2. Unique continuation principle

It is well known that an elliptic system with the same principal part has a unique continuation principle. For completeness we state this in the following lemma. It is easily proved by a standard Carleman estimate [11, 24] as in the single elliptic equation case.

Lemma 3.5. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\vec{U} = (U_1, \ldots, U_m) \in [H^1_{\mathrm{loc}}(\Omega)]^m$ *be a solution of*

$$\Delta \vec{U} + B(\nabla \vec{U}) + V(\vec{U}) = 0$$

in the distributional sense, where the lower-order operators B *and* V *are given by*

$$[B(\nabla \vec{U})]_k = \sum_{i=1}^{n} \sum_{j=1}^{m} a^k_{ij}(x) \frac{\partial U_j}{\partial x_i} \qquad \text{and} \qquad V(\vec{U})_k = \sum_{j=1}^{m} b^k_j(x) U_j \tag{3.11}$$

with coefficients $a^k_{ij}, b^k_j \in L^\infty(\Omega)$ *for* $k = 1, \ldots, m$*. Then* $\vec{U}$ *has a unique continuation principle in* Ω*.*

In order to prove the unique identifiability of elastic properties in our inverse problem, it is natural to begin by supposing that our solution $\vec{u}$ solves two elastic systems (2.3), each with distinct coefficients. Under this assumption we can show by subtracting that $\vec{u}$ satisfies a system of differential equations without a $\vec{u}_{tt}$ term. This is shown in the following lemma.

Lemma 3.6. *Assume that* $\rho_j \in \mathcal{C}^1(\bar{\Omega})$ *and* $\mu_j, \lambda_j \in \mathcal{C}^2(\bar{\Omega})$ *for* $j = 1, 2$ *satisfy* $\rho_j(x), \mu_j(x), \lambda_j(x) \geqslant \alpha_0 > 0$*. Let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a common solution for* $j = 1, 2$ *to the hyperbolic system*

$$\nabla(\lambda_j \nabla \cdot \vec{u}) + \nabla \cdot (\mu_j (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}})) = \rho_j \vec{u}_{tt} \qquad \text{in } \Omega \times (0, T). \tag{3.12}$$

Then $(\vec{u}, v, \vec{\omega}) := (\vec{u}, \nabla \cdot \vec{u}, \nabla \times \vec{u})$ *satisfies the following system:*

$$0 = \left\langle \frac{\mu}{\rho} \right\rangle \Delta \vec{u} + \left\langle \frac{\lambda + \mu}{\rho} \right\rangle \nabla v + \left\langle \frac{\nabla \lambda}{\rho} \right\rangle v + (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}) \left\langle \frac{\nabla \mu}{\rho} \right\rangle, \tag{3.13}$$

$$\begin{aligned} 0 = {} & \left\langle \frac{\lambda + 2\mu}{\rho} \right\rangle \Delta v + \left\langle \frac{1}{\lambda + 2\mu} \nabla \left(\frac{(\lambda + 2\mu)^2}{\rho} \right) \right\rangle \cdot \nabla v + \left(\nabla \cdot \left\langle \frac{\nabla \lambda}{\rho} \right\rangle \right) v \\ & - \left\langle \frac{1}{\mu} \nabla \left(\frac{\mu^2}{\rho} \right) \right\rangle \cdot \nabla \times \vec{\omega} + (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}) \cdot \left\langle \nabla \left(\frac{\nabla \mu}{\rho} \right) \right\rangle, \end{aligned} \tag{3.14}$$

$$\begin{aligned} 0 = {} & \left\langle \frac{\mu}{\rho} \right\rangle \Delta \vec{\omega} + \left\langle \nabla \left(\frac{\mu}{\rho} \right) \right\rangle \times (2\nabla v - \nabla \times \vec{\omega}) + \left\langle \lambda \nabla \left(\frac{1}{\rho} \right) \right\rangle \times \nabla v \\ & + \left\langle \nabla \left(\frac{1}{\rho} \right) \times \nabla \lambda \right\rangle v + \nabla \vec{\omega} \left\langle \frac{\nabla \mu}{\rho} \right\rangle + \left\langle \nabla \left(\frac{\nabla \mu}{\rho} \right)^{\mathrm{T}} \right\rangle \times (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}), \end{aligned} \tag{3.15}$$

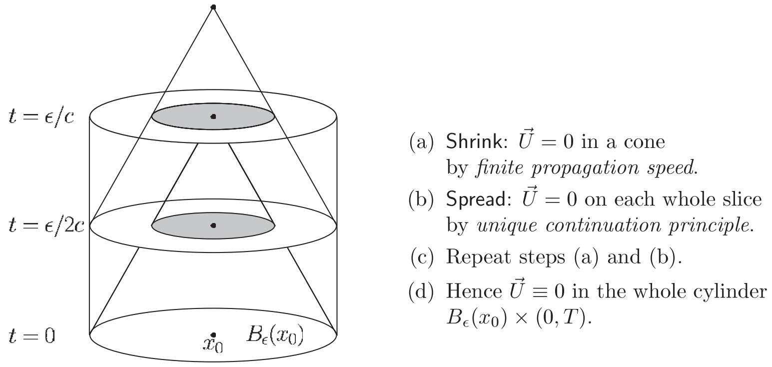
where $\langle \mathcal{F} \rangle := \mathcal{F}_1 - \mathcal{F}_2$ *for any indexed quantity* $\mathcal{F}_j$*. In the two-dimensional case, we identify* $\vec{u} = (u_1(x_1, x_2), u_2(x_1, x_2))^{\mathrm{T}}$ *by* $\vec{u} = (u_1(x_1, x_2), u_2(x_1, x_2), 0)^{\mathrm{T}}$ *so that all the above quantities are meaningful.*

Proof. From (3.12), using the fact that $\nabla \cdot (\nabla \vec{u}^{\mathrm{T}}) = \nabla v$ we easily get

$$\vec{u}_{tt} = \frac{\mu_j}{\rho_j} \Delta \vec{u} + \frac{\lambda_j + \mu_j}{\rho_j} \nabla v + \frac{\nabla \lambda_j}{\rho_j} v + (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}) \frac{\nabla \mu_j}{\rho_j}. \tag{3.16}$$

Taking the divergence of (3.16) and using the first and last identities in (3.1), we have

$$\begin{aligned} v_{tt} = {} & \frac{\lambda_j + 2\mu_j}{\rho_j} \Delta v + \frac{1}{\lambda_j + 2\mu_j} \nabla \left(\frac{(\lambda_j + 2\mu_j)^2}{\rho_j} \right) \cdot \nabla v + \nabla \cdot \left(\frac{\nabla \lambda_j}{\rho_j} \right) v \\ & - \frac{1}{\mu_j} \nabla \left(\frac{\mu_j^2}{\rho_j} \right) \cdot \nabla \times \vec{\omega} + (\nabla \vec{u} + \nabla \vec{u}^{\mathrm{T}}) \cdot \nabla \left(\frac{\nabla \mu_j}{\rho_j} \right). \end{aligned} \tag{3.17}$$



(a) Shrink: $\vec{U} = 0$ in a cone by *finite propagation speed.*
(b) Spread: $\vec{U} = 0$ on each whole slice by *unique continuation principle.*
(c) Repeat steps (a) and (b).
(d) Hence $\vec{U} \equiv 0$ in the whole cylinder $B_\epsilon(x_0) \times (0, T)$.

Figure 2. Illustration for the shrink and spread argument in theorem 3.8.

Taking the curl to (3.16) and using the identities (3.1), we have

$$\vec{\omega}_{tt} = \frac{\mu_j}{\rho_j}\Delta\vec{\omega} + \nabla\left(\frac{\mu_j}{\rho_j}\right) \times (2\nabla v - \nabla \times \vec{\omega}) + \nabla\left(\frac{1}{\rho_j}\right) \times \nabla(\lambda_j v) + \nabla\vec{\omega}\frac{\nabla\mu_j}{\rho_j} + \nabla\left(\frac{\nabla\mu_j}{\rho_j}\right)^{\mathrm{T}} \times (\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}). \tag{3.18}$$

Subtracting indexed equations in (3.16)–(3.18), respectively, (3.13)–(3.15) are easily obtained. □

If the leading coefficients $\langle\mu/\rho\rangle$ and $\langle\lambda + 2\mu/\rho\rangle$ in (3.13)–(3.15) are both away from zero, then we can establish a unique continuation principle, as was done in [1, 2, 8].

Theorem 3.7. *Assume that* $\rho_j \in \mathcal{C}^1(\bar{\Omega})$ *and* $\mu_j, \lambda_j \in \mathcal{C}^2(\bar{\Omega})$ *for* $j = 1, 2$ *satisfy* $\rho_j(x), \mu_j(x), \lambda_j(x) \geqslant \alpha_0 > 0$. *Let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a common solution for* $j = 1, 2$ *to the hyperbolic equations*

$$\nabla(\lambda_j \nabla \cdot \vec{u}) + \nabla \cdot (\mu_j(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})) = \rho_j \vec{u}_{tt} \qquad \text{in } \Omega \times (0, T).$$

Then in any open subset $D \subset \Omega$ *satisfying* $\min_D\{|\langle\mu/\rho\rangle|, |\langle\lambda + 2\mu/\rho\rangle|\} \geqslant \beta_0 > 0$, $\vec{U} := (\vec{u}, \nabla \cdot \vec{u}, \nabla \times \vec{u})$ *satisfies* $\Delta\vec{U} + B(\nabla\vec{U}) + V(\vec{U}) = 0$ *in* $D \times (0, T)$*, where B and V with coefficients* $a_{ij}^k, b_j^k \in L^\infty(D)$ *are defined in (3.11). Hence* $\vec{u}(\cdot, t_0)$ *has a unique continuation principle in D for any* $t_0 \in (0, T)$.

Proof. Since $\rho_j(x), \mu_j(x), \lambda_j(x) \geqslant \alpha_0 > 0$, $\rho_j \in \mathcal{C}^1(\bar{D})$ and $\mu_j, \lambda_j \in \mathcal{C}^2(\bar{D})$ for $j = 1, 2$, all the coefficients in (3.13)–(3.15) are $L^\infty(D)$. Since $|\langle\mu/\rho\rangle|, |\langle\lambda + 2\mu/\rho\rangle| \geqslant \beta_0 > 0$ in D, we can rewrite (3.13)–(3.15) as $\Delta\vec{U} + B(\nabla\vec{U}) + V(\vec{U}) = 0$ with $L^\infty(D)$ coefficients. Hence lemma 3.5 completes the proof. □

3.3. Shrink and spread argument

Now we are in a position to state the *shrink and spread argument*. As illustrated in figure 2, in any region where a solution (1) has a homogeneous initial condition, (2) has a finite propagation speed and (3) also satisfies a unique continuation principle should vanish for all time.

Theorem 3.8. *Let* $\vec{U} = (U_1, \ldots, U_m) \in [H^2(\Omega \times (0, T))]^m$ *and* $B_\epsilon(x_0) \subset \Omega$. *Assume that* $\vec{U}$ *satisfies the following assumptions:*

(a) $\vec{U}$ *has a homogeneous initial condition in the sense of a trace on* $B_\epsilon(x_0) \times \{t = 0\}$*:*

$$\vec{U}(\cdot, 0) = \vec{U}_t(\cdot, 0) = 0 \qquad \text{in } B_\epsilon(x_0).$$

(b) $\vec{U}$ *has a finite propagation speed in* $B_\epsilon(x_0) \times (0, T)$ *with the maximum speed* $c > 0$.

(c) For any $t_0 \in (0, T)$*, the trace* $\vec{U}(\cdot, t_0)$ *on* $B_\epsilon(x_0) \times \{t = t_0\}$ *has a unique continuation principle in* $B_\epsilon(x_0)$.

Then

$$\vec{U} \equiv 0 \qquad \text{in } B_\epsilon(x_0) \times (0, T).$$

Proof. By (a) and (b), we get $\vec{U} = 0$ in $\bigcup_{0<s<\epsilon/c} C_{\mathrm{s}}$, where $C_{\mathrm{s}} = B_{\epsilon - cs}(x_0) \times \{t = s\}$. Then using (c), we have $\vec{U} = 0$ in $B_\epsilon(x_0) \times (0, \epsilon/c)$. Applying (b) with

$$\vec{U}\left(x, \frac{\epsilon}{2c}\right) = \vec{U}_t\left(x, \frac{\epsilon}{2c}\right) = 0 \qquad \text{in } B_\epsilon(x_0)$$

as an initial condition, and (c) again, we get $\vec{U} = 0$ in $B_\epsilon(x_0) \times (0, 3\epsilon/2c)$. Iterating such procedures, we finally obtain $\vec{U} \equiv 0$ in $B_\epsilon(x_0) \times (0, T)$. □

4. Uniqueness of wave speeds in isotropic media

In this section we give our first set of uniqueness results for identifying wave speeds from interior displacement data. We begin with the scalar shear displacement case and show that the shear wave speed $c_{\mathrm{s}} = \sqrt{\mu/\rho}$ is uniquely identified from the interior displacement in any subregion where $u \neq 0$ for some time $t \in (0, T)$. The proof is based on our shrink and spread argument.

Theorem 4.1. *Assume that* $\rho_j \in \mathcal{C}^0(\bar{\Omega})$ *and* $\mu_j \in \mathcal{C}^1(\bar{\Omega})$ *for* $j = 1, 2$ *satisfy* $\rho_j(x), \mu_j(x) \geqslant \alpha_0 > 0$*. Let* $u \in H^2(\Omega \times (0, T))$ *be a common solution for* $j = 1, 2$ *to the hyperbolic equations*

$$\nabla \cdot (\mu_j(x) \nabla u(x, t)) = \rho_j(x) u_{tt}(x, t) \qquad \text{in } \Omega \times (0, T) \tag{4.1}$$

with the homogeneous initial condition

$$u(x, 0) = u_t(x, 0) = 0 \qquad \text{in } \Omega, \tag{4.2}$$

and satisfying either the same Dirichlet boundary condition

$$u(x, t) = f(x, t) \qquad \text{on } \partial\Omega \times (0, T), \tag{4.3}$$

or the same Neumann boundary condition

$$\mu_j(x) \nabla u(x, t) \cdot \nu(x) = g(x, t) \qquad \text{on } \partial\Omega \times (0, T), \tag{4.4}$$

where ν *is the outward normal to* $\partial\Omega$*. Then we have*

$$\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \qquad \text{in } \Omega \setminus \Omega_E,$$

where $\Omega_E := \bigcup\{V \subset \Omega$ *is an open set satisfying* $\|u\|_{L^2(V \times (0,T))} = 0\}$.

Remark. Intuitively, Ω_E is the subset of Ω where the wave has not yet travelled during the time $(0, T)$. See remark 4.5 for more details.

Proof. Let Ω be expressed by the union of disjoint subsets $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$, where

$$\Omega^0 := \{x \in \Omega : \mu_1(x)/\rho_1(x) = \mu_2(x)/\rho_2(x)\},$$
$$\Omega^\pm := \{x \in \Omega : \mu_1(x)/\rho_1(x) \gtrless \mu_2(x)/\rho_2(x)\}.$$

We will show that $\Omega^+ \cup \Omega^- \subset \Omega_E$: fix any point $x_0 \in \Omega^+$. Since $\mu_1/\rho_1 - \mu_2/\rho_2 \in \mathcal{C}^0(\bar{\Omega})$, and Ω^+ is an open subset of Ω, there exists an open ball $B_\epsilon(x_0) \subset \Omega^+$ on which we have

$$\alpha_1 \leqslant \frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2} \leqslant \alpha_2 \qquad \text{for some } \alpha_1, \alpha_2 > 0. \tag{4.5}$$

By theorem 3.4, u already has a finite propagation speed in $B_\epsilon(x_0) \times (0, T)$ with the maximum speed $c = \sup_{x \in B_\epsilon(x_0)} \sqrt{\mu_1(x)/\rho_1(x)}$. Multiplying (4.1) by $1/\rho_j$ and subtracting one from the other, we get that the trace $u(\cdot, t_0)$ for any $t_0 \in (0, T)$ solves the following elliptic equation:

$$\Delta u(x, t_0) + \left(\frac{\mu_1(x)}{\rho_1(x)} - \frac{\mu_2(x)}{\rho_2(x)}\right)^{-1} \left(\frac{\nabla\mu_1(x)}{\rho_1(x)} - \frac{\nabla\mu_2(x)}{\rho_2(x)}\right) \cdot \nabla u(x, t_0) = 0 \qquad \text{in } B_\epsilon(x_0).$$

From the Sobolev theory we get that $u(\cdot, t_0) \in H^{3/2}(B_\epsilon(x_0)) \subset H^1(B_\epsilon(x_0))$ for any fixed $t_0 \in (0, T)$. In addition, the smoothness assumptions on ρ_j and μ_j imply that all the coefficients in the above equation are in $L^\infty(B_\epsilon(x_0))$. Thus by lemma 3.5, $u(\cdot, t_0)$ has a unique continuation principle in $B_\epsilon(x_0)$ for any $t_0 \in (0, T)$. Finally, by theorem 3.8 using the homogeneous initial condition (4.2) we have $u \equiv 0$ in $B_\epsilon(x_0) \times (0, T)$. Hence $x_0 \in B_\epsilon(x_0) \subset \Omega_E$ and thus $\Omega^+ \subset \Omega_E$. Similarly we have $\Omega^- \subset \Omega_E$, implying that $\Omega \setminus \Omega_E \subset \Omega \setminus (\Omega^+ \cup \Omega^-) = \Omega^0$, which completes the proof. □

Now assume that the measured data is the vector displacement that satisfies the system of linear equations of elasticity. If λ/ρ is also given, then as in the scalar shear displacement case, the shear wave speed $c_s = \sqrt{\mu/\rho}$ is uniquely identified from the interior displacement $\vec{u}|_{\Omega \times (0,T)}$ in any subregion where $\vec{u} \neq 0$ for some time $t \in (0, T)$. Since the proof is parallel to that of theorem 4.1, only the outline is presented.

Theorem 4.2. *Assume that* $\rho_j \in \mathcal{C}^1(\bar{\Omega})$ *and* $\mu_j, \lambda_j \in \mathcal{C}^2(\bar{\Omega})$ *for* $j = 1, 2$ *satisfy* $\rho_j(x), \mu_j(x), \lambda_j(x) \geqslant \alpha_0 > 0$. *Let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a common solution for* $j = 1, 2$ *to the hyperbolic equations*

$$\nabla(\lambda_j \nabla \cdot \vec{u}) + \nabla \cdot (\mu_j(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})) = \rho_j \vec{u}_{tt} \qquad \text{in } \Omega \times (0, T) \tag{4.6}$$

with the homogeneous initial condition

$$\vec{u}(x, 0) = \vec{u}_t(x, 0) = 0 \qquad \text{in } \Omega, \tag{4.7}$$

and satisfying either the same Dirichlet boundary condition

$$\vec{u}(x, t) = \vec{f}(x, t) \qquad \text{on } \partial\Omega \times (0, T), \tag{4.8}$$

or the same Neumann boundary condition

$$[(\lambda_j(x)\nabla \cdot \vec{u})I + \mu_j(x)(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})]\nu(x) = \vec{g}(x, t) \qquad \text{on } \partial\Omega \times (0, T), \tag{4.9}$$

where ν *is the outward normal to* $\partial\Omega$. *If* $\lambda_1/\rho_1 = \lambda_2/\rho_2$ *in* Ω, *then we have*

$$\frac{\mu_1}{\rho_1} = \frac{\mu_2}{\rho_2} \qquad \text{in } \Omega \setminus \Omega_E,$$

where $\Omega_E := \bigcup\{V \subset \Omega$ *is an open set satisfying* $\|\vec{u}\|_{L^2(V \times (0,T))} = 0\}$.

Proof. Using the same arguments in the proof of theorem 4.1, it suffices to show that for any open ball $B_\epsilon(x_0) \subset \Omega^+ = \{x \in \Omega \mid \mu_1(x)/\rho_1(x) > \mu_2(x)/\rho_2(x)\}$ on which (4.5) is satisfied, $\vec{u}$ has a finite propagation speed in $B_\epsilon(x_0) \times (0, T)$ and $\vec{u}(\cdot, t_0)$ has a unique continuation principle in $B_\epsilon(x_0)$ for any $t_0 \in (0, T)$. Then we can apply theorem 3.8 with the homogeneous initial condition (4.7) to obtain $\vec{u} \equiv 0$ in $B_\epsilon(x_0) \times (0, T)$, implying $x_0 \in B_\epsilon(x_0) \subset \Omega_E$, which will complete the proof.

By theorem 3.3, $\vec{u}$ already has a finite propagation speed in $B_\epsilon(x_0) \times (0, T)$ with the maximum speed $c = \sup_{x \in B_\epsilon(x_0)} \sqrt{(\lambda_1(x) + 2\mu_1(x))/\rho_1(x)}$. From (4.5) and the fact that

$\lambda_1/\rho_1 = \lambda_2/\rho_2$, we have $\langle(\lambda + 2\mu)/\rho\rangle = 2\langle\mu/\rho\rangle \geqslant 2\alpha_1 > 0$ in $B_\epsilon(x_0)$, where again $\langle\mathcal{F}\rangle := \mathcal{F}_1 - \mathcal{F}_2$ for any indexed quantity $\mathcal{F}_j$. Therefore applying theorem 3.7, $\vec{u}(\cdot, t_0)$ has a unique continuation principle in $B_\epsilon(x_0)$ for any $t_0 \in (0, T)$, which completes the proof. □

In our application of interest, we are primarily interested in shear wave properties. Nevertheless we give hypotheses and uniqueness results that identify the compression wave speed. If the shear modulus μ is given and the Neumann boundary condition is specified (or λ is specified on the boundary with the Dirichlet boundary condition, see corollary 4.4), then the compression wave speed $c_{\mathrm{p}} = \sqrt{(\lambda + 2\mu)/\rho}$ is uniquely identified in any subregion where $\nabla \cdot \vec{u} \neq 0$ for some time $t \in (0, T)$. Note that the proof is essentially different from the previous ones.

Theorem 4.3. *Assume that* $\rho_j \in C^1(\bar{\Omega})$ *and* $\mu_j, \lambda_j \in C^2(\bar{\Omega})$ *for* $j = 1, 2$ *satisfy* $\rho_j(x), \mu_j(x), \lambda_j(x) \geqslant \alpha_0 > 0$. *Let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a common solution to the Neumann-type initial-boundary value problem (4.6), (4.7) and (4.9) for* $j = 1, 2$. *If* $\mu_1 = \mu_2$ *in* Ω, *then we have*

$$\frac{\lambda_1 + 2\mu_1}{\rho_1} = \frac{\lambda_2 + 2\mu_2}{\rho_2} \qquad \textit{in } \Omega \setminus \Omega_D$$

where $\Omega_D := \bigcup\{V \subset \Omega$ *is an open set satisfying* $\|\nabla \cdot \vec{u}\|_{L^2(V\times(0,T))} = 0\}$.

Remark. Intuitively, Ω_D is the subset of Ω where the compression wave has not yet travelled during the time $(0, T)$. See remark 4.5 for more details.

Proof. Let

$$\Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} := \{x \in \Omega : (\lambda_1 + 2\mu_1)(x)/\rho_1(x) \neq (\lambda_2 + 2\mu_2)(x)/\rho_2(x)\},$$

and

$$\Omega_{\langle\rho\rangle\neq 0} := \{x \in \Omega : \rho_1(x) \neq \rho_2(x)\}, \qquad \Omega_{\langle\rho\rangle=0} := \{x \in \Omega : \rho_1(x) = \rho_2(x)\}.$$

Here again $\langle\mathcal{F}\rangle := \mathcal{F}_1 - \mathcal{F}_2$ for any indexed quantity $\mathcal{F}_j$. It suffices to show that

$$\left(\frac{\lambda_1 + 2\mu_1}{\rho_1}(x) - \frac{\lambda_2 + 2\mu_2}{\rho_2}(x)\right)\nabla \cdot \vec{u}(x, t) = 0 \qquad \text{in } \Omega \times (0, T), \tag{4.10}$$

since this implies $\nabla \cdot \vec{u} = 0$ in $\Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \times (0, T)$, that is, $\Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \subset \Omega_D$. This will complete the proof.

Before proceeding further, we point out a useful observation which is

$$\vec{u} = 0 \qquad \text{in } (\Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \cap \Omega_{\langle\rho\rangle\neq 0}) \times (0, T). \tag{4.11}$$

This is easily verified by our shrink and spread argument in the following way: for any $x_0 \in \Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \cap \Omega_{\langle\rho\rangle\neq 0}$, since $\mu_1 = \mu_2$ and $\Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \cap \Omega_{\langle\rho\rangle\neq 0}$ is an open set, there exists an open ball $B_\epsilon(x_0) \subset \Omega_{\langle c_{\mathrm{p}}\rangle\neq 0} \cap \Omega_{\langle\rho\rangle\neq 0}$ on which $|\langle\mu/\rho\rangle|, |\langle(\lambda + 2\mu)/\rho\rangle| \geqslant \alpha_1 > 0$ in $B_\epsilon(x_0)$ for some $\alpha_1 > 0$. Hence as in the proof of theorem 4.2, applying theorems 3.3, 3.7 and then theorem 3.8 with the homogeneous initial condition (4.7), we have $\vec{u} \equiv 0$ in $B_\epsilon(x_0) \times (0, T)$. Thus (4.11) is obtained.

From (4.11) we immediately obtain that

$$\left(\frac{\lambda_1 + 2\mu_1}{\rho_1}(x) - \frac{\lambda_2 + 2\mu_2}{\rho_2}(x)\right)\nabla \cdot \vec{u}(x, t) = 0 \qquad \text{in } \Omega_{\langle\rho\rangle\neq 0} \times (0, T). \tag{4.12}$$

To complete the proof, it suffices to establish the same identity as in (4.12) for $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0} \times (0, T)$, where $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}$ denotes the interior of $\Omega_{\langle\rho\rangle=0}$, and we may assume $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0} \neq \emptyset$. If $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0} = \emptyset$,

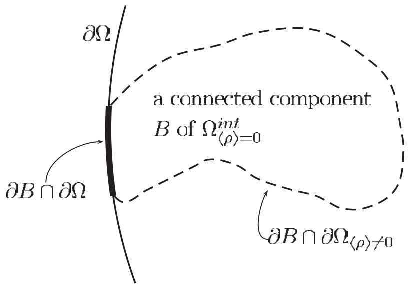


Figure 3. A typical configuration of a connected component B of $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}$ in the proof of theorem 4.3.

by the continuity of ρ_j and Baire's category theorem, we see that $\Omega_{\langle\rho\rangle\neq 0}$ is an open dense subset of Ω. Hence (4.10) is an immediate result from (4.12) and this completes the proof.

In the case when $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}\neq\emptyset$, by subtracting one from the other in (4.6), we get

$$\nabla((\lambda_1-\lambda_2)\nabla\cdot\vec{u})=0 \qquad \text{in } \Omega^{\mathrm{int}}_{\langle\rho\rangle=0}\times(0,T).$$

Thus for any connected component B of $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}$, we have

$$(\lambda_1-\lambda_2)(x)\nabla\cdot\vec{u}(x,t)=C_B(t) \qquad \text{in } B\times(0,T), \tag{4.13}$$

where C_B is independent of the spatial variable x. Now we derive the boundary condition for (4.13), see figure 3 for a typical configuration of B. We consider first the points in $\partial B\cap\partial\Omega$. Subtracting one from the other in the Neumann boundary condition (4.9), we get

$$(\lambda_1-\lambda_2)(x)\nabla\cdot\vec{u}(x,t)=0 \qquad \text{on } (\partial B\cap\partial\Omega)\times(0,T). \tag{4.14}$$

Now consider the points of ∂B contained in Ω. Since $\rho_1=\rho_2$ in ∂B, the coefficient of (4.12) is equal to $(\lambda_1-\lambda_2)/\rho_1$ on $\partial B\cap\partial\Omega_{\langle\rho\rangle\neq 0}$. Hence from (4.12) we have

$$(\lambda_1-\lambda_2)(x)\nabla\cdot\vec{u}(x,t)=0 \qquad \text{on } (\partial B\cap\partial\Omega_{\langle\rho\rangle\neq 0})\times(0,T). \tag{4.15}$$

Since $\partial B=(\partial B\cap\partial\Omega)\cup(\partial B\cap\partial\Omega_{\langle\rho\rangle\neq 0})$, from (4.14) and (4.15) we have

$$(\lambda_1-\lambda_2)(x)\nabla\cdot\vec{u}(x,t)=0 \qquad \text{on } \partial B\times(0,T). \tag{4.16}$$

From (4.13) and (4.16) we obtain $(\lambda_1-\lambda_2)(x)\nabla\cdot\vec{u}(x,t)=0$ in $B\times(0,T)$. Since B is any connected component of $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}$, and $\mu_1=\mu_2$, $\rho_1=\rho_2$ in $\Omega^{\mathrm{int}}_{\langle\rho\rangle=0}$, finally we have

$$\left(\frac{\lambda_1+2\mu_1}{\rho_1}(x)-\frac{\lambda_2+2\mu_2}{\rho_2}(x)\right)\nabla\cdot\vec{u}(x,t)=0 \qquad \text{in } \Omega^{\mathrm{int}}_{\langle\rho\rangle=0}\times(0,T). \tag{4.17}$$

Combining (4.12) and (4.17), we get (4.10) and this completes the proof. □

If we specify the Dirichlet boundary condition (4.8) instead of the Neumann boundary condition (4.9) in theorem 4.3, then *a priori* knowledge of λ on the boundary is required to obtain the same result. Because in the proof of theorem 4.3 the Neumann boundary condition (4.9) is used only to derive (4.14) which holds obviously when $\lambda_1=\lambda_2$ on $\partial\Omega$, we get the following corollary.

Corollary 4.4. *Under the same hypotheses* ($\mu_1=\mu_2$ *in* Ω) *in theorem 4.3, if (4.9) is substituted by the Dirichlet boundary condition (4.8) and, in addition,* $\lambda_1=\lambda_2$ *on* $\partial\Omega$ *is assumed, we have* $(\lambda_1+2\mu_1)/\rho_1=(\lambda_2+2\mu_2)/\rho_2$ *in* $\Omega\setminus\Omega_D$.

Remark 4.5. Roughly speaking, Ω_E (or Ω_D) represents a maximal subset of Ω which no wave (or no compression wave) has reached during the time $(0, T)$. Hence $\Omega \setminus \Omega_E$ and $\Omega \setminus \Omega_D$ represent the regions where the wave or the compression wave, respectively, have travelled during the time $(0, T)$. If $\vec{u}$ and $\nabla \cdot \vec{u}$ are continuous in $\Omega \times (0, T)$, then

$$\Omega_{\vec{u}\neq 0} := \{x \in \Omega : \vec{u}(x,t) \neq 0 \text{ for some } t \in (0,T)\},$$
$$\Omega_{\nabla\cdot\vec{u}\neq 0} := \{x \in \Omega : \nabla \cdot \vec{u}(x,t) \neq 0 \text{ for some } t \in (0,T)\}$$

are well defined and contained in $\Omega \setminus \Omega_E$ and $\Omega \setminus \Omega_D$, respectively.

5. Simultaneous identification in isotropic media

In the previous section, we showed the unique identifiability of the shear wave speed $c_{\mathrm{s}} = \sqrt{\mu/\rho}$ or the compression wave speed $c_{\mathrm{p}} = \sqrt{(\lambda + 2\mu)/\rho}$ under suitable assumptions. In fact, the two elastic parameters ρ and λ are uniquely identified under the same hypotheses of theorem 4.3. Moreover both ρ and μ are uniquely identified if we add the assumption that we are given the Neumann boundary condition in theorems 4.1 and 4.2. For the Dirichlet boundary condition, *a priori* knowledge of a certain elastic parameter on the boundary is required to guarantee a similar simultaneous unique identification. Note that Barbone *et al* [3, 4] make a similar observation for the *incompressible* static or dynamic sinusoidal excitation problems.

5.1. The Neumann case

If we specify the Neumann boundary condition (4.4) in theorem 4.1, not only the shear wave speed $c_{\mathrm{s}} = \sqrt{\mu/\rho}$ but also all the elastic parameters ρ and μ are uniquely identifiable. The proof is based on the unique identifiability of the shear wave speed, an energy estimate and careful use of the divergence theorem.

Theorem 5.1. *Under the same hypothesis on ρ_j and μ_j in theorem 4.1, let $u \in H^2(\Omega \times (0,T))$ be a common solution to the Neumann-type initial-boundary value problem (4.1), (4.2) and (4.4) for $j = 1, 2$. Then we have $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ in $\Omega \setminus \Omega_E$, where*

$$\Omega_E := \bigcup \{V \subset \Omega \textit{ is an open set satisfying } \|u\|_{L^2(V\times(0,T))} = 0\}.$$

Proof. Since we already know that $c_{\mathrm{s}}^2 = \mu_1/\rho_1 = \mu_2/\rho_2$ in $\Omega \setminus \Omega_E$ by theorem 4.1, it is sufficient to show that $\mu_1 = \mu_2$ in $\Omega \setminus \Omega_E$. Let $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$, where here

$$\Omega^0 := \{x \in \Omega : \mu_1(x) = \mu_2(x)\} \qquad \text{and} \qquad \Omega^{\pm} := \{x \in \Omega : \mu_1(x) \gtrless \mu_2(x)\}.$$

We will show that $\Omega^+ \cup \Omega^- \subset \Omega_E$. Then we have $\Omega \setminus \Omega_E \subset \Omega \setminus (\Omega^+ \cup \Omega^-) = \Omega^0$, which will complete the proof.

As in the derivation of (3.9), we have

$$0 = \int_0^{\mathrm{T}} \int_{\Omega^+} \int_0^s \{(\langle\rho\rangle|u_t|^2 + \langle\mu\rangle|\nabla u|^2)_t - 2\nabla \cdot (\langle\mu\rangle u_t \nabla u)\}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s$$

where $\langle\mathcal{F}\rangle := \mathcal{F}_1 - \mathcal{F}_2$ for any indexed quantity $\mathcal{F}_j$. Again using the homogeneous initial condition (4.2), we have

$$\int_0^{\mathrm{T}} \int_{\Omega^+} (\langle\rho\rangle|u_t|^2 + \langle\mu\rangle|\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}s = 2\int_0^{\mathrm{T}} \int_{\Omega^+} \int_0^s \nabla \cdot (\langle\mu\rangle u_t \nabla u)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s.$$

Applying the divergence theorem to the right-hand side with the Neumann boundary condition $\langle\mu\rangle\nabla u \cdot \nu = 0$ on $\partial\Omega \times (0, T)$, we have

$$\int_0^{\mathrm{T}} \int_{\Omega^+} (\langle\rho\rangle|u_t|^2 + \langle\mu\rangle|\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}s = 2\int_0^{\mathrm{T}} \int_{\partial\Omega^+\cap\partial\Omega} \int_0^s u_t \langle\mu\rangle \nabla u \cdot \nu\,\mathrm{d}t\,\mathrm{d}S_x\,\mathrm{d}s = 0. \tag{5.1}$$

Note that the divergence theorem holds even though $\partial\Omega^+$ might be irregular and non-rectifiable, since $\langle\mu\rangle$ vanishes on that possibly irregular boundary $\partial\Omega^+ \setminus \partial\Omega$ and $\partial\Omega$ is a $\mathcal{C}^2$ boundary [16]. On the other hand, since $\mu_1 - \mu_2 = c_s^2(\rho_1 - \rho_2)$ in $\Omega \setminus \Omega_E$ and $u_t = 0$ a.e. in $\Omega_E \times (0, T)$, we have $\langle\rho\rangle|u_t|^2 = (\langle\mu\rangle/c_s^2)|u_t|^2$ a.e. in $\Omega \times (0, T)$. Hence from (5.1) we get

$$\int_0^T \int_{\Omega^+} \langle\mu\rangle \left\{ \frac{1}{c_s^2}|u_t|^2 + |\nabla u|^2 \right\} \mathrm{d}x\, \mathrm{d}s = 0.$$

Since $\langle\mu\rangle > 0$ in Ω^+, using our standard argument and the homogeneous initial condition (4.2), we obtain $u = 0$ a.e. in $\Omega^+ \times (0, T)$. Thus we have $\Omega^+ \subset \Omega_E$. Similarly we have $\Omega^- \subset \Omega_E$, which completes the proof. □

If we specify the Neumann boundary condition (4.9) in theorem 4.2, both of the elastic parameters ρ and μ are uniquely identifiable. The proof is along the same line as that in theorem 5.1.

Theorem 5.2. *Under the same hypothesis on ρ_j, μ_j and λ_j in theorem 4.2, let $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ be a common solution to the Neumann-type initial-boundary value problem (4.6), (4.7) and (4.9) for $j = 1, 2$. If $\lambda_1/\rho_1 = \lambda_2/\rho_2$ in Ω, then we have $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ in $\Omega \setminus \Omega_E$, where*

$$\Omega_E := \bigcup \{V \subset \Omega \textit{ is an open set satisfying } \|\vec{u}\|_{L^2(V\times(0,T))} = 0\}.$$

Proof. Since we already know that $c_l^2 := \lambda_1/\rho_1 = \lambda_2/\rho_2$ and $c_s^2 = \mu_1/\rho_1 = \mu_2/\rho_2$ in $\Omega \setminus \Omega_E$ by theorem 4.2, it is sufficient to show that $\rho_1 = \rho_2$ in $\Omega \setminus \Omega_E$. Let $\Omega = \Omega^0 \cup \Omega^+ \cup \Omega^-$, where $\Omega^0 := \{x \in \Omega : \rho_1(x) = \rho_2(x)\}$ and $\Omega^\pm := \{x \in \Omega : \rho_1(x) \gtrless \rho_2(x)\}$. We will show that $\Omega^+ \cup \Omega^- \subset \Omega_E$. Then we have $\Omega \setminus \Omega_E \subset \Omega \setminus (\Omega^+ \cup \Omega^-) = \Omega^0$, which will complete the proof.

As in the derivation of (3.4), we have

$$0 = \int_0^T \int_{\Omega^+} \int_0^s \left\{ \left(\langle\rho\rangle|\vec{u}_t|^2 + \langle\lambda\rangle|\nabla\cdot\vec{u}|^2 + \frac{\langle\mu\rangle}{2}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2 \right)_t - 2\nabla\cdot(\langle\Sigma\rangle\vec{u}_t) \right\} \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}s,$$

where $\langle\Sigma\rangle := [(\lambda_1 - \lambda_2)(\nabla\cdot\vec{u})I + (\mu_1 - \mu_2)(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})]$. Using the homogeneous initial condition (4.7), we have

$$\begin{aligned} \int_0^T \int_{\Omega^+} &\left(\langle\rho\rangle|\vec{u}_t|^2 + \langle\lambda\rangle|\nabla\cdot\vec{u}|^2 + \frac{\langle\mu\rangle}{2}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2 \right) \mathrm{d}x\, \mathrm{d}s \\ &= 2\int_0^T \int_{\Omega^+} \int_0^s \nabla\cdot(\langle\Sigma\rangle\vec{u}_t)\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}s. \end{aligned} \tag{5.2}$$

From the fact that

$$\begin{aligned} &\lambda_1 - \lambda_2 = c_l^2(\rho_1 - \rho_2) && \text{in } \Omega \setminus \Omega_E, \\ &\mu_1 - \mu_2 = c_s^2(\rho_1 - \rho_2) && \text{in } \Omega \setminus \Omega_E, \\ &\nabla\vec{u} = 0 && \text{in } \Omega_E \times (0, T), \end{aligned} \tag{5.3}$$

we have $\langle\Sigma\rangle = \langle\rho\rangle[(c_l^2\nabla\cdot\vec{u})I + c_s^2(\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}})]$ in $\Omega \times (0, T)$. Since $\langle\rho\rangle = 0$ on the possibly irregular boundary $\partial\Omega^+ \setminus \partial\Omega$, we can apply the divergence theorem to the right-hand side of (5.2), which is therefore equal to

$$2\int_0^T \int_{\partial\Omega^+\cap\partial\Omega} \int_0^s (\langle\Sigma\rangle\vec{u}_t)\cdot\nu\, \mathrm{d}t\, \mathrm{d}S_x,\, \mathrm{d}s = 2\int_0^T \int_{\partial\Omega^+\cap\partial\Omega} \int_0^s (\langle\Sigma\rangle\nu)\cdot\vec{u}_t\, \mathrm{d}t\, \mathrm{d}S_x\, \mathrm{d}s = 0. \tag{5.4}$$

Here we have used the fact that $\langle\Sigma\rangle^{\mathrm{T}} = \langle\Sigma\rangle$ and $\langle\Sigma\rangle\nu = 0$ on $\partial\Omega \times (0, T)$, which is easily seen from the Neumann boundary condition (4.9). Again from (5.3) we have $\langle\lambda\rangle|\nabla\cdot\vec{u}|^2 = c_l^2\langle\rho\rangle|\nabla\cdot\vec{u}|^2$ and $\langle\mu\rangle|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2 = c_{\mathrm{s}}^2\langle\rho\rangle|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2$ in $\Omega \times (0, T)$. Thus from (5.2) and (5.4) we obtain

$$\int_0^{\mathrm{T}}\int_{\Omega^+}\langle\rho\rangle\left\{|\vec{u}_t|^2 + c_l^2|\nabla\cdot\vec{u}|^2 + \frac{c_{\mathrm{s}}^2}{2}|\nabla\vec{u} + \nabla\vec{u}^{\mathrm{T}}|^2\right\}\mathrm{d}x\,\mathrm{d}s = 0.$$

Since $\langle\rho\rangle > 0$ in Ω^+, using our standard argument and the homogeneous initial condition (4.7), we obtain $\vec{u} = 0$ a.e. in $\Omega^+ \times (0, T)$. Thus we have $\Omega^+ \subset \Omega_E$. Similarly we have $\Omega^- \subset \Omega_E$, which completes the proof. □

Although we concluded in theorem 4.3 that the compression wave speed c_{p} is uniquely identifiable, in fact both of the elastic parameters ρ and λ are uniquely identifiable. Since the proof is parallel to that of theorem 5.2, only the outline is given.

Theorem 5.3. *Under the same hypothesis on* ρ_j, μ_j *and* λ_j *in theorem 4.3, let* $\vec{u} \in [H^2(\Omega \times (0, T))]^n$ *be a common solution to the Neumann-type initial-boundary value problem (4.6), (4.7) and (4.9) for* $j = 1, 2$. *If* $\mu_1 = \mu_2$ *in* Ω, *then we have* $(\rho_1, \lambda_1) = (\rho_2, \lambda_2)$ *in* $\Omega \setminus \Omega_D$, *where*

$$\Omega_D := \bigcup\{V \subset \Omega \text{ is an open set satisfying } \|\nabla\cdot\vec{u}\|_{L^2(V\times(0,T))} = 0\}.$$

Proof. Let $\Omega^0 := \{x \in \Omega : \rho_1(x) = \rho_2(x)\}$ and $\Omega^\pm := \{x \in \Omega : \rho_1(x) \gtrless \rho_2(x)\}$ as in the previous proof. Since we already know by hypothesis that $\mu_1 = \mu_2$ in Ω and from theorem 4.3 that $c_{\mathrm{p}}^2 = (\lambda_1 + 2\mu_1)/\rho_1 = (\lambda_2 + 2\mu_2)/\rho_2$ in $\Omega \setminus \Omega_D$, we have

$$\begin{aligned} &\lambda_1 - \lambda_2 = c_{\mathrm{p}}^2(\rho_1 - \rho_2) && \text{in } \Omega \setminus \Omega_D, \\ &\mu_1 - \mu_2 = 0 && \text{in } \Omega, \\ &\nabla\cdot\vec{u} = 0 && \text{in } \Omega_D \times (0, T). \end{aligned} \tag{5.5}$$

Using (5.5) instead of (5.3) in the proof of theorem 5.2, we analogously obtain

$$\int_0^{\mathrm{T}}\int_{\Omega^+}\langle\rho\rangle\{|\vec{u}_t|^2 + c_{\mathrm{p}}^2|\nabla\cdot\vec{u}|^2\}\,\mathrm{d}x\,\mathrm{d}s = 2\int_0^{\mathrm{T}}\int_{\partial\Omega^+\cap\partial\Omega}\int_0^s (\langle\Sigma\rangle\nu)\cdot\vec{u}_t\,\mathrm{d}t\,\mathrm{d}S_x\,\mathrm{d}s = 0. \tag{5.6}$$

Since $\Omega \setminus \Omega_E \supset \Omega \setminus \Omega_D$, where Ω_E is defined in theorem 5.2, we will instead show that $\rho_1 = \rho_2$ in $\Omega \setminus \Omega_E$ because that result follows naturally from our previous arguments: since $\langle\rho\rangle > 0$ in Ω^+, using our standard argument and the homogeneous initial condition (4.7), we obtain $\vec{u} = 0$ a.e. in $\Omega^+ \times (0, T)$. Thus we have $\Omega^+ \subset \Omega_E$. Similarly we have $\Omega^- \subset \Omega_E$, which leads to $\Omega \setminus \Omega_E \subset \Omega \setminus (\Omega^+ \cup \Omega^-) = \Omega^0$. Hence $\rho_1 = \rho_2$ in $\Omega \setminus \Omega_E \supset \Omega \setminus \Omega_D$. From (5.5) we conclude that $\lambda_1 = \lambda_2$ in $\Omega \setminus \Omega_D$. This completes the proof. □

5.2. The Dirichlet case

Since the Neumann boundary condition is used only to derive (5.1), (5.4) and (5.6), the simultaneous unique identification still holds obviously under the Dirichlet boundary condition if a certain elastic parameter is specified on the boundary. The results are summarized as follows without proofs.

Corollary 5.4. *Under the same hypotheses in theorem 5.1, if the Neumann boundary condition (4.4) is substituted by the Dirichlet boundary condition (4.3) and, in addition, either* $\rho_1 = \rho_2$ *or* $\mu_1 = \mu_2$ *on* $\partial\Omega$, *then* $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ *in* $\Omega \setminus \Omega_E$.

Corollary 5.5. *Under the same hypotheses* ($\lambda_1/\rho_1 = \lambda_2/\rho_2$ *in* Ω) *in theorem 5.2, if the Neumann boundary condition (4.9) is substituted by the Dirichlet boundary condition (4.8) and, in addition, either* $\rho_1 = \rho_2$, $\mu_1 = \mu_2$ *or* $\lambda_1 = \lambda_2$ *on* $\partial\Omega$*, then* $(\rho_1, \mu_1) = (\rho_2, \mu_2)$ *in* $\Omega \setminus \Omega_E$.

Corollary 5.6. *Under the same hypotheses* ($\mu_1 = \mu_2$ *in* Ω) *in theorem 5.3, if the Neumann boundary condition (4.9) is substituted by the Dirichlet boundary condition (4.8) and, in addition,* $\lambda_1 = \lambda_2$ *on* $\partial\Omega$*, then* $(\rho_1, \lambda_1) = (\rho_2, \lambda_2)$ *in* $\Omega \setminus \Omega_D$.

In corollary 5.6, λ must be specified on the boundary, while any one of the elastic parameters may be specified in corollaries 5.4 and 5.5. That is because in this case ($\mu_1 = \mu_2$ in Ω) we use corollary 4.4. In that corollary under the Dirichlet boundary conditions the requirement that $\lambda_1 = \lambda_2$ on the boundary is the essential assumption.

It is natural to ask the question: in the Dirichlet case, is it possible to obtain the simultaneous unique identification without any additional assumptions? We will present counterexamples in the case of the scalar shear displacement, that is, it is impossible to remove the additional assumption in corollary 5.4.

Our counterexamples will be devised by travelling waves. Hence we begin with the following lemma which shows a structure of travelling wave solutions. This lemma will also be used for making counterexamples for the unique identifiability in anisotropic media.

Lemma 5.7. *Let* $U \in \mathcal{C}^2(\mathbb{R})$ *satisfy* $U(s) = 0$ *for* $s < 0$*, and let* $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ *satisfy* $|\nabla\varphi| > 0$ *in* $\mathbb{R}^n_{\varphi>0} := \{x \in \mathbb{R}^n : \varphi(x) > 0\}$. *Let* $\Omega \subset \mathbb{R}^n_{\varphi>0}$ *be an open connected* $\mathcal{C}^2$ *domain. If* $M \in [\mathcal{C}^1(\bar{\Omega})]^{n\times n}$ *and* $\rho \in \mathcal{C}^0(\bar{\Omega})$ *satisfy*

$$\nabla\varphi \cdot M\nabla\varphi = \rho \qquad \text{and} \qquad \nabla \cdot (M\nabla\varphi) = 0 \qquad \text{in } \Omega, \tag{5.7}$$

then the travelling wave $u(x, t) = U(t - \varphi(x)) \in \mathcal{C}^2(\bar{\Omega} \times [0, T])$ *satisfies*

$$\nabla \cdot (M(x)\nabla u(x, t)) = \rho u_{tt}(x, t) \qquad \text{in } \Omega \times (0, T) \tag{5.8}$$

with the homogeneous initial condition

$$u(x, 0) = u_t(x, 0) = 0 \qquad \text{on } \Omega, \tag{5.9}$$

and the Dirichlet boundary condition

$$u(x, t) = U(t - \varphi(x)) \qquad \text{on } \partial\Omega \times (0, T). \tag{5.10}$$

Moreover, the Neumann boundary data is given by

$$M(x)\nabla u(x, t) \cdot \nu(x) = -\dot{U}(t - \varphi(x))(M(x)\nabla\varphi(x)) \cdot \nu(x) \qquad \text{on } \partial\Omega \times (0, T), \tag{5.11}$$

where ν *is the outward normal to* $\partial\Omega$ *and* $\dot{U}$ *represents the derivative of* U.

Proof. Since $\nabla u = -\dot{U}\nabla\varphi$ and $u_{tt} = \ddot{U}$ where $\ddot{U}$ represents the derivative of $\dot{U}$, by (5.7) we get

$$\nabla \cdot (M(x)\nabla u(x, t)) = -\dot{U}\nabla \cdot (M\nabla\varphi) + \ddot{U}(\nabla\varphi \cdot M\nabla\varphi) = \rho\ddot{U} = \rho u_{tt} \qquad \text{in } \Omega \times (0, T),$$

which proves (5.8). By the construction of u, (5.9)–(5.11) are trivially satisfied. □

For the isotropic medium case ($M = \mu I$), (5.7) is equivalent to

$$\rho = \mu|\nabla\varphi|^2 \qquad \text{and} \qquad \nabla \cdot (\mu\nabla\varphi) = 0 \qquad \text{in } \Omega. \tag{5.12}$$

By virtue of lemma 5.7, the travelling wave $u(x, t) = U(t - \varphi(x))$ is the common solution to (5.8)–(5.10) for different ρ and μ that satisfy (5.12). Hence, the simultaneous identification of (ρ, μ) is generally impossible under the Dirichlet boundary condition unless we specify the boundary value of ρ or μ as in corollary 5.4. A concrete counterexample is presented in the following example.

Example 5.8. Let $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (x_1, \ldots, x_{n-1})$. Let us pick $\varphi(x) = x_n$, $\Omega = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$, fix $U \in \mathcal{C}^2(\mathbb{R})$ satisfying $U(s) = 0$ for $s < 0$ and choose any $\omega \in \mathcal{C}^1(\bar{\Omega})$ with $\omega(x) = \omega(\tilde{x}) > 0$ in $\bar{\Omega}$. Then the travelling wave solution $u(x,t) = U(t - x_n)$ solves (5.8)–(5.10) for $M = \mu I$, $\mu = \omega$ and $\rho = \omega$, which is easily verified by checking (5.12). That is, any $(\rho(x), \mu(x)) = (\omega(\tilde{x}), \omega(\tilde{x}))$ can be possible elastic parameters that assume the same shear displacement $u(x,t) = U(t - x_n)$ in $\mathbb{R}^n_+$ satisfying the same Dirichlet boundary condition.

6. Nonuniqueness in anisotropic media

In the previous sections, we considered various sufficient conditions for the unique identifiability of wave speeds, the simultaneous unique identification of all the elastic parameters and presented some counterexamples of simultaneous identification in isotropic media.

In anisotropic media, however, the shear tensor $M(x)$ appearing in (5.8) may not be uniquely identified, regardless of the type of specified boundary conditions, even though the density ρ is assumed to be known. Counterexamples will be constructed using travelling waves as in the previous section.

The simplest counterexample is analogous to example 5.8: let $\varphi(x) = x_n$ and $\Omega = \mathbb{R}^n_+$. Fix $U \in \mathcal{C}^2(\mathbb{R})$ and $\rho \in \mathcal{C}^1(\bar{\Omega})$ satisfying $U(s) = 0$ for $s < 0$ and $\rho(x) = \rho(\tilde{x}) > 0$, and choose any $\omega_k \in \mathcal{C}^1(\bar{\Omega})$ with $\omega_k(x) > 0$ in $\bar{\Omega}$ for $k = 1, \ldots, n-1$. Then $u(x,t) = U(t - x_n)$ solves (5.8)–(5.10) for $M(x) = \operatorname{diag}(\omega_1(x), \omega_2(x), \ldots, \omega_{n-1}(x), \rho(\tilde{x}))$, which is easily verified by checking (5.7). Moreover, from (5.11) the Neumann boundary data

$$M(x)\nabla u(x,t) \cdot \nu(x) = \rho(\tilde{x})\dot{U}(t - \varphi(x)) \qquad \text{on } \partial\Omega \times (0, T)$$

is independent of ω_k. Hence, as long as ρ is fixed, any $M = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_{n-1}, \rho)$ can be possible shear tensors that assume the same shear displacement $u(x,t) = U(t - x_n)$ in $\mathbb{R}^n_+$ satisfying the same Dirichlet and Neumann boundary conditions.

Now we will investigate the underlying structure between the wavefront function φ and the corresponding possible shear tensors M. For simplicity, we restrict ourselves to the case that $n = 2$ and $0 < \rho \in \mathcal{C}^1(\bar{\Omega})$ is assumed to be known.

Theorem 6.1. *Let $U \in \mathcal{C}^2(\mathbb{R})$ satisfy $U(s) = 0$ for $s < 0$ and $\varphi \in \mathcal{C}^2(\mathbb{R}^2)$ satisfy $|\nabla\varphi| > 0$ in $\mathbb{R}^2_{\varphi>0} := \{(x,y) \in \mathbb{R}^2 : \varphi(x,y) > 0\}$. Let $\Omega \subset \mathbb{R}^2_{\varphi>0}$ be an open connected $\mathcal{C}^2$ domain. Assume that $0 < \rho \in \mathcal{C}^1(\bar{\Omega})$ is known and $\eta, \omega \in \mathcal{C}^1(\bar{\Omega})$ satisfy*

$$\nabla\eta \cdot \nabla^{\perp}\varphi = -\nabla \cdot \left(\frac{\rho\nabla\varphi}{|\nabla\varphi|^2}\right) \qquad \text{and} \qquad \rho\omega > \eta^2|\nabla\varphi|^2, \tag{6.1}$$

where $\nabla^{\perp}\varphi = (-\varphi_y, \varphi_x)$. Then the symmetric positive-definite matrix

$$M = \frac{\rho}{|\nabla\varphi|^4}\begin{pmatrix} \varphi_x^2 & \varphi_x\varphi_y \\ \varphi_x\varphi_y & \varphi_y^2 \end{pmatrix} + \frac{\omega}{|\nabla\varphi|^2}\begin{pmatrix} \varphi_y^2 & -\varphi_x\varphi_y \\ -\varphi_x\varphi_y & \varphi_x^2 \end{pmatrix} + \frac{\eta}{|\nabla\varphi|^2}\begin{pmatrix} -2\varphi_x\varphi_y & \varphi_x^2 - \varphi_y^2 \\ \varphi_x^2 - \varphi_y^2 & 2\varphi_x\varphi_y \end{pmatrix} \tag{6.2}$$

makes the travelling wave $u(x,t) = U(t - \varphi(x)) \in \mathcal{C}^2(\bar{\Omega} \times [0, T])$ satisfy (5.8)–(5.10). In addition, the Neumann boundary condition

$$M\nabla u \cdot \nu = -\dot{U}\left(\rho\frac{\nabla\varphi \cdot \nu}{|\nabla\varphi|^2} + \eta\nabla^{\perp}\varphi \cdot \nu\right) \qquad \text{on } \partial\Omega \times (0, T)$$

is independent of ω.

Proof. By lemma 5.7, it suffices to check the condition (5.7). With respect to a new orthonormal coordinate system $\{\hat{e}_1, \hat{e}_2\} = \left\{\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla^\perp\varphi}{|\nabla\varphi|}\right\}$, (6.2) is represented by

$$\hat{M}(x, y) = PMP^{\mathrm{T}} = \begin{pmatrix} \rho|\nabla\varphi|^{-2} & \eta \\ \eta & \omega \end{pmatrix} \tag{6.3}$$

by using a transition matrix

$$P = \frac{1}{|\nabla\varphi|}\begin{pmatrix} \varphi_x & \varphi_y \\ -\varphi_y & \varphi_x \end{pmatrix}.$$

Hence we have $\nabla\varphi \cdot M\nabla\varphi = |\nabla\varphi|^2(\hat{e}_1 \cdot \hat{M}\hat{e}_1) = \rho$. From the fact that $\nabla \cdot (\nabla^\perp\varphi) = 0$ and the first assumption in (6.1), we have

$$\nabla \cdot (M\nabla\varphi) = \nabla \cdot \left(\frac{\rho\nabla\varphi}{|\nabla\varphi|^2} + \eta\nabla^\perp\varphi\right) = \nabla \cdot \left(\frac{\rho\nabla\varphi}{|\nabla\varphi|^2}\right) + \nabla\eta \cdot \nabla^\perp\varphi = 0.$$

Thus (5.7) is verified. The symmetric positive definiteness of M is immediately observed by the representation (6.3) and the second assumption in (6.1). □

Theorem 6.1 furnishes many counterexamples both under the Dirichlet and the Neumann boundary conditions: if we fix η and choose any $\omega > \eta^2|\nabla\varphi|^2/\rho$, we can construct many distinct anisotropic shear tensors M which assume the same shear displacement data $u|_{\Omega\times(0,T)}$ satisfying the same Neumann boundary condition. For the Dirichlet boundary condition, we need not even fix η. We conclude this paper by giving some concrete counterexamples with various wavefront functions φ. In the following examples, ρ is always assumed to be 1 for simplicity.

Example 6.2 (Linear wavefront). Let $\varphi(x, y) = ax+by$ for any constants satisfying $a^2+b^2 \neq 0$ and $\Omega \subset \{(x, y) \in \mathbb{R}^2 : ax + by > 0\}$ be an open connected $\mathcal{C}^2$ domain. Then for any real numbers ω and η satisfying $\omega > (a^2+b^2)\eta^2$, (6.1) is trivially satisfied and (6.2) is represented by

$$M = \frac{1}{a^2+b^2}\begin{pmatrix} \frac{a^2}{a^2+b^2} + b^2\omega - 2ab\eta & \frac{ab}{a^2+b^2} - ab\omega + (a^2-b^2)\eta \\ \frac{ab}{a^2+b^2} - ab\omega + (a^2-b^2)\eta & \frac{b^2}{a^2+b^2} + a^2\omega + 2ab\eta \end{pmatrix}.$$

For instance, picking $(a, b, \eta) = (1, 1, \frac{1}{4})$ and $\omega = 1, \frac{1}{2}$, or $\frac{1}{4}$, any of the following matrices:

$$M = \begin{pmatrix} 1/8 & 1/8 \\ 1/8 & 5/8 \end{pmatrix}, \qquad \begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \qquad \begin{pmatrix} 1/2 & -1/4 \\ -1/4 & 1 \end{pmatrix}$$

make the travelling wave $u(x, y, t) = U(t-(x+y))$ satisfy (5.8)–(5.10) and the same Neumann boundary condition $M\nabla u \cdot \nu = -\dot{U}(\frac{1}{4}, \frac{3}{4}) \cdot \nu$ on $\partial\Omega \times (0, T)$.

Example 6.3 (Circular wavefront). Let $\varphi(x, y) = x^2+y^2$ and $\Omega \subset \{(x, y) \in \mathbb{R}^2 : x^2+y^2 > 0\}$ be an open connected $\mathcal{C}^2$ domain. Then for any functions $\omega = \omega(x, y)$ and $\eta = \eta(x^2+y^2)$ satisfying $\omega(x, y) > 4(x^2+y^2)[\eta(x^2+y^2)]^2$, (6.1) is satisfied, since $\nabla\eta \cdot \nabla^\perp\varphi = 0$ and $\nabla \cdot (\nabla\varphi/|\nabla\varphi|^2) = 0$. Also (6.2) is represented by

$$M = \frac{1}{4(x^2+y^2)}\begin{pmatrix} \frac{x^2}{x^2+y^2} + 4y^2\omega - 8xy\eta & \frac{xy}{x^2+y^2} - 4xy\omega + 4(x^2-y^2)\eta \\ \frac{xy}{x^2+y^2} - 4xy\omega + 4(x^2-y^2)\eta & \frac{y^2}{x^2+y^2} + 4x^2\omega + 8xy\eta \end{pmatrix}.$$

For instance, picking $\eta = 0$ and $\omega = 1$ or $\frac{1}{4}(x^2+y^2)^{-1}$, either of the following matrices:

$$M = \frac{1}{4(x^2+y^2)}\begin{pmatrix} \frac{x^2}{x^2+y^2} + 4y^2 & \frac{xy}{x^2+y^2} - 4xy \\ \frac{xy}{x^2+y^2} - 4xy & \frac{y^2}{x^2+y^2} + 4x^2 \end{pmatrix}, \frac{1}{4(x^2+y^2)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

make $u(x, y, t) = U(t - (x^2 + y^2))$ satisfy (5.8)–(5.10) and the same Neumann boundary condition $M\nabla u \cdot \nu = -\dot{U}(\frac{x}{2}(x^2 + y^2)^{-1}, \frac{y}{2}(x^2 + y^2)^{-1}) \cdot \nu$ on $\partial\Omega \times (0, T)$. But these two matrices are essentially different from each other, as none of the entries of the first one are radially symmetric while all the entries in the second one are.

Example 6.4 (More general wavefront). Let $\varphi(x, y) = x + h(y)$ for $h \in \mathcal{C}^2(\mathbb{R})$ and $\Omega \subset \{(x, y) \in \mathbb{R}^2 : x + h(y) > 0\}$ be an open connected $\mathcal{C}^2$ domain. Then for any functions $\eta = \eta_0(x + h(y)) - h_y(y)/(1 + h_y^2(y))$ for any $\eta_0 \in \mathcal{C}^1(\bar{\Omega})$ and $\omega = \omega(x, y)$ satisfying $\omega(x, y) > (1 + h_y^2(y))\eta^2(x, y)$, (6.1) is satisfied, since

$$\nabla\eta \cdot \nabla^{\perp}\varphi = \left(\dot{\eta}_0, \dot{\eta}_0 h_y - \frac{\partial}{\partial y}\left[\frac{h_y}{1 + h_y^2}\right]\right) \cdot (-h_y, 1) = -\frac{\partial}{\partial y}\left[\frac{h_y}{1 + h_y^2}\right] = -\nabla \cdot \left[\frac{\nabla\varphi}{|\nabla\varphi|^2}\right].$$

Also (6.2) is represented by

$$M = \frac{1}{1 + h_y^2}\begin{pmatrix} \frac{1}{1+h_y^2} + h_y^2\omega - 2h_y\eta & \frac{h_y}{1+h_y^2} - h_y\omega + (1 - h_y^2)\eta \\ \frac{h_y}{1+h_y^2} - h_y\omega + (1 - h_y^2)\eta & \frac{h_y^2}{1+h_y^2} + \omega + 2h_y\eta \end{pmatrix}.$$

For instance, if we pick $h(y) = y + \frac{1}{2}\sin y$ and $\eta_0 = \frac{1}{2}$, then we have

$$\eta = \frac{(1 - h_y)^2}{2(1 + h_y^2)} = \frac{\cos^2 y}{16 + 8\cos y + 2\cos^2 y}.$$

From the fact that $(1 + h_y^2)\eta^2 = \frac{1}{4}(1 - h_y)^4/(1 + h_y^2)$ and $\frac{1}{2} \leqslant h_y(y) \leqslant \frac{3}{2}$, we can show

$$\frac{1}{4} > (1 + h_y^2)\eta^2 \qquad \text{and} \qquad \frac{1 + 2h_y^3 - h_y^4}{2h_y(1 + h_y^2)} > (1 + h_y^2)\eta^2.$$

Hence we can take $\omega = \frac{1}{4}$ or $\omega = (1 + 2h_y^3 - h_y^4)/(2h_y(1 + h_y^2))$. Then the resulting matrices are as follows, respectively:

$$M = \frac{1}{(8 + 4\cos y + \cos^2 y)^2}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \begin{pmatrix} \frac{1}{4}(2 - \cos y) & 0 \\ 0 & (2 + \cos y)^{-1} \end{pmatrix},$$

where

$$a_{11} = 24 + 12\cos y + 3\cos^2 y + \tfrac{1}{4}\cos^4 y,$$
$$a_{12} = 8 - 3\cos^2 y - \tfrac{5}{2}\cos^3 y - \tfrac{1}{2}\cos^4 y,$$
$$a_{22} = 24 + 20\cos y + 9\cos^2 y + 2\cos^3 y.$$

Both of them make $u(x, y, t) = U(t - (x + y + \frac{1}{2}\sin y))$ satisfy (5.8)–(5.10) and the same Neumann boundary condition $M\nabla u \cdot \nu = -\dot{U}(\frac{1}{4}(2 - \cos y), \frac{1}{2}) \cdot \nu$ on $\partial\Omega \times (0, T)$.

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