Inverse Problems: Recovery of BV Coefficients from Nodes

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Abstract

Consider the Sturm-Liouville problem on a finite interval with Dirichlet boundary conditions. Let the elastic modulus and the density be of bounded variation. Results for both the forward problem and the inverse problem are established. For the forward problem, new bounds are established for the eigenfrequencies. The bounds are sharp. For the inverse problem, it is shown that the elastic modulus is uniquely determined, up to one arbitrary constant, by a dense subset of the nodes of the eigenfunctions when the density is known. Similarly it is shown that the density is uniquely determined, up to one arbitrary constant, by a dense subset of the nodes of the eigenfunctions when the elastic modulus is known. Algorithms for finding piecewise constant approximates to the unknown elastic modulus or density are established and shown to converge to the unknown function at every point of continuity. Results from numerical calculations are exhibited.

Running head: Recovery of BV Coefficients

Subject Classification: 34B24, 34L15, 73D50
Section 1: Introduction

Consider the longitudinal vibrations of a rod. Consider the transverse vibrations of a string. Suppose that the density and/or the elastic modulus are variable and possess sharp discontinuities. We then ask two questions. The first is: Can an accurate estimate be made of the natural frequencies? Our answer is to give new bounds for the frequencies. The second question is: Can the density and/or the elastic modulus be determined from the nodal positions? We prove uniqueness results, present algorithms for finding the density or stiffness, and exhibit the results of numerical experiments.

The squares of the natural frequencies are the eigenvalues for a Sturm-Liouville boundary value problem. The asymptotic forms for these eigenvalues has a long history. If the density and the elastic modulus have integrable second derivatives the asymptotic forms are well known, see e.g. [B], [GL], [Ho], [L], [M], [PT], and more recently [C2] when the ends of the beam are fixed and [B], [GL], [IT], [Ho], [L], [M], [HMcL] when mixed boundary conditions are satisfied. In each case the $n$th eigenvalue is equal to a constant times $(n\pi)^2$ plus a term of order one. If the density and elastic modulus are not that smooth, then results are more recent. For the case where these coefficients have integrable first derivatives, see [HMcL], [CMcL], [A1], [A2]. There it is shown that the square root of the eigenvalue is a constant times $n\pi$ plus a small $o(1)$ term. If the coefficients have square integrable first derivatives, then the sequence of $o(1)$ terms is in $\ell^2$. Finally, if the coefficients have a finite number of discontinuities and have continuous or square integrable second derivatives between the discontinuities, results on the distribution of eigenvalues can be found in [MAL],[H], [Wi], or [C1]. For coefficients of bounded variation see also [A2] where an asymptotic form is not exhibited but it is shown that the eigenvalues are roots of a function with specified properties. The new feature when the elastic modulus and density can have discontinuities is that the square root of the eigenvalue does not, in general, approach a constant multiple of $n\pi$.

Here we assume that the stiffness and density are of bounded variation and hence can have a countable number of discontinuities. We exhibit four separate bounds for the
difference between a constant times the square root of the eigenvalue and \( n\pi \). The first bound applies when the density and stiffness coefficients are of bounded variation. We show that this bound is sharp for continuous functions of bounded variation. The second bound is for piecewise constant functions with a finite number of discontinuities. We demonstrate that this bound is sharp. The third bound is a combination of the first two. It applies when the stiffness and density are of bounded variation and have a finite number of discontinuities. The final bound applies when the stiffness and density have an infinite number of discontinuities. Numerical calculations are presented to support our results. This is done in Section 2.

The remaining sections of this paper address the inverse nodal problem. One-dimensional inverse nodal problems have previously been considered in [McL], [HMcL1], [HMcL2], [S], [ST]. There it is shown that under sufficient smoothness assumptions for the coefficients, a single coefficient is uniquely determined up to one arbitrary constant by a dense subset of nodes. See also [BS] for uniqueness results when the boundary conditions contain the eigenvalue parameter. Further, it is announced in [HMcL1] that two sufficiently smooth coefficients can both be uniquely determined, up to two arbitrary constants, by a dense subset of the nodes. Here we establish uniqueness results for two cases where the coefficients are not so smooth. In each case either the elastic modulus or the density, but not both, is of bounded variation. The other coefficient is constant. It could also be a known function of bounded variation. It is shown that the unknown variable coefficient is uniquely determined, up to one arbitrary constant, at every point of continuity by a dense subset of adjacent pairs of nodal positions. This is done in Section 3. In Section 4, simple algorithms for computing piecewise constant approximations for the unknown variable coefficients are presented. Prior algorithms are contained in [HMcL1], [HMcL2], [S] and [ST] under sufficient smoothness assumptions. Here it is shown that the piecewise constant approximations converge to the desired (bounded variation) coefficients at each point of continuity. The data that is used for each piecewise constant approximation is one natural frequency and all the nodal positions for the corresponding mode shape. This data can be obtained by performing the following experiment: Excite a rod longitudinally at a natural frequency. Measure that frequency. Then scan the rod with a lazer. At each point,
measure the Doppler shift in the backscatter. The places on the rod where the Doppler shift is minimized are the nodes. There is a similar experiment for obtaining the data for a vibrating string.

In the remaining section, Section 5, numerical computations that illustrate the accuracy of the algorithms are presented. Other examples to show how well the piecewise constant approximations estimate the variable coefficient when the coefficient is of bounded variation were previously presented in [HMcL3].
Section 2: Bounds for the Eigenvalues

Consider the longitudinal vibration of a rod. The rod has fixed ends and both the elasticity coefficient, \( p \), and density, \( \rho \), are variable. Let \( m \) be a positive constant and require that \( 0 < m < p, \rho \). Let \( p, \rho \in BV[0,L], \) continuous from the right and at \( x = L \). Then we obtain the natural frequencies, \( \omega_n/2\pi \), and the corresponding mode shapes, \( y_n \), from the eigenvalues, \( \omega_n^2 \), and the corresponding eigenfunctions, \( y_n, n = 1, 2, \ldots \), for

\[
(py_x)_x + \omega^2py = 0, \quad 0 \leq x \leq L, \quad (1)
\]

\[
y(0) = y(L) = 0. \quad (2)
\]

The goal of this section is to establish new bounds for the \( nth \) eigenfrequency \( \omega_n \). We state six theorems. Four theorems give the bounds and two additional theorems establish that the bounds are sharp. We also establish a crude bound for the distance between consecutive nodes or zeros of \( y_n \).

We say that \( y \) is a solution of (1) if \( y \) is absolutely continuous, \( py_x \) is absolutely continuous and the differential equation is satisfied a.e.. It is known, see [At], that this eigenvalue problem has a sequence of eigenvalues \( 0 < \omega_1^2 < \omega_2^2 < \ldots \) with \( \lim_{n \to \infty} \omega_n^2 = \infty \). For each eigenvalue, \( \omega_n^2 \), the corresponding eigenfunction \( y_n \) has exactly \( n - 1 \) interior zeros, \( x_j^n, j = 1, \ldots, n - 1 \) ordered with \( x_j^n < x_{j+1}^n, j = 1, 2, \ldots, (n - 2), n = 1, 2, \ldots \).

From [RN, p.15] we can express

\[
\ell np = (\ell np)_c + \sum_{i=1}^{\infty} \alpha_i H(x - z_i)
\]

\[
\ell np = (\ell np)_c + \sum_{i=1}^{\infty} \beta_i H(x - z_i)
\]

where \( (\ell np)_c \) and \( (\ell np)_c \) are continuous on \( 0 \leq x \leq L \), \( H(x) \) is the Heaviside function

\[
H(x) = \begin{cases} 
0 & x < 0 \\
1 & x \geq 0,
\end{cases}
\]

\( \sum_{i=1}^{\infty} |\alpha_i| < \infty, \quad \sum_{i=1}^{\infty} |\beta_i| < \infty \). The points, \( z_i \), are all distinct and satisfy \( 0 < z_i < L, i = 1, 2, \ldots \). Since \( p \) and \( \rho \) may not have discontinuities at the same place, we allow that \( \alpha_i = 0 \) or \( \beta_i = 0 \), but not both, for each \( i \).
Our method of proof for the first bound for the eigenvalues will be to select a sequence of approximating functions for $\ell np$ and $\ell n\rho$ that converge uniformly to $\ell np$ and $\ell n\rho$, respectively. We will establish the bound for each of the approximating functions and then take the limit to obtain the final result. Our choice for approximating functions is as follows. Let $\epsilon > 0$. Choose $k_\epsilon$ so that $\sum_{i=k_\epsilon+1}^{\infty} |\alpha_i| < \epsilon/2$, $\sum_{i=k_\epsilon+1}^{\infty} |\beta_i| < \epsilon/2$. Choose piecewise linear functions, ($\ell np_{ac\epsilon}$) and ($\ell n\rho_{ac\epsilon}$) which interpolate ($\ell np_c$) and ($\ell n\rho_c$), respectively, where the interpolating mesh is the same for both functions, includes $z_1, \ldots, z_{k_\epsilon}$, and so that the uniform bounds

$$| (\ell np)_c - (\ell np)_{ac\epsilon} | < \epsilon/2, \quad | (\ell n\rho)_c - (\ell n\rho)_{ac\epsilon} | < \epsilon/2$$

hold for all $x \in [0, L]$. Note that each ($\ell np)_{ac\epsilon}$ and ($\ell n\rho)_{ac\epsilon}$ is absolutely continuous. Further, we will call

$$(\ell np)_\epsilon = (\ell np)_{ac\epsilon} + \sum_{i=1}^{k_\epsilon} \alpha_i H(x - z_i), \quad (\ell n\rho)_\epsilon = (\ell n\rho)_{ac\epsilon} + \sum_{i=1}^{k_\epsilon} \beta_i H(x - z_i) \quad (3)$$

$\epsilon$-approximations to $\ell np$ and $\ell n\rho$, respectively, and choose $\epsilon > 0$ sufficiently small so that the $\epsilon$-approximations are always greater than $m$.

We now state our first bound for the eigenfrequencies. The proof described briefly above will be given in a series of steps.

**Theorem 1** (Bounds for the eigenvalues): Let $0 < m < p, \rho$ for all $0 \leq x \leq L$. Suppose $p, \rho \in BV[0, L]$ is continuous from the right and at $x = L$. Let $\omega_n^2$ be the $n$th eigenvalue, $n = 1, 2, \ldots$ for (1) – (2). Then

$$| \omega_n \int_0^L \sqrt{\rho / p} dt - n\pi | \leq \frac{1}{4} V(\ell npp)$$

where $V(\ell npp)$ is the total variation of $\ell npp$.

In order to prove this theorem we require the following lemma. Here we present this lemma without proof, use it to prove Theorem 1, and then prove it after we have presented all of our theorems on the eigenvalue bounds.

**Lemma 1**: Let $0 < m < p_\epsilon, p, \rho_\epsilon, \rho$ for all $0 \leq x \leq L$. Let $p_\epsilon, p, \rho_\epsilon, \rho \in BV[0, L]$ be continuous from the right and at $x = L$. Suppose $| p_\epsilon - p | < \epsilon, | \rho_\epsilon - \rho | < \epsilon$ for all
$x \in [0, L]$. Then the eigenvalues $\omega_n^2(p, \rho)$ of (1) − (2) and the eigenvalues, $\omega_n^2(p_\varepsilon, \rho_\varepsilon)$, of (1) − (2) with $p, \rho$ replaced by $p_\varepsilon, \rho_\varepsilon$ satisfy $\lim_{\varepsilon \to 0} \omega_n^2(p_\varepsilon, \rho_\varepsilon) = \omega_n^2(p, \rho), n = 1, 2, …. $

**Proof of Theorem 1:**

We remark that when a function, $f$, is in $BV[0, L]$ then $\log f$ (when $f > 0$) and $\exp f$ are also in $BV[0, L]$.

**Step 1:** We first consider the special case where $\ell np = (\ell np)_{ac} + \alpha_1 H(x - z_1)$ and $\ell n\rho = (\ell n\rho)_{ac} + \beta_1 H(x - z_1)$ where $(\ell np)_{ac}$ and $(\ell n\rho)_{ac}$ are absolutely continuous and their derivatives are in $L^\infty(0, L)$. Note that in this case the product $\ell np\rho = (\ell np\rho)_{ac} + \gamma_1 H(x - z_1)$, where $\gamma_1 = \alpha_1 + \beta_1$ and where $(\ell np\rho)_{ac}$ is absolutely continuous on $[0, L]$. Further

$$V(\ell np\rho) = |(\ell np\rho)(z_1) - \lim_{x \to z_1^-(\ell np\rho)(x)}| + \int_0^L |[(\ell np\rho)_{ac}]_x| \, dx$$

$$= |\gamma_1| + \int_0^L |[(\ell np\rho)_{ac}]_x| \, dx,$$

see [RN, p.14, p.26].

Let $y_n$ be the eigenfunction corresponding to $\omega_n^2$. Now on each of the subintervals, $[0, z_1)$ and $(z_1, L]$, we have that $y_n \in C^1$, $py_{n,x} \in C^1$ and $p, \rho$ are absolutely continuous with derivatives in $L^\infty$. On each of these subintervals, make the modified Pr"ufer transformation, see, e.g. [AT, p.210, p.216],

$$\omega_n(p\rho)^{1/2}y_n = r \sin \theta,$$

$$py_{n,x} = r \cos \theta,$$

with $r \geq 0$ to obtain

$$\theta_x = \omega_n \sqrt{\frac{\rho}{p}} + \frac{1}{4} \left( \frac{p_x}{p} + \frac{\rho_x}{\rho} \right) \sin 2\theta, \quad a.e. \quad (5)$$

Note that here the function $r$ is positive since the function $y_n$ is a non-trivial solution. This follows since $r = 0$ would imply both $y_n$ and $py_{n,x}$ are zero at the same point. To identify the nodal positions, observe that the eigenfunction $y_n$ is zero iff $\theta$ is a multiple of $\pi$. Further, since $p_x/p, \rho_x/\rho \in L^\infty(0, L)$, if $\theta$ is a multiple of $\pi$ at any point in one of the subintervals then $\theta_x > 0$ a.e. in a neighborhood of that point. Now $y_n(0) = 0$ implies that
we can choose \( \theta(0) = 0 \) for the initial condition for \( \theta \). We determine \( \theta \) in the rest of the interval by solving the differential equation in \([0, z_1]\) and \((z_1, L)\) with the jump condition at \( z = z_1 \) determined as follows. Let \( r_1^-, \theta_1^-, (p \rho)_1^- \) be the limits of \( r, \theta, p \rho \) as \( x \to z_1^- \) and \( r_1^+, \theta_1^+, (p \rho)_1^+ \) be the limits of \( r, \theta, p \rho \) as \( x \to z_1^+ \). Then the requirement that \( y_n \) and \( p y_{n,x} \) be continuous at \( x = z_1 \) implies the matching conditions

\[
\frac{r_1^+ \sin \theta_1^+}{\sqrt{(p \rho)_1^+}} = \frac{r_1^- \sin \theta_1^-}{\sqrt{(p \rho)_1^-}}, \quad (6)
\]

\[
r_1^+ \cos \theta_1^+ = r_1^- \cos \theta_1^-.
\]

From these formulas we can conclude that \( \cos \theta_1^- \) and \( \cos \theta_1^+ \) have the same sign and \( \sin \theta_1^- \) and \( \sin \theta_1^+ \) have the same sign. Hence \( \theta_1^- \) and \( \theta_1^+ \) are both in the same quadrant. We choose \( \theta^+ \) uniquely by requiring that \( | \theta_1^+ - \theta_1^- | < \pi/2 \).

We now establish that \( \theta(L) = n\pi \) and we establish the bound for \( \omega_n \). There are three cases to consider.

**Case 1**: Suppose \( \sin \theta_1^- = 0 \). Then, since \( r \neq 0 \), we must have \( \sin \theta_1^+ = 0 \) and hence \( \theta_1^+ = \theta_1^- \) and \( \theta \) is continuous at \( z_1 \). Further \( \theta_x > 0 \) a.e. in a neighborhood of \( z_1 \). Integrating (5) from 0 to \( z_1 \) and from \( z_1 \) to \( L \) and taking the sum, we obtain

\[
\theta(L) - \theta(0) = \theta_1^+ - \theta_1^- + \omega_n \int_0^L \sqrt{\frac{\rho}{p}} \, dx + \frac{1}{4} \left[ \int_0^{z_1} [(\ell npp)_{ac}]_x \sin 2\theta \, dx + \int_{z_1}^L [(\ell npp)_{ac}]_x \sin 2\theta \, dx \right].
\]

Now, \( y_n \) has exactly \( n - 1 \) zeros in \( 0 < x < L \) and is zero iff \( \sin \theta = 0 \). The function \( \theta \) is continuous in \( 0 \leq x \leq L \), \( \theta(0) = 0 \) and \( \theta_x > 0 \) a.e. in neighborhoods of the points where \( \theta = \ell \pi \) and \( \ell \) is an integer. Hence \( \theta \) takes on the values \( \ell \pi \) exactly once in \( 0 < x < L \) for each \( \ell = 1, 2, \ldots, n - 1 \) and we must have \( \theta(L) = n\pi \). This implies the inequalities

\[
| n\pi - \omega_n \int_0^L \sqrt{\frac{\rho}{p}} \, dx | \leq \frac{1}{4} \left[ \int_0^{z_1} | [(\ell npp)_{ac}]_x | \, dx + \int_{z_1}^L | [(\ell npp)_{ac}]_x | \, dx \right] \leq \frac{1}{4} V (\ell npp),
\]

and the theorem is established in this case.

**Case 2**: Suppose \( \cos \theta_1^- = 0 \). We can show as in Case 1 that \( \theta \) is continuous at \( x = z_1 \) and the bound holds.

**Case 3**: Suppose \( \sin \theta_1^- \neq 0 \) and \( \cos \theta_1^- \neq 0 \). Then also \( \sin \theta_1^+ \neq 0 \) and \( \cos \theta_1^+ \neq 0 \). This is
the most interesting case since in this case $\theta$ is not continuous at $z_1$. Now $\theta_1^-$ and $\theta_1^+$ are in the same quadrant and $\theta_1^+$ is chosen uniquely so that $|\theta_1^+ - \theta_1^-| < \pi/2$. Consequently, the closed interval with end points $\theta_1^-$ and $\theta_1^+$ does not contain a multiple of $\pi$. Again, $\theta_\omega > 0$ a.e. in a neighborhood of a point where $\theta$ is a multiple of $\pi$, $y_n$ has $n - 1$ zeros in $(0, L)$ and $y_n$ is zero iff $\sin \theta$ is zero. Hence, $\theta$ assumes each of the values $\ell\pi$, $\ell = 1, ..., n - 1$ exactly once in $0 < x < L$ and $\theta(L) = n\pi$.

Now integrate (5) over the intervals $[0, z_1)$ and $(z_1, L]$ and sum to obtain
\[
n\pi - \omega_n \int_0^L \sqrt{\frac{p}{P}} \, dx = \theta_1^+ - \theta_1^- + \frac{1}{4} \left[ \int_0^{z_1} [(\ell npp)_ac] \sin 2\theta \, dx + \int_{z_1}^L [(\ell npp)_ac] \sin 2\theta \, dx \right]. \tag{8}
\]
We can achieve the desired bound for the absolute value of the left hand side of this equation if we can show that $|\theta_1^+ - \theta_1^-| \leq \frac{1}{4} |\ell n(pp)_1^+ - \ell n(pp)_1^-|$. To do this, we write the set of inequalities
\[
|\theta_1^+ - \theta_1^-| = \left| \int_{\tan \theta_1^-}^{\tan \theta_1^+} \frac{1}{1 + t^2} \, dt \right| \\
\leq \left| \int_{\tan \theta_1^-}^{\tan \theta_1^+} \frac{1}{2 |t|} \, dt \right| \\
= \frac{1}{4} |\ell n(pp)_1^+ - \ell n(pp)_1^-| \tag{9}
\]
Combining (8) and (9) we obtain
\[
\left| n\pi - \omega_n \int_0^L \sqrt{\frac{p}{P}} \, dx \right| \leq \frac{1}{4} |\ell n(pp)^+_1 - \ell n(pp)^-_1| + \frac{1}{4} \int_0^L |[(\ell npp)_ac] | \, dx = \frac{1}{4} V(\ell npp).
\]
and the theorem is established for this case.

Step 2: Here we consider the special case where $\ell np = (\ell np)_{ac} + \sum_{i=1}^k \alpha_i H(x - z_i)$ and $\ell n p = (\ell np)_{ac} + \sum_{i=1}^k \beta_i H(x - z_i)$, where $k > 1$, and where $(\ell np)_{ac}$ and $(\ell np)_{ac}$ are absolutely continuous in $[0, L]$ with derivatives in $L^\infty(0, L)$. Without loss of generality assume that $z_i < z_{i+1}$, $i = 1, ..., k - 1$. Let $z_0 = 0$ and $z_{k+1} = L$. Then on each of the subintervals $[z_0, z_1)$, $(z_i, z_{i+1})$, $i = 1, ..., k - 1$, and $(z_k, z_{k+1}]$, make the modified Prüfer transformation (4) to obtain equation (5). To solve equation (5) we begin by using the initial condition $\theta(0) = 0$. Solve the differential equation on each subinterval with the correct jump condition at each $z_i$, $i = 1, ..., k$ determined as follows. Let $r_i^-, \theta_i^-, (pp)_i^-$ be the limits of
Let \( r, \theta, \rho \) be the limits of \( r, \theta, \rho \) as \( x \to z_i^- \) and \( r_i^+, \theta_i^+, (\rho \rho)_i^+ \) be the limits of \( r, \theta, \rho \) as \( x \to z_i^+, i = 1, \ldots, k \). Then \( r_i^+, \theta_i^+, (\rho \rho)_i^+ \) are determined by the matching conditions

\[
\begin{align*}
\frac{r_i^+ \sin \theta_i^+}{\sqrt{(\rho \rho)_i^+}} &= \frac{r_i^- \sin \theta_i^-}{\sqrt{(\rho \rho)_i^-}}, \\
\frac{r_i^+ \cos \theta_i^+}{\sqrt{(\rho \rho)_i^+}} &= \frac{r_i^- \cos \theta_i^-}{\sqrt{(\rho \rho)_i^-}}.
\end{align*}
\]

For each \( i, i = 1, \ldots, k \), \( \theta_i^+, \theta_i^- \) are in the same quadrant and we choose \( \theta_i^+ \) uniquely so that \( |\theta_i^+ - \theta_i^-| < \pi/2 \). We now want to conclude that \( \theta(L) = n\pi \). To do this we first apply similar arguments to those in Step 1 to say that: (i) if \( \theta_i^- \) is an integer multiple of \( \pi/2 \), then \( \theta_i^+ = \theta_i^- \); (ii) if \( \theta_i^- \) is not an integer multiple of \( \pi/2 \), then the closed interval with endpoints \( \theta_i^- \) and \( \theta_i^+ \) does not contain a multiple of \( \pi \); (iii) there is exactly \( n - 1 \) zeros in \((0, L)\) and \( y_n = 0 \) iff \( \theta \) is not an integer multiple of \( \pi \), we have the desired result: \( \theta(L) = n\pi \).

To obtain the bound, we integrate (5) over the intervals \([z_0, z_1], (z_i, z_{i+1}), i = 1, \ldots k-1, \) and \((z_k, z_{k+1})\). Then take the sum to get

\[
\theta(L) - \theta(0) - \omega_n \int_0^L \sqrt{\frac{\rho}{p}} \, dx = \sum_{i=1}^{k} [\theta_i^+ - \theta_i^-] + \frac{1}{4} \left[ \sum_{i=1}^{k} \int_{z_i}^{z_i+1} [\ell n(\rho \rho)]_{ac} \, x \sin 2\theta \, dx \right].
\]

(12)

For each \( i, i = 1, \ldots, k \), there are two cases. For one case \( \theta_i^- \) is an integer multiple of \( \pi/2 \). In this case \( \theta_i^+ = \theta_i^- \) and we have easily that

\[
|\theta_i^+ - \theta_i^-| \leq \frac{1}{4} |\ell n(\rho \rho)_i^+ - \ell n(\rho \rho)_i^-|.
\]

For the second case \( \theta_i^- \) is not an integer multiple of \( \pi/2 \). Then a similar set of inequalities as those given in (9) yields

\[
|\theta_i^+ - \theta_i^-| \leq \frac{1}{4} |\ell n(\rho \rho)_i^+ - \ell n(\rho \rho)_i^-|.
\]

Hence

\[
\left| n\pi - \omega_n \int_0^L \sqrt{\frac{\rho}{p}} \, dx \right| \leq \frac{1}{4} \sum_{i=1}^{k} |\ell n(\rho \rho)_i^+ - \ell n(\rho \rho)_i^-| + \frac{1}{4} \left[ \sum_{i=1}^{k} |(\ell n\rho \rho ac)_x\, dx \right] \]

\[
= \frac{1}{4} V(\ell n\rho \rho).
\]
and the theorem is established for this case.

Step 3: We now prove the full theorem using the results from Step 1 and Step 2. We recall the notation in (3). Let

\[(\ell np)_\varepsilon = (\ell np)_{ac\varepsilon} + \sum_{i=1}^{k_\varepsilon} \alpha_i H(x - z_i)\]

\[(\ell \rho p)_\varepsilon = (\ell \rho p)_{ac\varepsilon} + \sum_{i=1}^{k_\varepsilon} \beta_i H(x - z_i)\]

and that \((\ell np\rho)_\varepsilon = (\ell np)_\varepsilon + (\ell \rho p)_\varepsilon\). Assume \(\varepsilon\) is small enough so that \(p_\varepsilon = \exp[(\ell np)_\varepsilon]\), \(\rho_\varepsilon = \exp[(\ell \rho p)_\varepsilon]\) satisfy \(0 < m < p_\varepsilon, \rho_\varepsilon\). Let \((\omega_n^\varepsilon)^2\) be the \(n\)th eigenvalue for (1) - (2) when \(p, \rho\) are replaced by \(p_\varepsilon, \rho_\varepsilon\). By Lemma 1, taking square roots, we have \(\lim_{\varepsilon \to 0} \omega_n^\varepsilon = \omega_n\). Let \(K_\varepsilon = \int_0^L \sqrt{\rho_\varepsilon/p_\varepsilon} dx\) and \(K = \int_0^L \sqrt{\rho/p} dx\). Clearly \(\lim_{\varepsilon \to 0} K_\varepsilon = K\). Thus given \(\varepsilon_1 > 0\) there exists \(\varepsilon_0\) such that for \(\varepsilon < \varepsilon_0\)

\[| K_\varepsilon \omega_n^\varepsilon - K \omega_n | < \varepsilon_1.\]

Let \(\varepsilon < \varepsilon_0\). Then, by applying Steps 1 and 2 we have

\[| K \omega_n - n\pi | \leq | K \omega_n - K_\varepsilon \omega_n^\varepsilon | + | K_\varepsilon \omega_n^\varepsilon - n\pi | \leq \varepsilon_1 + \frac{1}{4} V((\ell np\rho)_\varepsilon). \quad (13)\]

We now show that for each \(\varepsilon\), \(V((\ell np\rho)_\varepsilon) \leq V(\ell np\rho)\). Since \(\ell n(pp) \in BV[0, L]\) we can write

\(\ell n(pp) = (\ell np)_{c} + \sum_{i=1}^{\infty} \gamma_i H(x - z_i),\)

and then

\[(\ell np\rho)_\varepsilon = (\ell np\rho)_{ac\varepsilon} + \sum_{i=1}^{k_\varepsilon} \gamma_i H(x - z_i).\]

The function \((\ell np\rho)_c\) is continuous on \([0, L]\) and \(\sum_{i=1}^{\infty} | \gamma_i | < \infty\). Each \((\ell np\rho)_{ac\varepsilon} = (\ell np)_{ac\varepsilon} + (\ell \rho p)_{ac\varepsilon}\) is piecewise linear on a mesh \(\{x_i\}_{i=0}^{n_\varepsilon}\), with \(x_0 = 0, x_{n_\varepsilon} = L\), and interpolates \((\ell np\rho)_c\) on the same mesh. Each \((\ell np\rho)_{ac\varepsilon}\) is absolutely continuous in \([0, L]\) with derivative in \(L^\infty\). We require that the mesh \(\{x_i\}_{i=0}^{n_\varepsilon}\) include the points \(z_1, \ldots, z_{k_\varepsilon}\). We then have the equation
\[ V(\ln p\rho) \epsilon = \sum_{i=1}^{n} |(\ln p\rho)_{\epsilon}(x^{-}_i) - (\ln p\rho)_{\epsilon}(x^{+}_i-1)| + \sum_{i=1}^{n} |(\ln p\rho)_{\epsilon}(x^{+}_i) - (\ln p\rho)_{\epsilon}(x^{-}_i)| \]

\[ = \sum_{i=1}^{n} |(\ln p\rho)_{\epsilon}(x^{-}_i) - (\ln p\rho)_{\epsilon}(x^{+}_i-1)| + \sum_{i=1}^{k} |(\ln p\rho)(z_i) - \lim_{x \to z_i}(\ln p\rho)(x)| \]

\[ = \sum_{i=1}^{n} |(\ln p\rho)_{\epsilon}(x^{-}_i) - (\ln p\rho)_{\epsilon}(x^{+}_i-1)| + \sum_{i=1}^{k} |(\ln p\rho)(z_i) - \lim_{x \to z_i}(\ln p\rho)(x)| \]

\[ \leq V(\ln p\rho) \]

Inserting this inequality into (13) we obtain

\[ |K \omega_n - n \pi| \leq \epsilon_1 + \frac{1}{4} V(\ln p\rho) \]

for all \( \epsilon_1 > 0 \). Let \( \epsilon_1 \to 0 \) and the proof is complete.

It remains to prove Lemma 1. We will do this after presenting our next three theorems and after presenting some numerical experiments.

Our bound in Theorem 1 is sharp for functions \( p, \rho \) that are continuous and of bounded variation. To show this we construct an explicit example. This is done in Appendices A and B. There we construct a specific 'Cantor' function in Appendix A and use it in Appendix B to establish the following theorem.

**Theorem 2:** There exist pairs of continuous coefficients, \( p_q, \rho_q \in BV[0,1] \), and eigenvalues, \( (\omega_{n_q})^2 \), for (1)-(2) that satisfy

\[ K = \int_0^1 \sqrt{\rho_q/p_q} \, dx = 1, \quad \frac{1}{4} V(\ln p_q \rho_q) = \frac{1}{2}, \]

for all \( q = 1, 2, ... \) and

\[ |K \omega_{n_q} - n_q \pi| - \frac{1}{4} V(\ln p_q \rho_q) \leq \frac{4\pi^2}{92^{q+2}}, \]

implying
Now consider the case where $p$ and $\rho$ can have discontinuities. We observe in our numerical experiments, with piecewise constant $p$ and $\rho$, that the bound is often not sharp. In our computations it was possible to obtain relative errors, i.e. values of
\[
|\omega_n \int_0^L \sqrt{\rho/p} dx - n\pi| - \frac{1}{4} V(\ell n p\rho)\times \left[\frac{1}{4} V(\ell n p\rho)\right]^{-1},
\]
as small as $10^{-4}$ or smaller. To do this, however, the jumps at the discontinuities, while not zero, might have to be extremely small. Below we give histograms for three more typical examples. In each example the interval $[-A, A]$ where $A = \max_n |\omega_n K - n\pi|$, $K = \int_0^L \sqrt{\rho/p} dx$ is divided into 20 subintervals. The vertical dashed lines, at $\pm B$, mark the bound $B = \frac{1}{4} V(\ell n p\rho)$.

To compute we used the following formula. Letting $\rho \equiv 1$, $z_0 = 0$, $z_3 = L$, $p(x) = b_i^2$ for $z_{i-1} \leq x < z_i$, $i = 1, 2, 3$, and $p(L) = b_3^2$, then $\omega$ satisfies
\[
\sin(\omega K) = \frac{b_2 - b_1}{b_2 + b_1} \sin \left(\omega \left( K - 2 \frac{z_1 - z_0}{b_1} \right) \right) + \frac{b_2 - b_3}{b_2 + b_3} \sin \left(\omega \left( K - 2 \frac{z_3 - z_2}{b_3} \right) \right) + \frac{b_2 - b_1}{b_2 + b_1} \cdot \frac{b_2 - b_3}{b_2 + b_3} \sin \left(\omega \left( K - 2 \frac{z_2 - z_1}{b_2} \right) \right).
\]
Setting $b_2 = b_3$ gives the formula for one discontinuity, see [H], [Wi]. In all our examples, $z_1 = (1/2) - \sqrt{2}/8$, $z_2 = z_1 + 2/\sqrt{8} + 2/\sqrt{5}$, and $L = z_2 + 1 - 1/\sqrt{5}$.

Figure 1 corresponds to $b_1 = 1$, $b_2 = b_3 = 2$. For this case $\frac{1}{4} V(\ell n p\rho) = B = .34657$.
and the largest difference $|\omega_n K - n\pi|$ is $A = .33984$.

Figure 1. Histogram for 60,000 values of $\omega_n K - n\pi$ where

$$A = .33984 \text{ and } B = .34657.$$  

Figures 2 and 3 are histograms for cases where $p$ has two discontinuities. Figure 2 has $b_1 = 1$, $b_2 = 2.1$, $b_3 = 2$, so that the second jump discontinuity is quite small. Here $A = .38713$, $B = .39536$ and the histogram takes on a bimodal shape. In Figure 3, $b_1 = 1$, $b_2 = 4$, $b_3 = 2$ so that the difference between the second and third discontinuities is increased. Here $A = .98328$, $B = 1.03972$ and the subintervals with the largest numbers of differences, $\omega_n K - n\pi$, are closer together.

Figure 2. Histogram for 60,000 values of $\omega_n K - n\pi$ where

$$A = .38713, \text{ and } B = .39536.$$
Figure 3. Histogram for 60,000 values of \( \omega_n K - n\pi \) where A = .98328, and B = 1.03972.

These computations suggest that the bound can be improved when \( p, \rho \) are piecewise constant functions. In fact we can prove

**Theorem 3:** (Bounds for the eigenvalues). Let \( p, \rho \) be piecewise constant positive functions continuous from the right and satisfying \( 0 < m < p, \rho \) for all \( 0 \leq x \leq L \). Let \( \omega_n^2 \) be the \( n \)th eigenvalue, \( n = 1, 2, ..., \) for (1)-(2). Let \( z_i \) be the positions of the discontinuities and let \( (p\rho)_i^+ = \lim_{x \to z_i^+} (p\rho), (p\rho)_i^- = \lim_{x \to z_i^-} (p\rho), i = 1, 2, ..., k \). Then

\[
|\omega_n \int_0^L \frac{\sqrt{p}}{p} \, dx - n\pi| \leq \sum_{i=1}^k \text{arcsin} \left| \frac{(p\rho)_i^+ - (p\rho)_i^-}{(p\rho)_i^+ + (p\rho)_i^-} \right|
\]

where arcsin refers to the principal value.

**Proof of Theorem 3:** Following the arguments given in the proof of Theorem 1, using the Prüfer transformation (4) and letting \( \theta_i^+ = \lim_{x \to z_i^+} \theta, \theta_i^- = \lim_{x \to z_i^-} \theta \), we observe that

\[
|\omega_n \int_0^L \sqrt{p} \, dx - n\pi| \leq \sum_{i=1}^k |\theta_i^+ - \theta_i^-|
\]

where
\[ |\theta_i^+ - \theta_i^-| = \left| \int_{\tan \theta_i^-} \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right] \frac{1}{1 + t^2} dt \right| \]

\[ = \left| \arctan \left[ \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{1/2} \tan \theta_i^- \right] - \arctan(\tan \theta_i^-) \right| \]

\[ < \left| \arctan \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{1/4} - \arctan \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{-1/4} \right| \]

\[ = \left| \arcsin \left( \left\{ \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{1/2} - 1 \right\} / \left\{ \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{1/2} + 1 \right\} \right) \right| \]

\[ = \left| \arcsin \left( \left\{ \left[ (p\rho)_i^+ \right]^{1/2} - \left[ (p\rho)_i^- \right]^{1/2} \right\} / \left\{ \left[ (p\rho)_i^+ \right]^{1/2} + \left[ (p\rho)_i^- \right]^{1/2} \right\} \right) \right| \]

\[ < \left| \ln \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right]^{1/4} \right| \]

when \((p\rho)_i^+ \neq (p\rho)_i\). Here \(\arctan\) and \(\arcsin\) refer to principal values. From the next to the last line in (14) the bound in the statement of the theorem is established. 

\[ \bullet \]

**Remark:** Since \(pp\) is of bounded variation the last inequality in (14) yields

\[ \sum_{i=1}^{k} \arcsin \left[ \left| \sqrt{(p\rho)_i^+} - \sqrt{(p\rho)_i^-} \right| / \left( \sqrt{(p\rho)_i^+} + \sqrt{(p\rho)_i^-} \right) \right] < \frac{1}{4} \sum_{i=1}^{k} \left| \ln \left[ \frac{(p\rho)_i^+}{(p\rho)_i^-} \right] \right| < \infty. \]

establishing that Theorem 3 gives a tighter bound than Theorem 1 for the case where \(p, \rho\) are piecewise constant functions.

It is important to say also that, for piecewise constant functions, Theorem 3 is best possible. This is first suggested by our numerical examples. Letting

\[ A_0 = \sum_{i=1}^{k} \arcsin \left[ \left| \sqrt{(p\rho)_i^+} - \sqrt{(p\rho)_i^-} \right| / \left( \sqrt{(p\rho)_i^+} + \sqrt{(p\rho)_i^-} \right) \right], \]

where \(k\) is the number of discontinuities, we observe the following: for the case \(b_1 = 1, b_2 = b_3 = 2\) then \(| A - A_0 | < 10^{-9}\); for the case \(b_1 = 1, b_2 = 2.1, b_3 = 3\) then \(| A - A_0 | < 10^{-5}\); and for the case \(b_1 = 1, b_2 = 4, b_3 = 2\), then \(| A - A_0 | < 10^{-4}\). In fact we observe more.
For example, for the case \( b_1 = 1, b_2 = 4, b_3 = 2 \) the maximum distance between the points \( \{ \omega_n K - n\pi \}_{n=1}^\infty \) is less than \( 10^{-4} \). This suggests the result we show in Appendix C: for piecewise constant \( p, \rho \) the set of differences \( \{ \omega_n \int_0^L \sqrt{\rho/p} \, dx - n\pi \}_{n=1}^\infty \) can be dense in the interval \([-A_0, A_0]\). Specifically we establish:

**Theorem 4:** Let \( p \equiv 1, z_0 = 0 < z_1 < z_2 < \ldots < z_{k+1} = L \) and \( 0 < \rho = b_i^2 \) for \( z_{i-1} \leq x < z_i, i = 1, 2, \ldots, k + 1 \) with \( \rho(L) = b_{k+1}^2 \). Suppose the set \( \{ b_i(z_i - z_{i-1}) \}_{i=1}^{k+1} \) is rationally independent. Then the set of differences

\[
\left\{ \omega_n \int_0^L \sqrt{\rho/p} \, dx - n\pi \right\}_{n=1}^\infty
\]

is dense in the interval \([-A_0, A_0]\).

We can combine the results of Theorems 1, 2, and 3 to obtain the best possible bound for the eigenvalues of (1)-(2) when \( p, \rho \) are of bounded variation and have a finite number of discontinuities.

**Theorem 5:** Let \( 0 < m < p, \rho \) for all \( 0 \leq x \leq L \). Suppose \( p, \rho \in BV[0, L] \) are continuous from the right and at \( x = L \) and have a finite number of discontinuities. Let \( z_i, i = 1, 2, \ldots, k \) be distinct and be the positions of the discontinuities of the product \( pp \). Express

\[
pp(x) = (pp)_c(x) + \sum_{i=1}^k \left[ (pp)_i^+ - (pp)_i^- \right] H(x - z_i),
\]

where \( (pp)_c \) is a continuous function. Let \( \omega_n^2 \) be the \( n \)th eigenvalue, \( n = 1, 2, \ldots \) for (1)-(2). Then

\[
| \omega_n \int_0^L \sqrt{\rho/p} - n\pi | \leq \frac{1}{4} V(\ell n(pp)_c)
\]

\[
+ \sum_{i=1}^k \arcsin \left[ \left| \sqrt{(pp)_i^+} - \sqrt{(pp)_i^-} \right| / \left( \sqrt{(pp)_i^+} + \sqrt{(pp)_i^-} \right) \right].
\]

**Remark:** If \( p, \rho \in BV[0, L] \) have an infinite number of discontinuities then an inequality analogous to the one in Theorem 5 can be easily established. One simply writes

\[
p = p_c + \sum_{i=1}^\infty \left[ p_i^+ - p_i^- \right] H(x - z_i),
\]
\[
\rho = \rho_c + \sum_{i=1}^{\infty} [\rho_i^+ - \rho_i^-] H(x - z_i),
\]

\[
p_k = p_c + \sum_{i=1}^{k} [p_i^+ - p_i^-] H(x - z_i),
\]

\[
\rho_k = \rho_c + \sum_{i=1}^{k} [\rho_i^+ - \rho_i^-] H(x - z_i),
\]

\[
(pp)_k = (pp)_c + \sum_{i=1}^{k} [(pp)_i^+ - (pp)_i^-] H(x - z_i),
\]

(15)

where \( p_c, \rho_c, (pp)_c \) are all continuous and where \( z_1, z_2, \ldots \) are all distinct and are the positions of the discontinuities of \( p, \rho \). Then from Theorem 5 we directly conclude that

\[
|\omega_n(p_k, \rho_k) \int_0^L \sqrt{\rho_k/p_k - n\pi} | \leq \frac{1}{4} V(\ell n(pp)_c)
\]

\[
+ \sum_{i=1}^{k} \arcsin \left[ \frac{\sqrt{(pp)_i^+} - \sqrt{(pp)_i^-}}{\sqrt{(pp)_i^+} + \sqrt{(pp)_i^-}} \right].
\]

for all \( k = 1, 2, \ldots \).

Taking the limit in the above equation as \( k \to \infty \) and using Lemma 1 establishes the following theorem

**Theorem 6:** Let \( 0 < m < p, \rho \) for all \( 0 \leq x \leq L \). Suppose \( p, \rho \in BV[0, L] \) are continuous from the right and at \( x = L \) and have an infinite number of discontinuities. Let \( z_i, i = 1, 2, \ldots \), be distinct and be the positions of the discontinuities of the product \( pp \). Express

\[
pp(x) = (pp)_c(x) + \sum_{i=1}^{\infty} [(pp)_i^+ - (pp)_i^-] H(x - z_i),
\]

where \( (pp)_c \) is a continuous function. Let \( \omega_n^2 \) be the \( n \)th eigenvalue, \( n = 1, 2, \ldots \) for (1)-(2).

Then

\[
|\omega_n \int_0^L \sqrt{\rho/p - n\pi} | \leq \frac{1}{4} V(\ell n(pp)_c)
\]

\[
+ \sum_{i=1}^{\infty} \arcsin \left[ \frac{\sqrt{(pp)_i^+} - \sqrt{(pp)_i^-}}{\sqrt{(pp)_i^+} + \sqrt{(pp)_i^-}} \right].
\]

We now present the proof of Lemma 1.
Proof of Lemma 1: It is known from [At] that for (1) - (2) there exists an infinite set of eigenvalues $0 < \omega_1^2 < \omega_2^2 < \ldots$ with $\lim_{n \to \infty} \omega_n^2 = \infty$. The eigenspace corresponding to each eigenvalue $\omega_n^2$ has dimension one, $n = 1, 2, \ldots$. Label the eigenfunction, $y_n$, that corresponds to $\omega_n^2$ and satisfies $\int_0^L p y_n^2 dx = 1$.

Using the definition of $p_\varepsilon, \rho_\varepsilon$ in the statement of the lemma we first show that the eigenvalues, $\omega_n^2(p, \rho), \omega_n^2(p_\varepsilon, \rho_\varepsilon), \varepsilon > 0$ can be determined as extremal values of the Rayleigh quotients

$$\mathcal{R}(p, \rho, y) = \frac{\int_0^L p(y_x)^2 dx}{\int_0^L \rho y^2 dx}, \quad \mathcal{R}(p_\varepsilon, \rho_\varepsilon, y) = \frac{\int_0^L p_\varepsilon(y_x)^2 dx}{\int_0^L \rho_\varepsilon y^2 dx}.$$ 

To see this we begin as follows. Define two Hilbert spaces. The first is $H^1_{0,p,\rho}(0, L)$, the completion of $C^\infty_0(0, L)$ with respect to the norm $\|u\|_{p,\rho}^1 = [\int_0^L pu_x^2 dx + \int_0^L pu^2 dx]^{1/2}$. We will use the equivalent norm $\|u\|_{p}^1 = [\int_0^L pu_x^2 dx]^{1/2}$; the corresponding inner product is $(u, v)_p^1 = \int_0^L pu_x v_x dx$. The second Hilbert space is $L^2_\rho(0, L) = \{u \mid \int_0^L \rho u^2 dx < \infty\}$ with norm $\|u\|_\rho^2 = [\int_0^L \rho u^2 dx]^{1/2}$ and inner product $(u, v)_\rho = \int_0^L \rho uv dx$.

The eigenfunctions $y_n$ and the corresponding products $py_{n,x}$ are absolutely continuous with $y_n(0) = y_n(1) = 0$, $n = 1, 2, \ldots$. Hence $y_n \in H^1_{0,p,\rho}(0, L)$, $n = 1, 2, \ldots$; the set $\{y_n\}_{n=1}^\infty$ is a complete orthonormal set in $L^2_\rho(0, L)$, see [At]; further

$$(y_n, y_m)^1_p = \begin{cases} 0 & n \neq m \\ \omega_n^2 & n = m, \end{cases}$$

and $\{y_n/\omega_n\}_{n=1}^\infty$ is a complete, orthonormal basis in $H^1_{0,p,\rho}$.

Now let $y \in H^1_{0,p,\rho}(0, L)$ be arbitrary with $y \neq 0$. Then there exists $\{d_n\}_{n=1}^\infty$ such that

$$y = \sum_{n=1}^\infty d_n y_n$$

with convergence both in $L^2_\rho(0, L)$ and in $H^1_{0,p,\rho}(0, L)$ implying

$$\sum_{n=1}^\infty d_n^2 = \int_0^L \rho y^2 dx \quad \text{and} \quad \sum_{n=1}^\infty \omega_n^2 d_n^2 = \int_0^L p y_x^2 dx.$$ 

Thus we can evaluate the Rayleigh quotient as

$$\mathcal{R}(p, \rho, y) = \frac{\sum_{n=1}^\infty \omega_n^2 d_n^2}{\sum_{n=1}^\infty d_n^2}.$$
This is minimized when $R(p, \rho, y) = \omega^2_n$ and the minimum is achieved only when
\[
d_n = \begin{cases} 
1 & n = 1 \\
0 & n \neq 1.
\end{cases}
\]
Hence
\[
\omega^2_1 = \inf_{y \in H^1_{0,p,\rho}(0,L)} R(p, \rho, y).
\]
A similar argument shows that
\[
\omega^2_k = \inf_{y \in H^1_{0,p,\rho}(0,L), (y, y_j)_\rho = 0, j = 1, \ldots, k-1} R(p, \rho, y).
\]
Finally, let $\ell_1, \ldots, \ell_{k-1}$ be $k-1$ linearly independent, bounded linear functionals defined on $L^2_\rho$. Then let $d_1, \ldots, d_k$ be constants such that $w = \sum_{j=1}^k d_j y_j$ and $\ell_i(w) = 0$, $i = 1, \ldots, k-1$. Then
\[
\inf_{y \in H^1_{0,p,\rho}(0,L), \ell_j(y) = 0, j = 1, \ldots, k-1} R(p, \rho, y) \leq \inf_{y \in H^1_{0,p,\rho}(0,L)} R(p, \rho, y) = \frac{\sum_{j=1}^k d_j^2 \omega^2_j}{\sum_{j=1}^k d_j^2} \leq \omega^2_k.
\]
Hence we arrive at the Rayleigh Ritz Principle for the eigenvalues for the problem (1) - (2)
\[
\sup_{\ell_1, \ldots, \ell_{k-1}} \inf_{y \in H^1_{0,p,\rho}(0,L), \ell_j(y) = 0, j = 1, \ldots, k-1} R(p, \rho, y) = \omega^2_k
\]
Here we have used the fact that (16) holds and that each $(y, y_j)_\rho, j = 1, \ldots, k-1$ is a bounded linear functional on $H^1_{0,p,\rho}(0,L)$. See [W] for more discussion of this Rayleigh-Ritz method.

Our lemma now follows from the inequalities
\[
\left[ \frac{1 - \frac{\varepsilon}{m}}{1 + \frac{\varepsilon}{m}} \right] \frac{f_0^L py^2 dx}{f_0^L \rho y^2} \leq \frac{f_0^L \rho_\varepsilon y^2 dx}{f_0^L \rho \varepsilon y^2} \leq \left[ \frac{1 + \frac{\varepsilon}{m}}{1 - \frac{\varepsilon}{m}} \right] \frac{f_0^L py^2 dx}{f_0^L \rho y^2 dx}.
\]
which imply, after applying the Rayleigh Ritz Principle,
\[
\left[ \frac{1 - \frac{\varepsilon}{m}}{1 + \frac{\varepsilon}{m}} \right] \omega^2_n(p, \rho) \leq \omega^2_n(p_\varepsilon, \rho_\varepsilon) \leq \left[ \frac{1 + \frac{\varepsilon}{m}}{1 - \frac{\varepsilon}{m}} \right] \omega^2_n(p, \rho).
\]
Let $\varepsilon \to 0$ and the lemma is proved.

In the next two sections we will require the following rough estimate of the distance between adjacent zeros of the eigenfunctions, $y_n$, for (1) - (2). In the following lemma we
show that when \( n \to \infty \), the distance between adjacent zeros of \( y_n \) goes to zero. Before we state the lemma we remind the reader that a function of bounded variation is always bounded.

**Lemma 2**: Define \( p, \rho \in BV[0,L] \) continuous from the right and at \( x = L \). Let \( m, M \) be constants with \( 0 < m < p, \rho < M \) for \( 0 \leq x \leq L \). Let \( \omega_n^2, y_n \) be the \( n \)th eigenvalue, eigenfunction pair for (1) - (2) with \( x_j^n, j = 1, \ldots, n-1 \), the consecutive zeros of \( y_n \) in \((0,L)\). Letting \( x_0^n = 0, x_n^n = L \), we have

\[
| x_{j+1}^n - x_j^n | \leq \frac{\pi}{\omega_n} \sqrt{\frac{M}{m}}, \quad j = 0, \ldots, n-1.
\]

**Proof of Lemma 2**: To begin select a zero, \( x_j^n, j = 0, \ldots, n-1 \) of \( y_n \). Extend the coefficients \( p, \rho \) to be \( p(L) \) and \( \rho(L) \) for \( x > L \). Consider the two initial value problems

\[
(py_x)_x + \omega_n^2 \rho y = 0, \quad x_j^n < x, \quad y(x_j^n) = 0, \quad y_x(x_j^n) = 1,
\]

and

\[
M v_{xx} + \omega_n^2 m v = 0, \quad x_j^n < x, \quad v(x_j^n) = 0, \quad v_x(x_j^n) = 1.
\]

The solution of (17) - (18) is a constant times \( y_n \) in the interval \([x_j^n, L]\). Letting \( x_a = x_j^n + (\pi/\omega_n)\sqrt{M/m} \), we can solve (19) - (20) explicitly to show that its solution, \( v(x) \), satisfies \( v(x_j^n) = 0, v(x_a) = 0 \), and \( v > 0 \) for \( x_j^n < x < x_a \). Here we have suppressed the dependence of \( x_a \) on \( n \) and \( j \).

The conclusion of Lemma 2 can now be shown by contradiction. We assume, contrary to what we want to show, that \( y_n(x) > 0 \) for \( x \in (x_j^n, x_a) \). The Picone identity,
see [1, p.226],
\[
\left( \frac{py_{n,x}v^2}{y_n} - Mvv_x \right)_x + p\left( v_x - \frac{vy_{n,x}}{y_n} \right)^2 + (M - p)v^2_x + \omega^2_n(\rho - m)v^2 = 0.
\]
is valid, a.e., in \([x^a_n, x_a]\). Since \((py_{n,x}v^2/y_n) - Mvv_x\) is absolutely continuous in \([x^a_n, x_a]\), we can integrate the identity from \(x^a_n\) to \(x_a\) to obtain
\[
\int_{x^a_n}^{x_a} \left\{ p\left[ v_x - \frac{vy_{n,x}}{y_n} \right]^2 + (M - p)v^2_x + \omega^2_n(\rho - m)v^2 \right\} dx = 0.
\]
Then the contradiction follows from the facts that \(m < p, \rho < M\) and \(0 < v\) in \((x^a_n, x_a)\); the conclusion of Lemma 2 follows.

Section 3: Uniqueness Theorems

In this section we present uniqueness results for the inverse nodal problem for two special cases of problem (1) - (2). For the first case \(p \equiv 1\) and \(\rho\) is variable and unknown on \([0, L]\).

For the second case, \(\rho \equiv 1\) and \(p\) is variable and unknown on \([0, L]\). We show for each case that the unknown variable coefficient, \(p\) or \(\rho\), is determined uniquely (up to a multiplicative constant) at every point of continuity by a dense subset of nodal positions. If \(p\) and \(\rho\) are smooth, see [McL], [HMcL1] and [HMcL2], then the dense subset is completely arbitrary. In this rough coefficient case our assumptions are stronger. We require that if \(x^n_j\) is in the dense subset then either \(x^n_{j-1}\) or \(x^n_{j+1}\) is also in the dense subset. We present the following two uniqueness results.

**Theorem 7:** In (1) - (2) let \(p \equiv 1, \rho \in BV[0, L]\), continuous from the right and at \(x = L\). Suppose there exists a constant \(m\) with \(0 < m < \rho\) for \(0 \leq x \leq L\). Then \(\rho\) is uniquely determined up to one multiplicative constant by a dense subset of pairs of nodal positions, \(x^n_j, x^n_{j+1}\).

**Proof of Theorem 7:** Let \(\rho_1, \rho_2\) both satisfy the hypotheses of the theorem. Suppose that for a dense subset of pairs of nodal positions \(x^n_j(\rho_1) = x^n_j(\rho_2)\) and \(x^n_{j+1}(\rho_1) = x^n_{j+1}(\rho_2)\). We will show that there exists a constant \(R > 0\) such that \(\rho_1/\rho_2 \equiv R\) at every point of continuity of \(\rho_1\) and \(\rho_2\).
To do this let $\tilde{x} \in [0, L]$ be a point of continuity of both $\rho_1$ and $\rho_2$. Let $[x_{n'}^{j(n')}, x_{n'}^{j(n')+1}]$, $n' = 1, 2, \ldots$, be a subset of the closed intervals defined by the given pairs of nodal positions and satisfying $\tilde{x} = \lim_{n' \to \infty} x_{n'}^{j(n')}$. For each $n'$, $n' = 1, 2, \ldots$ let $w_{n'}, y_{n'}$ be the $n$th eigenfunctions for the problem (1) - (2) with $p \equiv 1$ and $\rho = \rho_1$ or $\rho = \rho_2$ respectively. Then $\omega_{n'}^2(\rho_1), w_{n'}$ and $\omega_{n'}^2(\rho_2), y_{n'}$ are the first eigenvalue, eigenfunction pairs for

$$w_{xx} + \omega^2 \rho_1 w = 0,$$

$$w(x_j) = w(x_{j+1}) = 0,$$  (21)

and

$$y_{xx} + \omega^2 \rho_2 y = 0,$$

$$y(x_j) = y(x_{j+1}) = 0,$$  (23)

respectively. Here we have suppressed the dependency of the nodal positions on $n'$.

Substitute $w = w_{n'}$ in (21) and $y = y_{n'}$ in (23). Multiply equation (21) by $y_{n'}$ and equation (23) by $w_{n'}$. Subtract the two equations, integrate from $x_j$ to $x_{j+1}$ and divide by $\omega_{n'}^2(\rho_1)$. The result is

$$\int_{x_j}^{x_{j+1}} \left[ \rho_1 - \frac{\omega_{n'}^2(\rho_2)}{\omega_{n'}^2(\rho_1)} \rho_2 \right] w_{n'} y_{n'} dx = 0.$$  

Without loss we may assume that $w_{n'} y_{n'} > 0$ for $x_j < x < x_{j+1}$. Hence there exists $x'_j; x''_j \in (x_j, x_{j+1})$ with

$$\left[ \rho_1 - \frac{\omega_{n'}^2(\rho_2)}{\omega_{n'}^2(\rho_1)} \rho_2 \right] \bigg|_{x=x'_j} \geq 0$$

$$\left[ \rho_1 - \frac{\omega_{n'}^2(\rho_2)}{\omega_{n'}^2(\rho_1)} \rho_2 \right] \bigg|_{x=x''_j} \leq 0$$

Recall that $j = j(n')$. Since $\rho$ is in $BV[0, L]$ it is bounded above. Lemma 2 can be applied to show that $x_{j(n')}^{n'} - x_{j(n')+1}^{n'} \to 0$ as $n' \to \infty$. Hence $x'_j \to \tilde{x}$, $x''_j \to \tilde{x}$ as $n' \to \infty$. From Theorem 1 we conclude that $\lim_{n' \to \infty} \omega_{n'}^2(\rho_2)/\omega_{n'}^2(\rho_1)$ exists. Label the limit, $R$. Then in both of the above two inequalities let $n' \to \infty$ to obtain

$$(\rho_1 - R \rho_2)(\tilde{x}) = 0.$$  

24
This shows that $\rho_1 - R\rho_2 = 0$ at every point of continuity of $\rho_1$ and $\rho_2$. Since $\rho_1$ and $\rho_2$ each can have only a countable number of discontinuities and are both continuous from the right, the equation, $\rho_1 - R\rho_2 = 0$, holds for all $x \in [0, L]$. The theorem is proved. •

**Theorem 8:** In (1) and (2) let $\rho \equiv 1$, $p \in BV[0, L]$, continuous from the right and at $x = L$. Suppose there exists a constant $m$ with $0 < m < p$ for $0 \leq x \leq L$. Then $p$ is uniquely determined, up to one multiplicative constant, from a dense subset of pairs of nodal positions $x_j^n, x_{j+1}^n$.

**Proof of Theorem 8:** Let $p_1, p_2$ satisfy the hypotheses of the theorem. Suppose that for a dense subset of pairs of nodal positions $x_j^n(p_1) = x_j^n(p_2)$ and $x_{j+1}^n(p_1) = x_{j+1}^n(p_2)$. To establish the theorem we will show that there exists a constant $R > 0$ such that $p_1/p_2 \equiv R$ at every point of continuity of $p_1$ and $p_2$.

To do this let $\tilde{x} \in [0, L]$ be a point of continuity of both $p_1$ and $p_2$. Let $[x_{j(n')}^n, x_{j(n')+1}^n]$ be a subset of the closed intervals defined by the given pairs of nodal positions and satisfying $\tilde{x} = \lim_{n' \to \infty} x_{j(n')}^n$. For each $n'$, let $w_{n'}, y_{n'}$ be the $n'$th eigenfunctions for the problem (1) - (2) with $\rho \equiv 1$ and $p = p_1$ or $p = p_2$, respectively. Then $\omega_{n'}^2(p_1), w_{n'}$ and $\omega_{n'}^2(p_2), y_{n'}$ are the first eigenvalue, eigenfunction pairs for the eigenvalue problems

\begin{align*}
(p_1 w_x)_x + \omega^2 w &= 0, \tag{25} \\
 w(x_j) &= w(x_{j+1}) = 0, \tag{26}
\end{align*}

and

\begin{align*}
(p_2 y_x)_x + \omega^2 y &= 0, \tag{27} \\
y(x_j) &= y(x_{j+1}) = 0, \tag{28}
\end{align*}

respectively. Here again we have suppressed the dependency of the nodal positions on $n'$. Substitute $w = w_{n'}$ in (25) and $y = y_{n'}$ in (27). Then multiply (25) by $w_{n'}$ and (27) by $[w_{n'}/y_{n'}] \times [\omega_{n'}^2(p_1)/\omega_{n'}^2(p_2)]$, subtract the first resultant equation from the second and integrate from $x_j$ to $x_{j+1}$. This yields the equation

\[
\int_{x_j}^{x_{j+1}} \left[ p_1 - \frac{\omega_{n'}^2(p_1)}{\omega_{n'}^2(p_2)} p_2 \right] (w_{n',x})^2 \, dx + \int_{x_j}^{x_{j+1}} \left[ \frac{\omega_{n'}^2(p_1)}{\omega_{n'}^2(p_2)} p_2 \right] \left[ \frac{w_{n',x,y_{n'}} - y_{n',x} w_{n'}}{y_{n'}} \right]^2 \, dx = 0.
\]
Again we delay the justification that the ratio, \( w'_n/y_n' \), is bounded, and hence the integration steps are justified, until the end of our proof. The second of the above two integrals is non-negative. Hence there exists \( x'_j \in (x_j, x_{j+1}) \) with

\[
\left( p_1 - \frac{\omega^2_n(p_1)}{\omega^2_n(p_2)} p_2 \right) (x'_j) \leq 0.
\]

To obtain a complimentary inequality multiply (25) by \( y^2_n/w_n' \) and (27) by \( y_n' \times [\omega^2_n(p_1)/\omega^2_n(p_2)] \). Subtract the first resultant equation from the second and integrate from \( x_j \) to \( x_{j+1} \). This yields the equation

\[
\int_{x_j}^{x_{j+1}} \left[ p_1 - \frac{\omega^2_n(p_1)}{\omega^2_n(p_2)} p_2 \right] (y_n', x)^2 \, dx - \int_{x_j}^{x_{j+1}} p_1 \left[ \frac{y_n', x w_n' - w_n', x y_n'}{w_n'} \right]^2 \, dx = 0.
\]

Again we delay the justification that the ratio, \( y_n'/w_n' \), is bounded. The second of the above two integrals is non-negative. Hence there exists \( x''_j \in (x_j, x_{j+1}) \) with

\[
\left( p_1 - \frac{\omega^2_n(p_1)}{\omega^2_n(p_2)} p_2 \right) (x''_j) \geq 0.
\]

Now \( x''_j \to \bar{x} \) and \( x'_j \to \bar{x} \) as \( n' \to \infty \). Let \( R = \lim_{n' \to \infty} \omega^2_n(p_1)/\omega^2_n(p_2) \). Then taking the limit as \( n' \to \infty \) we have

\[
p_1 = Rp_2
\]

at every point of continuity of \( p_1 \) and \( p_2 \). Since \( p_1 \) and \( p_2 \) can have only a countable number of discontinuities and are both continuous from the right, the theorem will be proved once we have shown that the ratios, \( w'_n/y_n' \) and \( y_n'/w_n' \) are bounded.

The outline of this last demonstration is as follows. Let \( x_j < x < x_{j+1} \). Recall that \( p_1 y_n', x \) and \( p_2 w_n', x \) are absolutely continuous and bounded away from zero in sufficiently small neighborhoods of any zero of \( y_n \) and \( w_n \) respectively. Further there exist \( x_j < x', x'' < x \) satisfying

\[
y_n'(x) = \int_{x_j}^{x} y_{n', s} ds = \int_{x_j}^{x} \frac{1}{p_2} p_2 y_{n', s} ds = p_2 y_{n', x}(x') \int_{x_j}^{x} \frac{1}{p_2} ds,
\]

\[
w_n'(x) = \int_{x_j}^{x} w_{n', s} ds = \int_{x_j}^{x} \frac{1}{p_1} p_1 w_{n', s} ds = p_1 w_{n', x}(x'') \int_{x_j}^{x} \frac{1}{p_1} ds,
\]

from which it follows that \( w_n'/y_n' \) is bounded as \( x \to x_j^+ \). A similar argument applies for the ratio \( y_n'/w_n' \) and when \( x \to x_{j+1}^- \).
The proof is complete.

Remark 1: We remark that the constant $R$ in our proofs can never be determined by the nodal positions. The reason for this is that we can multiply either $p$ or $\rho$ by a constant and then if $p$ is changed we would multiply the eigenvalues by the same constant or if $\rho$ is changed we would divide the eigenvalues by the constant. In either case the eigenvalues would all change but the eigenfunctions, and hence the nodal positions, would all remain the same.

Remark 2: In our uniqueness theorems we do not require that the eigenvalues be given as data. However, if the eigenvalues are known for (1) - (2) when either $p \equiv 1$ and $\rho$ is variable or $\rho \equiv 1$ and $p$ is variable then the multiplicative constant can be determined. In fact the algorithms in the next section can be adapted to show how to uniquely determine one variable coefficient at each point of continuity from both the eigenvalues and a dense subset of pairs of adjacent nodal positions.

Remark 3: It is possible to obtain a dense set of pairs of nodal positions by choosing exactly one pair from each eigenfunction. See [McL] for this construction for a similar problem.

Remark 4: In our proofs of Theorems 7 and 8 when either $p$ or $\rho$ are assumed known we choose to assume that they are equal to the constant, 1. The same proof would hold if $p$ in Theorem 7 and $\rho$ in Theorem 8 are assumed to be known, positive, functions of bounded variation.

Section 4: Algorithms

In this section we present two sets of algorithms for determining piecewise constant approximations to rough (bounded variation) coefficients from nodal position data. In each of the algorithms the data is an eigenvalue, $\omega_n^2$, and all of the nodal positions, $x_j^n$, $j = 1, \ldots, n - 1$.
for the corresponding mode shape or eigenfunction. We choose this data because it can all be measured when a rod is excited longitudinally or a string is excited transversely at a natural frequency. An experiment for obtaining these measurements is described in the introduction.

Our algorithms yield piecewise constant approximates to the unknown coefficient. The piecewise constant approximates have a special property: when we substitute the $n$th approximate in (1) - (2) the corresponding $n$th eigenvalue, $\omega_n^2$, and nodal positions, $x_j^n$, $j = 1, ... n - 1$ are exactly the given data.

In our presentation of the algorithms we let $x_0^n = 0$ and $x_n^n = L$ and then suppress the dependence of the nodal positions on $n$. Now consider the case where $p \equiv 1$. Then the piecewise constant approximate for the coefficient $\rho$ in problem (1) - (2) is given by the algorithm:

**Algorithm A:**

\[
\rho^n = \begin{cases} 
\rho_{j+1}^n = \frac{\pi^2}{\omega_n^2(x_{j+1}^n - x_j^n)} x_j^n \leq x < x_{j+1}^n, j = 0, \ldots, n - 2 \\
\rho_n = \frac{\pi^2}{\omega_n^2(L - x_{n-1}^n)} x_{n-1} \leq x \leq x_n = L 
\end{cases}
\]

Each $\rho^n \in BV[0, L]$. The following convergence result holds.

**Theorem 9:** In (1) - (2) let $p \equiv 1$, $\rho \in BV[0, L]$, continuous from the right and at $x = L$. Suppose there exists a constant $m$ with $0 < m < \rho$ for $0 \leq x \leq L$. Let the $n$th eigenvalue $\omega_n^2$ and the $n - 1$ nodal positions $x_j^n$, $j = 1, \ldots, n - 1$ for the $n$th eigenfunction be given. Define $\rho^n$ as in Algorithm A. Then $\rho^n \rightarrow \rho$ pointwise at every point of continuity of $\rho$. Furthermore, the total variation $V(\rho^n) \leq 2V(\rho)$ for all $n$.

**Proof of Theorem 9:** Let $y_n$ be the $n$th eigenfunction for (1) - (2). Let $v_n$ be the $n$th eigenfunction for

\[
v_{xx} + \omega^2 \rho^n v = 0. \tag{29}
\]

\[
v(0) = v(L) = 0. \tag{30}
\]

Note that (1) - (2) with $p \equiv 1$ and (29) - (30) have the same $n$th eigenvalue, $\omega_n^2$, and the same nodal positions for the $n$th eigenfunction. Because of this property, we can apply the
same identities as in the proof of the uniqueness theorem, Theorem 7.

We first show the pointwise convergence. Let \( \tilde{x} \in [0, L) \) be a point of continuity of \( \rho \).
For each \( n \) select \( j = j(n) \) so that \( \tilde{x} \in [x_{j(n)}^n, x_{j(n)+1}^n) \). Arguing as in the proof of Theorem 7, except that now \( \omega_n(\rho) = \omega_n(\rho^n) \) there exists \( x_j', x_j'' \in (x_j, x_{j+1}) \) with

\[
\rho(x_j') \leq \rho^n(\tilde{x}) \leq \rho(x_j'').
\]

[We remind the reader that in our notation we suppress the dependence of \( x_j, x_{j+1}, x_j' \) and \( x_j'' \) on \( n \).] Now let \( n \to \infty \). We have \( x_j' \to \tilde{x}, x_j'' \to \tilde{x} \) and \( \rho(\tilde{x}) = \lim_{n \to \infty} \rho(\tilde{x}) \leq \lim_{n \to \infty} \rho(x_j') = \rho(\tilde{x}) \).

Hence \( \rho^n(\tilde{x}) \to \rho(\tilde{x}) \). The argument when \( \tilde{x} = L \) is a point of continuity is similar.

It remains to show \( V(\rho^n) \leq 2V(\rho) \) for all \( n \). Clearly

\[
V(\rho^n) = \sum_{j=1}^{n-1} |\rho_{j+1}^n - \rho_j^n|.
\]

Now for each \( j = 1, 2, \ldots, n - 1 \), we know that there exist \( x_j', x_j'' \in (x_j, x_{j+1}) \) and \( x_{j+1}', x_{j+1}'' \in (x_{j+1}, x_{j+2}) \) with

\[
\rho(x_j') \leq \rho^n_j \leq \rho(x_j'')
\]

and

\[
\rho(x_{j+1}') \leq \rho^n_{j+1} \leq \rho(x_{j+1}'')
\]

respectively. This implies that we can choose \( s_j \in (x_j, x_{j+1}) \) and \( t_j \in (x_{j+1}, x_{j+2}) \), \( j = 1, \ldots, n - 1 \) satisfying

\[
|\rho_{j+1}^n - \rho_j^n| \leq |\rho(s_j) - \rho(t_j)|.
\]

Then

\[
V(\rho^n) = \sum_{j=1}^{n-1} |\rho_{j+1}^n - \rho_j^n| \leq \sum_{j=1}^{n-1} |\rho(s_j) - \rho(t_j)|
\]

\[
= \sum_{j \text{ odd}} |\rho(s_j) - \rho(t_j)| + \sum_{j \text{ even}} |\rho(s_j) - \rho(t_j)|
\]

\[
\leq 2V(\rho)
\]
This completes the proof.

The piecewise constant approximate for problem (1) - (2) when $\rho \equiv 1$ and $p$ is variable is given by the algorithm:

Algorithm B:

\[
\begin{align*}
p_n &= \begin{cases} 
p_{j+1}^n = \frac{\omega_n^2 (x_{j+1} - x_j)^2}{\pi^2} & x_j \leq x < x_{j+1}, j = 0, \ldots, n-2 \\
p_n^n = \frac{\omega_n^2 (x_n - x_{n-1})^2}{\pi^2} & x_{n-1} \leq x \leq x_n = L.
\end{cases}
\end{align*}
\]

Each $p^n \in BV[0, L]$. The following convergence result holds.

**Theorem 10:** In (1) - (2) let $\rho \equiv 1$, $p \in BV[0, 1]$, continuous from the right and at $x = L$. Suppose there exists a constant $m$ with $0 < m < p$ for $0 \leq x \leq L$. Let the $n$th eigenvalue $\omega_n^2$ and the $n-1$ nodal positions $x_j^n$, $j = 1, \ldots, n-1$ for the $n$th eigenfunction be given. Define $p^n$ as in Algorithm B. Then $p^n \to p$ pointwise at every point of continuity of $p$. Furthermore the total variation $V(p^n) \leq 2V(p)$ for all $n$.

**Proof of Theorem 10:** The proof is very similar to the proof of Theorem 9.

**Remark:** In Theorem 9, it can be shown that there exists an $M > 0$ so that each $\rho^n$ satisfies $|\rho^n| < M$ and $\rho^n \to \rho$ a.e. In Theorem 10, it can be shown that there exists $M > 0$ so that each $p^n$ satisfies $|p^n| < M$ and $p^n \to p$ a.e. Hence by the Lebesgue dominated convergence theorem $\rho^n \to \rho$ and $p^n \to p$ in $L^q(0, L)$ for any $q, 1 \leq q < \infty$. However, no convergence rate can be given.

**Section 5: Numerical Experiments**

To test algorithms A, B we must calculate the eigenvalues $\omega_n^2$ and the nodal positions $x_j^n$ for $(py, x) + \omega^2 \rho y = 0$ with $y(0) = y(L) = 0$. To do this we use the Prüfer transformation (4). We assume that $p$, $\rho$ are piecewise smooth functions and use the classical fourth order Runge-Kutta method to solve equation (5) between the discontinuities with 4000 points in each subinterval. In all our experiments $L = 1$. To compute the jump in $\theta$ in a stable manner we consider two cases. If $|\sin(\theta^-)| < |\cos(\theta^-)|$ then
\[
\theta^+ = \theta^- + \tan^{-1} \left[ \sqrt{\frac{(p\rho)^+}{(p\rho)^-}} \tan(\theta^-) \right] - \tan^{-1} \left[ \tan(\theta^-) \right];
\]

if \( |\cos(\theta^-)| \leq |\sin(\theta^-)| \), we use

\[
\theta^+ = \theta^- + \frac{\sin(\theta^-)}{|\sin(\theta^-)|} \left( \cos^{-1} \left[ \frac{\cos(\theta^-)}{\sqrt{\frac{(p\rho)^+}{(p\rho)^-} \sin^2(\theta^-) + \cos^2(\theta^-)}} \right] - \cos^{-1} \left[ \cos(\theta^-) \right] \right).
\]

The eigenfrequency \( \omega_n \) is obtained by solving \( \theta(L) = n\pi \) by the IMSL subroutine ZBRENT. To determine the nodes \( x_n \) we solve (5) once more with \( \omega = \omega_n \) and determine successive meshpoints between which \((\theta + \pi/2) \mod \pi - \pi/2\) changes sign. Since both \( \theta \) and \( \theta_x \) are available at the mesh points we find the nodes by inverse interpolation.

We consider two classes of problems

\[
(p y_x)_x + \omega^2 y = 0 \tag{31}
\]
\[
y_{xx} + \omega^2 \rho y = 0, \tag{32}
\]

each with Dirichlet boundary conditions. In Table 1 we give the eigenfrequency and the nodes for the 10th eigenfunction for (31) and (32) when \( p = f_1 \) or \( \rho = f_1 \) where

\[
f_1(x) = \begin{cases} 
\sqrt{\frac{1}{4} + 2x} & 0 \leq x < 0.375 \\
\sqrt{3 - 2x} & 0.375 \leq x \leq 1.
\end{cases}
\]

Note that the nodes are close when \( p \) is small or \( \rho \) is large.
Now let $e^n = \rho^n - \rho$ and $e^n = p^n - p$ be the errors in algorithms A and B, respectively. We see from Theorems 9 and 10 that $e^n \to 0$ a.e. and $|e^n| \leq 2M$. As discussed in the remark following Theorem 10, it follows that $\|e^n\|_{L^q(0,1)} \to 0$ as $n \to \infty$ for any $q \in [1, \infty)$. But for $p, \rho \in BV[0, L]$, no convergence rate can be given. To study this problem numerically we set (suppressing the dependence of the nodal positions on $n$) $\hat{x}_j = (x_j + x_{j-1})/2$ and use the discrete $L^1$ norm, the discrete $L^2$ norm and the discrete maximum norm

$$
\|e\|_q = \left( \sum_{j=1}^{n} (x_j - x_{j-1}) |e(\hat{x}_j)|^q \right)^{1/q}, \quad q = 1, 2
$$

$$
\|e\|_\infty = \max_{1 \leq j \leq n} |e(\hat{x}_j)|.
$$

In Table 2 we present the errors when $p = f_1$ and $n = 5, 10, 20, 40$. The results for $\rho = f_1$ are similar. The main contribution comes from the interval that contains the jump. To see this we give the truncated errors, obtained by disregarding the interval where $p$ jumps. The data indicates quadratic convergence away from the discontinuity, which is consistent with our observations in [HMcL2].

<table>
<thead>
<tr>
<th></th>
<th>$p = f_1$</th>
<th>$\rho = f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{10}$</td>
<td>31.667945</td>
<td>30.340327</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.074607</td>
<td>0.131772</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.157087</td>
<td>0.248017</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0.246160</td>
<td>0.355741</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0.340969</td>
<td>0.441698</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0.450408</td>
<td>0.528447</td>
</tr>
<tr>
<td>$x_6$</td>
<td>0.568081</td>
<td>0.617180</td>
</tr>
<tr>
<td>$x_7$</td>
<td>0.682145</td>
<td>0.708204</td>
</tr>
<tr>
<td>$x_8$</td>
<td>0.792357</td>
<td>0.801927</td>
</tr>
<tr>
<td>$x_9$</td>
<td>0.898426</td>
<td>0.898916</td>
</tr>
</tbody>
</table>

Table 1. Two problems with 9 nodes
\[
\begin{array}{|c|cccc|}
\hline
 & n = 5 & n = 10 & n = 20 & n = 40 \\
\hline
\text{Discrete } L^1 \text{ error} & 0.046 & 0.031 & 0.011 & 0.0074 \\
\text{Discrete } L^2 \text{ error} & 0.086 & 0.089 & 0.046 & 0.044 \\
\text{Discrete } L^\infty \text{ error} & 0.18 & 0.27 & 0.20 & 0.27 \\
\text{Truncated } L^1 \text{ error} & 0.0052 & 0.0014 & 0.00035 & 0.000090 \\
\text{Truncated } L^2 \text{ error} & 0.0068 & 0.0017 & 0.00044 & 0.00011 \\
\text{Truncated } L^\infty \text{ error} & 0.014 & 0.0042 & 0.0012 & 0.00031 \\
\hline
\end{array}
\]

Table 2. Errors in reconstruction of \( p = f_1 \).

For the remaining examples we use the discrete \( L^1 \) norm since it is least sensitive to the discontinuity. Here is the list of test functions:

\[
\begin{align*}
f_2(x) & = x + \frac{1}{2} + H\left(x - \frac{1}{2}\right) \\
f_3(x) & = e^x + H\left(x - \frac{1}{2}\right) \\
f_4(x) & = 1 + \frac{1}{2} \sin[10\pi(x - \frac{1}{2})] + H\left(x - \frac{1}{2}\right) \\
f_5(x) & = \begin{cases} 
1 - x & 0 \leq x < 0.45 \\
\frac{3}{2} - x + \frac{1}{2} \sin[10\pi(x - \frac{1}{2})] & 0.45 \leq x < 0.50 \\
\frac{5}{2} - x + \frac{1}{2} \sin[10\pi(x - \frac{1}{2})] & 0.50 \leq x < 0.55 \\
3 - x & 0.55 \leq x \leq 1.00.
\end{cases}
\end{align*}
\]

In Tables 3 and 4 we give the discrete \( L^1 \) errors in the reconstruction of \( p = f_i \) and \( \rho = f_i \) for \( i = 2, 3, 4, 5 \). The jump at \( x = 1/2 \) is 1 in all cases. In general the errors for \( p \) are larger than the errors for \( \rho \). The function \( f_4 \) is particularly difficult because it oscillates and has large second derivatives. Note that \( f_5(x) \) is differentiable at \( x = 0.45 \) and \( x = 0.55 \).
Table 3. Discrete $L^1$ errors for different elasticity coefficients.

<table>
<thead>
<tr>
<th></th>
<th>$n = 5$</th>
<th>$n = 10$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = f_2$</td>
<td>0.0052</td>
<td>0.016</td>
<td>0.011</td>
<td>0.0056</td>
</tr>
<tr>
<td>$\rho = f_3$</td>
<td>0.0019</td>
<td>0.00084</td>
<td>0.0016</td>
<td>0.0049</td>
</tr>
<tr>
<td>$\rho = f_4$</td>
<td>0.18</td>
<td>0.057</td>
<td>0.018</td>
<td>0.0047</td>
</tr>
<tr>
<td>$\rho = f_5$</td>
<td>0.011</td>
<td>0.026</td>
<td>0.020</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

Table 4. Discrete $L^1$ errors for different densities.

Finally we comment on the accuracy of the computed eigenvalues. To check this we have used two problems that can be transformed into each other by a Liouville transformation and thus have the same eigenvalues. We define these problems as follows. Let $a = \ell n(13/8)$ and set

$$p(x) = \begin{cases} 
\frac{3}{4} u^{-1} & 0 \leq x < \frac{1}{2}, \\
 a (u + u^{-1}) & \frac{1}{2} \leq x \leq 1 
\end{cases} \quad u = \frac{1}{2} + x$$

$$\rho(x) = \begin{cases} 
\frac{2}{3} (1 + 6x)^{-1/2} & 0 \leq x < \frac{1}{2} \\
 a \left(v^{1/2} + v^{-1/2}\right) & \frac{1}{2} \leq x < 1, \\
 v = \frac{16}{15} \exp(2ax) - 1 
\end{cases}$$

The computed eigenfrequencies for the two problems agree to 12 significant digits. We list four of the computed eigenfrequencies in Table 5 together with the discrete $L^1$ errors for the corresponding approximate for $p$ and the corresponding approximate of $\rho$ given directly above. Once again, the errors in the approximation for $\rho$ are slightly smaller than the errors in the approximation for $p$.

<table>
<thead>
<tr>
<th></th>
<th>$n = 5$</th>
<th>$n = 10$</th>
<th>$n = 20$</th>
<th>$n = 40$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_n$</td>
<td>15.771784</td>
<td>31.505064</td>
<td>62.997018</td>
<td>125.987236</td>
</tr>
<tr>
<td>$L^1$ error for $p$</td>
<td>0.020</td>
<td>0.0015</td>
<td>0.00070</td>
<td>0.00050</td>
</tr>
<tr>
<td>$L^1$ for $\rho$</td>
<td>0.027</td>
<td>0.00070</td>
<td>0.00018</td>
<td>0.000048</td>
</tr>
</tbody>
</table>
Table 5. Inverse nodal problems with same eigenvalues.

Acknowledgment

The authors thank L.E. Andersson, R.L. Anderssen and E.T. Trubowitz for useful discussions. The research of J.R. McLaughlin was partially supported by NSF grants VPW-8902067, DMS-9401700 and ONR grants NOOO14-91J-1166, N)))14-96-1-0349. The research of O.H. Hald was partially supported by NSF grant DMS-9503482.
References


**Appendix A**

In this appendix we construct continuous, increasing functions $G_p(x)$, $p = 1, 2, \ldots$ and $G(x)$ on $0 \leq x \leq 1$ satisfying

$$\lim_{p \to \infty} \| G_p - G \|_{\infty} = 0.$$ 

Each $G_p$ will be piecewise continuously differentiable and have total variation, $V(G_p) = 1$; $G$ is in the class of Cantor functions with $V(G) = 1$. In addition, we establish a set of positive integers $n_q$ with the properties

$$\lim_{q \to \infty} \int_0^1 \sin (2\pi n_q x) \, dG = -1, \quad \text{and} \quad \int_0^x \sin (2\pi n_q x) \, dG \leq 0, \ 0 \leq x \leq 1. \quad (A.1)$$

Our goal is to use this result in Appendix B to show that the bound in Theorem 1 is best possible when $p(x)$ and $\rho(x)$ are continuous and of bounded variation.

Our construction is explicit. Similar results, using the concept of $N$ set instead of explicit construction, can be found in Zygmund, [Z], for Fourier cosine coefficients. Here we use an explicit construction as it makes our application in Appendix B easier.
To construct the example, let

\[ \xi_1 = \frac{4}{9}, \quad \xi_k = \frac{1}{9^k}, \quad k = 2, 3, \ldots, \]

\[ \eta_k = \frac{1}{4} - \frac{1}{2} \xi_k, \quad k = 1, 2, \ldots, \]

\[ R_1 = \eta_1, \quad R_p = \eta_1 + \sum_{k=2}^{p} \eta_k \prod_{j=1}^{k-1} \xi_j = \frac{1}{4} - \frac{1}{4} \sum_{k=1}^{p-1} \prod_{j=1}^{k} \xi_j - \frac{1}{2} \prod_{j=1}^{p} \xi_j, \quad p = 2, 3, \ldots \]

Let \( \ell_0^p = 0, \quad \ell_{2p+1}^p = 1 \) and let \( \ell_1^p < \ell_2^p < \ldots < \ell_{2p}^p \) be the ordered elements in

\[ \mathcal{L}_p = \left\{ R_p + \frac{1}{2} \left[ \varepsilon_1 + \xi_1 \varepsilon_2 + \ldots + \varepsilon_p \prod_{k=1}^{p-1} \xi_k \right] \middle| \varepsilon_j = 0, 1, \quad j = 1, 2, \ldots, p \right\}, \quad p = 2, \ldots \]

where we note that \( \ell_j^p + 2^{p-1} = \frac{1}{2} + \ell_j^p, \quad j = 1, \ldots, 2^{p-1} \). Now define

\[ F_p = \begin{cases} 
0, & \ell_0^p \leq x \leq \ell_1^p, \\
\frac{j}{2^p}, & \ell_j^p + \Pi_{k=1}^{p} \xi_k \leq x \leq \ell_{j+1}^p, \quad j = 1, \ldots, 2^p, \\
\frac{(x - \ell_j^p)}{2^p \Pi_{k=1}^{p} \xi_k} + \frac{j - 1}{2^p}, & \ell_j^p \leq x \leq \ell_{j+1}^p + \Pi_{k=1}^{p} \xi_k, \quad j = 1, \ldots, 2^p. 
\end{cases} \]

Using \( F_p \) we construct

\[ G_p = \begin{cases} 
1 - F_p, & 0 \leq x \leq \frac{1}{2} \\
F_p, & \frac{1}{2} < x \leq 1 
\end{cases} \quad (A.2) \]

and note that \( G_p \) is continuous at \( x = \frac{1}{2} \) with \( G_p(\frac{1}{2}) = \frac{1}{2} \) and that \( G_p(x) = G_p(1 - x) \). Further \( V(G_p) = 1, \quad p = 1, 2, \ldots \) Let

\[ G(x) = \lim_{p \to \infty} G_p. \quad (A.3) \]

Clearly, \( V(G) = 1 \). Further for each positive integer, \( q \), let \( n_q = 9^{q(q+1)/2} \). Then we can establish

**Theorem A:**

\[ - \int_0^x dG \leq \int_0^x \sin(2\pi n_q x) dG \leq - \left( 1 - \frac{8\pi^2}{9^{2q+2}} \right) \int_0^x dG, \]

\[ \limsup_{n \to \infty} \int_0^1 \sin(2\pi n x) dG = -1. \]
Proof of Theorem A: Set \( p = q + 1 \) and let \( \ell_j^{q+1}, j = 1, 2, ..., 2^{q+1} \) be the collection of points in \( L_{q+1} \). Then the function \( G \) is decreasing on each of the subintervals

\[
\left\{ \ell_j^{q+1} < x < \ell_j^{q+1} + \prod_{k=1}^{q+1} \xi_k \right\}_{j=1}^{2^q}
\]

and increasing on

\[
\left\{ \ell_j^{q+1} < x < \ell_j^{q+1} + \prod_{k=1}^{q+1} \xi_k \right\}_{j=2^{q+1}}^{2^{q+1}}.
\]

Let

\[
J_q = \bigcup_{j=1}^{2^{q+1}} \left( \ell_j^{q+1}, \ell_j^{q+1} + \prod_{k=1}^{q+1} \xi_k \right).
\]

The complement of \( J_q \) in \([0,1]\) is a finite collection of closed disjoint intervals. On each of these intervals the function \( G \) is constant.

Hence

\[
\int_0^1 \sin(2\pi n_q x) dG = \sum_{j=1}^{2^{q+1}} \int_{\ell_j^{q+1}}^{\ell_j^{q+1} + \prod_{k=1}^{q+1} \xi_k} \sin(2\pi n_q x) dG.
\]

The goal is to obtain a bound on the integrals on the right side of the above inequality.

Let an arbitrary \( x \) in the interval

\[
\ell_j^{q+1} < x < \ell_j^{q+1} + \prod_{k=1}^{q+1} \xi_k, \ j = 1, 2, ..., 2^{q+1}
\]

be represented by

\[
x = \ell_j^{q+1} + \alpha \prod_{k=1}^{q+1} \xi_k, \ 0 < \alpha < 1.
\]

Since \( \prod_{k=1}^{q+1} \xi_k = 4/9(q+1)/2 \), we have

\[
2\pi n_q x = \frac{\pi}{2} + 2\pi(n_q) \left[ \left( \alpha - \frac{1}{2} \right) \prod_{k=1}^{q+1} \xi_k + \frac{1}{2} \ell_1 \right]
\]

\[
+ 2\pi \times \text{integer}
\]

\[
= \frac{\pi}{2} + \pi \epsilon_1 \times \text{odd integer} + \frac{4\pi(2\alpha - 1)}{9^{q+1}}
\]

\[
+ 2\pi \times \text{integer}.
\]
Now $|2\alpha - 1| \leq 1$. Hence for the case $0 < \ell_j^{q+1} < \frac{1}{2} (\epsilon_1 = 0)$ we obtain
\[
1 - \frac{8\pi^2}{q^{2q+2}} \leq \sin(2\pi n_q x) \leq 1.
\]
Similarly, for $\frac{1}{2} < \ell_j^{q+1} < 1(\epsilon_1 = 1)$
\[-1 \leq \sin(2\pi n_q x) \leq -(1 - \frac{8\pi^2}{q^{2q+2}}).
\]
Hence,
\[-\int_0^x dG \leq \int_0^x \sin(2\pi n_q x) dG \leq -(1 - \frac{8\pi^2}{q^{2q+2}}) \int_0^x dG.
\]
The statement in the theorem follows.

**Appendix B**

In this appendix we show that for $p, \rho$ continuous and in $BV[0, 1]$ the bound in Theorem 1 is best possible. We do this by considering the specific example where $p = \rho = a(x)$; that is
\[
(au)_x + \omega^2 au = 0, \quad 0 \leq x \leq 1, \quad (B.1)
\]
\[u(0) = u(1) = 0 \quad (B.2)
\]
We will use the example of Appendix A to exhibit a sequence of continuous functions $a_q \in BV[0, 1]$ satisfying
\[
\lim_{q \to \infty} \left| \omega_{n_q} - n_q \pi \right| - \frac{1}{2} V(\ell n_q) = 0.
\]
where $(\omega_{n_q})^2$ is the $n_q$th eigenvalue of (B.1),(B.2) with $a$ replaced by $a_q$.

To begin we prove the following technical lemma.

**Lemma B.1:** Let $G(\phi)$ and $n_q, q = 1, 2, \ldots$ be as defined in Appendix A. Define
\[
\omega_{n_q} = n_q \pi - \frac{1}{2} \int_0^1 \sin(2n_q \pi \phi) dG(\phi). \quad (B.3)
\]
Then the integral equation
\[
\phi(x) = \frac{1}{n_q \pi} \left\{ \omega_{n_q} x + \frac{1}{2} \int_0^\phi \sin(2n_q \pi \tilde{\phi}) dG(\tilde{\phi}) \right\}
\]
has a strictly increasing solution \(\phi_q(x)\) satisfying \(\phi_q(0) = 0, \phi_q(1) = 1\).

**Proof of Lemma B.1:** Let
\[
x(\phi) = \frac{1}{\omega_{n_q}} \left[ n_q \pi \phi - \frac{1}{2} \int_0^\phi \sin(2n_q \pi \tilde{\phi}) dG(\tilde{\phi}) \right].
\]
The function \(x(\phi)\) is a continuous, strictly increasing function of \(\phi\) with piecewise continuous derivative satisfying \(x(0) = 0, x(1) = 1\). The inverse, \(\phi(x)\), of \(x(\phi)\) satisfies (B.4) together with \(\phi(0) = 0, \phi(1) = 1\). Label \(\phi(x)\) as \(\phi_q(x)\) and the lemma is proved.

Having established Lemma B.1 we can immediately construct a sequence of continuous functions \(a = a_q(x)\) having \((\omega_{n_q})^2\) as their \(n_q\)th eigenvalues, \(q = 1, 2, \ldots\)

**Lemma B.2:** Let
\[
a_q(x) = \exp \left[ G(\phi_q(x)) \right].
\]
Then the \(n_q\)th eigenvalue for the eigenvalue problem
\[
(a_x u)_x + \omega^2 a_q u = 0, \quad 0 \leq x \leq 1,
\]
\[
u(0) = u(1) = 0
\]
is \((\omega_{n_q})^2\), defined in (B.3).

**Proof of Lemma B.2:** Construct the Prüfer transformation (4) for the eigenvalue problem (B.6). The integral equation for \(\theta\) becomes
\[
\theta(x) = \omega x + \frac{1}{2} \int_0^x \sin 2\theta(\tilde{x})dG(\phi_q(\tilde{x})).
\]
Letting \(\theta = n_q \pi \phi_q\) and \(\omega = \omega_{n_q}\), then the lemma follows from (B.4).

We can now quickly prove the desired theorem, that is, that for continuous functions of bounded variation, \(p = q = a(x)\), the bound in Theorem 1 is best possible.
**Theorem B:** The \( n_q \)th eigenvalue, \( (\omega_{n_q})^2 \), for

\[
(a_q u_x)_x + \omega^2 a_q u = 0, \quad 0 \leq x \leq 1
\]

\[
u(0) = u(1) = 0,
\]

\( q = 1, 2, \ldots \) satisfies

\[
\left| \omega_{n_q}(a_q) - n_q \pi \right| - \frac{1}{2} V(\ell n a_q) \leq \frac{4\pi^2}{9^{q+2}}
\]

implying

\[
\lim_{q \to \infty} \left| \omega_{n_q} - n_q \pi \right| - \frac{1}{2} V(\ell n a_q) = 0.
\]

**Proof of Theorem B:** Observe that

\[
\left| \omega_{n_q} - n_q \pi \right| - \frac{1}{2} V(\ell n a_q) = -\frac{1}{2} \int_0^1 \sin(2 n_q \pi \phi) dG(\phi) - \frac{1}{2} V(\ell n a_q)
\]

and that, from Lemma A.1,

\[-\frac{4\pi^2}{9^{q+2}} \leq -\frac{1}{2} \int_0^1 \sin(2 n_q \pi \phi) dG(\phi) - \frac{1}{2} V(\ell n a_q) \leq 0.
\]

The statement of the theorem follows.

**Remark:** Using the same function, \( G \), we can also construct continuous functions of bounded variation \( p_q(x) \) and \( \rho_q(x) \) for the eigenvalue problems

\[
(p_q u_x)_x + \omega^2 u = 0, \quad 0 \leq x \leq 1,
\]

\[
u(0) = u(1) = 0,
\]

or

\[
u_{xx} + \omega^2 \rho_q(x) u = 0, \quad 0 \leq x \leq 1,
\]

\[
u(0) = u(1) = 0,
\]

respectively, to show that for these cases as well the bound in Theorem 1 is best possible.
Appendix C

In this appendix we establish Theorem 4 of Section 2. Without loss of generality we consider

\[ u_{xx} + \omega^2 \rho u = 0, \quad 0 \leq x \leq L, \]  

(B.7)

\[ u(0) = u(L) = 0 \]  

(B.8)

where, with \( 0 = z_0 < z_1 < z_2 < ... < z_{k+1} = L \), we define

\[ \rho(z) = b_i^2 > 0, \quad z_{i-1} \leq z < z_i, \quad i = 1, ..., k + 1 \]

\[ \rho(L) = b_{k+1}^2. \]

Let \( a_i = b_{i+1}/b_i \). We first establish the following lemma.

**Lemma C**: Suppose \( \{b_i(z_i - z_{i-1})\}_{i=1}^{k+1} \) is a rationally independent set. Then given \( \epsilon > 0 \) and \( \eta \epsilon [-1, 1] \) there exists an eigenvalue \( (\omega_n)^2 \) for (B.7), (B.8) satisfying

\[ \left| \omega_n \int_0^L \sqrt{\rho} \, dx - n\pi - \eta \sum_{i=1}^k \arcsin \left( \frac{|a_i - 1|}{a_i + 1} \right) \right| < \epsilon, \]

(B.9)

where \( \arcsin \) refers to the principal value.

**Proof of Lemma C**: To obtain our result we return to the Prüfer transformation and use equation (5) for \( \theta \) in each interval \( z_{i-1} < x < z_i, \quad i = 1, ..., k + 1 \). We require \( \theta(0, \omega) = 0 \) and that \( \theta \) is continuous as a function of \( x \) everywhere in \([0, L]\) except at each \( x = z_i, \quad i = 1, ..., k \). Letting \( \theta_i^- = \lim_{x \to z_i^-} \theta(x, \omega) \) and \( \theta_i^+ = \lim_{x \to z_i^+} \theta(x, \omega) \) the jump condition at each \( x = z_i, \quad i = 1, ..., k \) is determined by (6), (7) with \( \theta_i^+ \) and \( \theta_i^- \) chosen to be in the same quadrant \( i = 1, ..., k \). For each \( \omega \) this yields

\[ \theta_x(x, \omega) = \omega b_i, \quad z_{i-1} < x < z_i, \quad i = 1, ..., k + 1, \]

(B.10)

\[ \theta(0, \omega) = 0, \]

(B.11)

with

\[ a_i \tan \theta_i^- = \tan \theta_i^+, \quad i = 1, 2, ..., k. \]

(B.12)
Further, \( \omega \) is an eigenvalue provided that \( \theta(L, \omega) \) is a positive integer multiple of \( \pi \).

We divide the proof into three cases: either \( \eta > 0 \), \( \eta < 0 \) or \( \eta = 0 \), and without loss assume \( a_i \neq 1 \), \( i = 1, \ldots, k \). Suppose first that \( \eta > 0 \). Our immediate goal is to find a real number \( \tilde{\omega} \) that will satisfy inequality (C.3), with \( \omega_n \) replaced by \( \tilde{\omega} \), and that we will ultimately show is close to \( \omega_n \). We proceed as follows. Select \( \delta_1, \ldots, \delta_k \) to satisfy

\[
\arctan(a_i \delta_i) - \arctan(\delta_i) = \eta \arcsin \left( \frac{|a_i - 1|}{a_i + 1} \right),
\]

where \( \arctan \) refers to the principal value. For each \( i \) there are two solutions of this equation. We choose the solution closest to the origin. Now apply Kronecker’s Theorem from number theory, see [HW, p.380], to find \( \tilde{\omega} > \frac{3\pi}{2 \min_{i=1,2,\ldots,k+1} (b_i(z_i - z_{i-1}))} \) and integers \( p_1, \ldots, p_{k+1} \) satisfying

\[
\tilde{\omega} b_1 z_1 - p_1 \pi + \arctan(\delta_1) = \epsilon_1, \quad (B.13)
\]

\[
\tilde{\omega} b_i (z_i - z_{i-1}) - p_i \pi + \arctan(\delta_i) - \arctan(a_{i-1} \delta_{i-1}) = \epsilon_i, \quad i = 2, \ldots, k, \quad (B.14)
\]

\[
\tilde{\omega} b_{k+1} (L - z_k) - p_{k+1} \pi - \arctan(a_k \delta_k) = \epsilon_{k+1}, \quad (B.15)
\]

where \( |\epsilon_i| < \epsilon^2, i = 1, \ldots, k \). Adding all of the above equations and letting \( n = \sum_{i=1}^{k+1} p_i \) we obtain

\[
\left| \tilde{\omega} \int_0^L \sqrt{\rho} \, dx - n \pi - \eta \sum_{i=1}^{k} \arcsin \left( \frac{|a_i - 1|}{a_i + 1} \right) \right| = \left| \sum_{i=1}^{k+1} \epsilon_i \right| < (k + 1) \epsilon^2.
\]

It remains to establish that the \( n \)th eigenvalue \( \omega_n \) is sufficiently close to \( \tilde{\omega} \).

Letting \( \tilde{\epsilon} = \epsilon/(2 \int_0^L \sqrt{\rho} \, dx) \) we can show that such an eigenvalue exists if we can show that the expressions

\[
\theta(L, \tilde{\omega} + \tilde{\epsilon}) - n \pi \quad \text{and} \quad \theta(L, \tilde{\omega} - \tilde{\epsilon}) - n \pi
\]

have opposite sign. To do this we proceed sequentially and choose \( \epsilon \) sufficiently small when needed. Replace \( \omega \) in (B.10)-(B.11) by \( \tilde{\omega} \pm \tilde{\epsilon} \) and define positive constants

\[
C_i = \sum_{j=1}^i b_j (z_j - z_{j-1}) \prod_{\ell=j}^i \frac{a_\ell (1 + \delta_\ell^2)}{1 + a_\ell^2 \delta_\ell^2}, \quad i = 1, 2, \ldots, k.
\]

Using (B.10), (B.11) and (B.13) and after making a Taylor expansion we obtain

\[
\theta^-_i = (\tilde{\omega} \pm \tilde{\epsilon}) b_1 z_1 = p_1 \pi - \arctan(\delta_1) \pm \tilde{\epsilon} b_1 z_1 + 0(\epsilon^2),
\]

\[
\arctan(a_1 \tan \theta^-_1) = - \arctan(a_1 \delta_1) \pm C_1 \tilde{\epsilon} + 0(\epsilon^2)
\]

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implying, since (B.12) is satisfied with $\theta^+_1$ and $\theta^-_1$ in the same quadrant, that

$$\theta^+_1 = p_1 \pi - \arctan(a_1 \delta_1) \pm C_1 \tilde{\epsilon} + 0(\epsilon^2).$$  \hspace{1cm} (B.16)$$

Similarly using (B.10), (B.14) and (B.16) we obtain

$$\theta^-_2 = + \theta^+_1 (\tilde{\omega} \pm \tilde{\epsilon}) b_2 (z_2 - z_1)$$

$$= (p_1 + p_2) \pi - \arctan(\delta_2) \pm (C_1 + b_2 (z_2 - z_1)) \tilde{\epsilon} + 0(\epsilon^2),$$

$$\arctan(a_2 \tan \theta^-_2) = - \arctan(a_2 \delta_2) \pm C_2 \tilde{\epsilon} + 0(\epsilon^2).$$

Since $\theta^-_2$ and $\theta^+_2$ are in the same quadrant, it follows from (B.12) that

$$\theta^+_2 = (p_1 + p_2) \pi - \arctan(a_2 \delta_2) \pm C_2 \tilde{\epsilon} + 0(\epsilon^2)$$

$$\vdots$$

$$\theta^+_k = (p_1 + \ldots + p_k) \pi - \arctan(a_k \delta_k) \pm C_k \tilde{\epsilon} + 0(\epsilon^2).$$

Now calculate $\theta(L, \tilde{\omega} \pm \tilde{\epsilon}) = \theta^+_k + b_{k+1} (L - z_k)$ using (B.15).

$$\theta(L, \tilde{\omega} \pm \tilde{\epsilon}) = (p_1 + \ldots + p_k) \pi - \arctan(a_k \delta_k) \pm C_k \tilde{\epsilon} + 0(\epsilon^2)$$

$$+ p_{k+1} \pi + \arctan(a_k \delta_k) \pm \tilde{\epsilon} b_{k+1} (L - z_k)$$

$$= n \pi \pm \tilde{\epsilon} [b_{k+1} (L - z_k) + C_k] + 0(\epsilon^2)$$

implying that, for sufficiently small $\epsilon$, $\theta(L, \tilde{\omega} + \tilde{\epsilon}) - n \pi$ and $\theta(L, \tilde{\omega} - \tilde{\epsilon}) - n \pi$ have opposite sign. Hence the eigenvalue $\omega_n$ is in the interval $(\tilde{\omega} - \tilde{\epsilon}, \tilde{\omega} + \tilde{\epsilon})$.

The cases $\eta < 0$ and $\eta = 0$ are handled similarly. The lemma is proved. \hspace{1cm} $\bullet$

The following theorem is a direct corollary of Lemma C.

**Theorem C (Theorem 4 in Section 2):** Let $p \equiv 1, z_0 = 0 < z_1 < z_2 < \ldots < z_{k+1} = L$ and $0 < \rho = b_i^2$ for $z_{i-1} \leq x < z_i, i = 1, 2, \ldots, k + 1$, with $\rho(L) = b_{k+1}^2$. Suppose the set $\{b_i(z_i - z_{i-1})\}_{i=1}^{k+1}$ is rationally independent. Then the set of differences

$$\left\{ \omega_n \int_0^L \sqrt{\rho/p} \, dx - n \pi \right\}_{n=1}^{\infty}$$

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is dense in the interval \([-A_0, A_0]\) where

\[ A_0 = \sum_{i=1}^{k} \arcsin \left( \frac{|a_i - 1|}{a_i + 1} \right). \]