

Recovery of a vertically stratified seabed in shallow water*

Joyce R. McLaughlin[†]

Shixiao Wang[‡]

Abstract

We describe an algorithm for finding a vertically stratified seabed in shallow water from acoustic data. Numerical reconstructions are given.

Introduction: The goal is to find an approximate sound speed profile for a vertically stratified seabed in shallow water. The data is the pressure field measured in the water column where it is assumed that the sound speed is known. The pressure field is created by a sound wave source located in the water column. In our method we transform the pressure field and from this transform we extract partial spectral data for a half space problem. The approximate sound speed in the seabed is then obtained as a solution to an inverse scattering problem.

Our results build on the work of Frisk, Rajan, et al, [3],[10], [8], [9] and Henkin and Novikov, [5], [6]. We briefly review this previous work. In Frisk, Rajan, et al, the authors begin with the same experiment and make the same transform of the data. The bound state information, i.e. eigenvalues, is extracted for a half space inverse scattering problem. No information about the eigenfunctions and no data from the reflection coefficient for the continuous spectrum is used. Then, with sound source frequencies at 200 HZ to 400 HZ they show that with their limited data set and under the assumption that the sound speed is piecewise constant, the inverse problem does not produce a unique solution. This is to be expected; however, it is shown that there is a wide range of variability in the possible solutions of the inverse problem. In [7] the starting point is the half line inverse scattering problem. It is assumed that the Weyl function for the half line problem is measured at a finite number of points. A Gelfand-Levitan, [4], integral equation method is employed to solve the half line inverse scattering problem. Only bound state information is used; it is obtained from a Padé approximation to the spectral function. In [5] again the Gelfand-Levitan method is used, only bound state data is employed, and estimates of the error are given in inverse powers of the sound source frequency. Note that as the sound source frequency increases the number of bound states increases.

Here we take a different approach. The data is naturally given to us as the bound states, some information about the corresponding eigenfunctions, and the reflection coefficient for part of the continuous spectrum. The reflection coefficient is a rational function of the Weyl function. Our method to find the soundspeed approximates uses: (1) the bound states, i.e. eigenvalues; (2) the value of the Weyl function for each eigenfunction calculated at a point in the water column; and (3) the reflection coefficient, or equivalently the Weyl function, also calculated at a point in the water column for part of the continuous

*This work was partially funded by ONR #N00014-96-1-0349.

[†]Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180, mclauj@rpi.edu

[‡]Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180, wangs3@rpi.edu

spectrum. The numerical procedure is based on the trace formula, given in [2], and Darboux transformations, similar but not the same as those given in [2]. Since we have only partial spectral data we cannot recover a unique soundspeed function. However, we show by our numerical computations that we obtain a good approximation. In a later paper we will prove an estimate for the difference between the true sound speed and our approximate.

Specifically, we show that our calculated soundspeed approximate becomes better and better as the sound source frequency increases. This is in direct correlation with the fact that the number of bound states increases as the sound source frequency increases and also the length of the subinterval of the continuous spectrum, where we know the reflection coefficient, increases. Two other features that we observe are also worth noting. First, the contribution to the solution of the reflection coefficient from the continuous spectrum varies according to the choice of the sound source frequency. If the sound source frequency has been decreased so that a bound state has just been eliminated then the contribution of the reflection coefficient is highest. Further in this particular case, we must take special care with the algorithm, artificially decreasing the soundspeed in the water column to ensure that information contained in the reflection coefficient is not lost.

Mathematical model and description of the data set: In our model, we represent the soundspeed as $c(z)$. The water column, $0 < z < z_d$, has constant sound speed, c_0 , and the seabed soundspeed is the constant, c_∞ , at sufficient depth, $z > z_\infty > z_d$. The density, ρ_0 , is constant throughout. In addition the sound source is assumed located in the water column at z_0 and has frequency ω . The pressure is measured at z_1 also in the water column. The pressure, $p(r, z, z_0)$ satisfies the circularly symmetric Helmholtz equation together with the pressure release boundary condition and Sommerfeld radiation condition,

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} + k^2(z) \right) p(r, z, z_0) = -2 (\delta(r)/r) \delta(z - z_0), \quad p(r, 0, z_0) = 0$$

$$\lim_{\sqrt{r^2+z^2} \rightarrow \infty} (\sqrt{r^2+z^2}) \left(\frac{\partial p(r, z, z_0)}{\partial \sqrt{r^2+z^2}} - ik_\infty p(r, z, z_0) \right) = 0$$

Taking then the zero order Hankel transformation one obtains a depth dependent Green's function, $g(k_r, z, z_0)$ when k_r , the transform variable, is real satisfying

$$(1) \quad g(k_r, z, z_0) = i \left\{ e^{i\sqrt{k_0^2 - k_r^2} |z - z_0|} - e^{i\sqrt{k_0^2 - k_r^2} (z + z_0)} \right. \\ \left. + R_b e^{2i\sqrt{k_0^2 - k_r^2} z_s} \left(e^{-i\sqrt{k_0^2 - k_r^2} (z + z_0)} - e^{-i\sqrt{k_0^2 - k_r^2} |z - z_0|} \right) \right\} \\ \div \left\{ \sqrt{k_0^2 - k_r^2} \left(1 + R_b e^{2i\sqrt{k_0^2 - k_r^2} z_s} \right) \right\}$$

where $R_b = R_b(\sqrt{k_\infty^2 - k_r^2})$ is the reflection coefficient at $z = z_s$, $0 < z_0 < z_1 < z_s < z_d$. This implies that, with $k(z) = \omega/c(z)$, $k_\infty = \omega/c_\infty$, $k_0 = \omega/c_0$, for the problem

$$(2) \quad \left(\frac{d^2}{dz^2} + k^2(z) - k_r^2 \right) f(k_r, z) = 0 \quad f(k_r, z) = e^{i\sqrt{k_\infty^2 - k_r^2} z}, \quad z > z_\infty$$

where k_r is considered as the spectral parameter, the solution is

$$(3) \quad f(k_r, z) = \beta(k_r) \left(e^{i\sqrt{k_0^2 - k_r^2} (z - z_s)} + R_b e^{-i\sqrt{k_0^2 - k_r^2} (z - z_s)} \right), \quad \text{for } 0 < z \leq z_s.$$

Further $g(k_r, z, z_0)$ has a pole as a function of k_r precisely where the boundary value problem comprised of (2) together with the boundary condition $f(k_r, 0) = 0$ has a bound state. Finally we can obtain the value of the Weyl function, $f'(k_r, z_s)/f(k_r, z_s)$, for each discrete spectral point, $k_r^2 = k_{r\ell}^2, \ell = 1, 2, \dots, n$ where $k_{r1}^2 > k_{r2}^2 > \dots > k_{rn}^2 > k_\infty^2$ and for the real continuous spectrum, $0 < k_r < k_\infty$, from $R_b\left(\sqrt{k_\infty^2 - k_r^2}\right)$.

Algorithm for Solution: Our method is as follows. First, we employ Darboux type transformations to remove the bound states from our half line problem. Our transformations are related to but not the same as the one in [2, p.180, 181]. Our method is similar to the method given in [2] in that we always remove the largest bound state, or equivalently the one with the nonzero eigenfunction (on $(0, \infty)$). However in our method when we remove, e.g., k_{r1}^2 the coefficient $k^2(z)$ is changed to

$$-k^2(z, -1) = k^2(z) - 2k_{r1}^2 + 2\left(\frac{f'_1(k_{r1}, z)}{f_1(k_{r1}, z)}\right)^2$$

where $f_1(k_n, z)$ is the solution of (2) satisfying $f_1(k_{r1}, 0) = 0$. Letting $f(k_r, z, -j)$, be the solution of (2) and $k^2(z, -j)$ the new coefficient in the differential equation after j bound states have been removed, the information we carry forward at each step is: (1) the Weyl Function $f'(k_r, z_s, -j)/f(k_r, z_s, -j)$ for $k_r = k_{r\ell}, \ell = 1, 2, \dots, n$ and $0 < k_r < k_\infty$; and (2) $k^2(z, -j)$ for $0 < z \leq z_s$. This step is achievable since at the start we know the soundspeed in the water column. For lack of space we do not give the transformation at every step here; however, see [6]. Second, we use $k^2(z, -n)$, for $0 < z \leq z_s$ and $f'(k_r, z_s, -n)/f(k_r, z_s, -n)$ to calculate $f'(k_r, 0, -n)/f(k_r, 0, -n)$ and hence the reflection coefficient, $R\left(\sqrt{k_\infty^2 - k_r^2}, -n\right)$ at $z = 0$ for $0 < k_r < \infty$ as

$$(4) \quad R\left(\sqrt{k_\infty^2 - k_r^2}, -n\right) = \frac{1 + i\sqrt{k_\infty^2 - k_r^2} f'(k_r, 0, -n)/f(k_r, 0, -n)}{1 - i\sqrt{k_\infty^2 - k_r^2} f'(k_r, 0, -n)/f(k_r, 0, -n)}.$$

However we note that our Darboux type transformation can introduce a $1/z^2$ singularity at the origin. In our third step, we extend $k^2(z, -n)$ to be k_∞^2 on $-\infty < z < 0$ and artificially alter $k^2(z, -n)$ near the origin to remove the singularity. This extension to the whole line may introduce a single bound state for the whole line problem. We cannot calculate the position of this bound state. The possible introduction of a bound state for the whole line problem and the fact that we cannot calculate such a bound state is not a consequence of the particular Darboux transformation we have chosen. Further, to eliminate this possibility then, the value of $k^2(z, -n)$ may be further decreased, artificially, near the origin.

The fourth step is to use $k(z, -n)$ for $0 < z \leq z_s$ and $f'(k_r, z_s, -n)/f(k_r, z_s, -j)$ to calculate $f'(k_r, 0, -n)/f(k_r, 0, -n)$ and hence the reflection coefficient, $R\left(\sqrt{k_\infty^2 - k_r^2}, -n\right)$ at $z = 0$ for $0 < k_r < k_\infty$ as in (4). Again this is achievable since we know $k(z, -n)$ in the water column. Then use the trace formula, see [2], to simultaneously solve

$$k_\infty^2 - k_a^2(z, -n) = \frac{2i}{\pi} \int_{-k_\infty}^{k_\infty} \mu R(\mu, -n) h_2^2(\mu, z) d\mu,$$

$$\left(\frac{d^2}{dz^2} - (k_\infty^2 - k_a^2(z, -n)) + \mu^2\right) h_2(\mu, z) = 0, \text{ where } h_2(\mu, z) = e^{-i\mu z}, -\infty < z < 0,$$

for the approximate, $k_a^2(z, -n)$, and $h_2(\mu, z)$. Note that we have used $R(\mu) = \bar{R}(\mu)$ and that we have let $\mu^2 = k_\infty^2 - k_r^2$. Note further that here we can make a comparison between

the known $k^2(z, -n)$ and the calculated $k_a^2(z, -n)$ in the water column. If the comparison is not favorable then this indicates that, in Step 2, a bound state for the whole line problem has been introduced; thus if the comparison is not favorable then in Step 2 a further reduction of $k^2(z, -n)$ near the origin is required.

Numerical reconstructions: We exhibit two figures. In Figure 1 we show the recovery of $c_a(z)$ where $\omega = 1000Hz$ and the true $c(z)$ is piecewise constant. For this value of ω there are seven bound states. In Figure 2 we show the recovery of $c_a(s)$ when $\omega = 330Hz$. At this frequency there are three bound states. In Figure 2 we also show the approximate, $c_b(z)$, obtained with the bound state information only (eigenvalues and the corresponding Weyl function at $z = z_s$ for each bound state). The algorithm for $c_b(z)$ is substantially

easier than the full algorithm described here; the bound state only algorithm involves only the last step of the full algorithm by beginning with $k = k_\infty$ for $0 < z < \infty$ and using our Darboux type transformations to put in the bound states. Finally, in Figure 3 when $\omega = 1000Hz$ we show the recovery of $c_a(z, -7) = \omega/k_a(z, -7)$ determined by the trace formula and partial continuous spectrum data only and the recovery of $c_a(z, -5)$, $c_a(z, -3)$ and $c_a(z)$ where we use partial continuous spectrum data together with two, four, or seven bound states, respectively.

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