Formulas for Finding Coefficients from Nodal Lines

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Introduction

We ask the question: What can be determined about a vibrating system from the positions of nodal lines? The question is answered for a rectangular membrane. We give a formula for finding a potential from the nodal line positions. A uniqueness result is presented. Analogous one-dimensional results are presented; three dimensional results are announced. A case study which shows the effect of structural damping on measurements is described.

Two experiments can be performed to both motivate the choice of nodal position data for the inverse problem and to motivate the mathematical difficulties in the two dimensional case. One experiment is a vibrating beam, the other a vibrating plate. The vibrating beam is driven at one end and free at the other. This experiment is performed twice, once for a homogeneous beam and once for a beam with mass added in a small subregion of the beam. The changes in natural frequencies, which are lowered, and changes in the nodal positions, which move toward the added mass, can be measured. For the second experiment, a plate driven at the center and free on the edges is excited at several different natural frequencies. Sand is distributed on the plate. At each frequency the sand accumulates along the corresponding nodal lines. This experiment is similar to the Chladni experiments, see [17]. One observes that the connected domains defined by the nodal lines are frequently long, curved strips with occasional smaller enclosed domains.

These experiments, especially the plate experiment, are demonstration experiments. One can ask then: Can the nodal lines or nodal points be measured accurately? One method to do this is to direct a lazer at the vibrating surface. The Doppler shift in the backscatter is measured. The lines, or points, where the Doppler shift is minimized are the nodal lines.

The mathematical results in higher dimensions build on previous one dimensional results. These results, obtained by McLaughlin and Hald, [11], [6], [7], [8], present formulas, uniqueness results, and numerical calculations. In the second section we briefly present a few of these results. The model we choose is the longitudinal motion of a beam in the case where both the elasticity coefficient, p, and the density, ρ, may have discontinuities. Two formulas are given. One gives a piecewise constant approximation to the elasticity coefficient when the density is constant. This formula converges to the elasticity coefficient at every point of continuity. The other formula gives a piecewise constant approximation to the density when the elasticity coefficient is constant. This formula converges to the density at every point of continuity. To establish our results a bound on the square of the eigenvalues is established for the case where both p and ρ are of bounded variation. This work is presented in the first section.

For the two dimensional problem, see [9] and [12], we consider a rectangular membrane. There is a force on the membrane which depends linearly on the displacement. The amplitude of the force is unknown. We solve the inverse problem: find the amplitude, q, from nodal line positions. We show that at a dense set of points q can be approximated by the difference of two eigenvalues. To establish this result there are two difficulties. One is that in order to establish perturbation results for the eigenvalues and eigenfunctions, we must solve a small divisor problem. This problem is solved by establishing criteria for eigenvalues to be well-separated, thus making it pos;sible to bound terms that contain the small divisors. Almost all eigenvalues satisfy the criteria and perturbation results are established for only those eigenvalues. The second difficulty is that even though the nodal domains can be long, thin, curved domains, we must cut these domains to define small approximate nodal domains. The method for solving each of these problems, as well as the presentation of the formula for q is given in the third section. The three dimensional results are discussed briefly in Section 3 as well.

We discuss briefly the effect of structural damping on nodal point measurement in the last section.

The One Dimensional Bounded Variation Problem

We consider the mathematical model for the longitudinal vibrations of a beam with fixed ends. The elasticity coefficient $p > 0$ and density $\rho > 0$ are of bounded variation.

$$\begin{aligned} (p\,u_x)_x + \lambda \rho u &= 0\,, \qquad 0 \,<\, x \,<\, L, \quad p, \rho \in BV[0,L], \\ u(0) \,=\, u(L) &= 0. \end{aligned} \tag{1}$$

This problem has a set of eigenvalues satisfying $0 < (\omega_1)^2 < (\omega_2)^2 < ....$ The nth eigenfunction has exactly n-1 nodes which we label in increasing order, x_j^n, $j = 1, ..., n-1$.

To establish our piecewise constant approximations to p and ρ we require an estimate for $\omega_n(p,\rho)$. Letting $V(f)$ represent the total variation of a function f of bounded variation, we have shown, see [8],

<u>Theorem 1</u>: *Let* $p > 0$, $\rho > 0$ *satisfy* p, $\rho \in BV[0,L]$. *Then the eigenvalues for* (1),$\{[\omega_n(p,\rho)]^2\}_{n=1}^{\infty}$, *obey the bound*

$$\left| \omega_n(p,\rho) \int_0^L \sqrt{\frac{p}{\rho}} dx \quad - \quad n\pi \right| \;\leq\; \frac{1}{4} V(\ln p\rho).$$

Our goal is to find piecewise constant approximations to either p or ρ from measurements of the eigenvalues and nodal positions. To do this, let $x_0^n = 0$ and $x_n^n = L$ and define

$$\rho_n \;=\; \left[\frac{\pi}{\omega_n (x_j^n - x_{j-1}^n)} \right]^2 \qquad \text{when} \qquad x_{j-1}^n \leq x < x_j^n, \qquad j = 1, ..., n$$

together with $\rho_n(L) = \rho_n(x_{n-1}^n)$. Define

$$p_n \;=\; \left[\frac{\omega_n \,(x_j^n \,-\, x_{j-1}^n)}{\pi} \right]^2 \qquad \text{when} \qquad x_{j-1}^n \leq x < x_j^n \quad j = 1, ..., n$$

together with $p_n(L) = p_n(x_{n-1}^n)$. Then we have shown in [8].

<u>Theorem 2</u>: *Let* $p = 1$, $\rho > 0$ *and* $\rho \in BV[0,L]$ *in* (1). *Then* ρ_n *converges pointwise to* ρ *at every point of continuity.*

<u>Theorem 3</u>: *Let* $\rho = 1$, $p > 0$ *and* $p \in BV[0,L]$ *in* (1). *Then* p_n *converges pointwise to* p *at every point of continuity.*

Results of numerical experiments are presented in [8].

The Two Dimensional Smooth Potential Problem

Here we consider the mathematical model for a rectangular vibrating membrane fixed on the boundary; the rectangle $R = [0, \pi/a] \times [0, \pi]$ with a^2 chosen to be irrational. Letting $q \in C_0^\infty(R)$, the mathematical model for the eigenvalue problem is then

$$\begin{aligned} -\Delta u + qu &= \lambda u, \qquad x \in R, \\ u &= 0, \qquad x \in \partial R. \end{aligned} \tag{2}$$

The object is to find q from the nodal line positions of the eigenfunctions for this problem.

Before stating our main results, we establish notation for the $q = 0$ problem. Letting $\alpha = (an, m)$ we define the lattice L to be

$$L = \{\alpha = (an, m) |\ n, m = 1, 2, 3, ...\}.$$

Then the eigenvalue and normalized eigenfunction pairs for the $q = 0$ problem can be naturally indexed by $\alpha \in L$ as

$$\lambda_{\alpha 0} = |\alpha|^2 = a^2 n^2 + m^2, \quad u_{\alpha 0} = \frac{2\sqrt{a}}{\pi} \sin anx \sin my.$$

Our theory requires that almost all of the eigenvalues $\{\lambda_{\alpha 0}\}_{\alpha \in L}$ be well-separated. To be well -separated the eigenvalues must satisfy five conditions. We describe three of these conditions in words here. The first condition is that there is a small interval about $\lambda_{\alpha 0} = |\alpha|^2$ which contains no other eigenvalues of the $q = 0$ problem. The length of the interval decreases slowly as $|\alpha| \to \infty$. The second condition is that for lattice points β near α the corresponding eigenvalues, $\lambda_{\beta 0} = |\beta|^2$, are a large distance from $\lambda_{\alpha 0}$. This distance, $\lambda_{\beta 0} - \lambda_{\alpha 0}$, increases rapidly as $|\alpha| \to \infty$. The third condition is that the number of oscillations of $u_{\alpha 0}$ in the x and y directions are comparable. All five conditions are stated explicitly in the Appendix at the end of this paper.

We do not show our main results for all irrational a^2 but only for those values which are poorly approximated by rational numbers. Our condition is almost the same as that given by Moser, [13]. That is, fixing $A_0 > 1$ and $0 < \delta < \frac{1}{2}$, we require $a \in V$ where

$$V = \Big\{ a \;\;|\;\; 1 < a < A_0 \quad \text{and there exists} \quad K > 0 \quad \text{so that for all}$$

$$\text{integers} \quad p, q > 0, \quad | a^2 - \frac{p}{q} | > \frac{K}{q^{2+\delta}} \Big\},$$

We note that $meas\, V = A_0 - 1$. Further we can identify specific $a \in V$. By a theorem by Roth, [14], if a^2 is irrational and algebraic then $a \in V$.

We can now state our main theorems. We do not give our sharpest possible results here but give a presentation which makes our ideas more accessible to the reader. Letting $⨍ q = \frac{2\sqrt{a}}{\pi} \int_R q$ be the integral average of q we can show, see [9], the uniqueness theorem,

Theorem 4: Let $a \in V$ *and* $q \in C_0^\infty(R)$. *The translated potential* $q - ⨍ q$ *in* (2) *is uniquely determined by a subset of nodal lines of the eigenfunctions.*

The uniqueness theorem above is a corollary of the following theorem which gives a formula for approximating q.

Theorem 5: Let $a \in V$. *Let* $q \in C_0^\infty(R)$. *There exists an infinite set* $L(a) \setminus M(a)$ *and a dense set of* $x' \in R$ *with corresponding subsets* $\Omega'_\alpha \subset R$ *defined by the nodal lines, and indexed by* $\alpha \in L(a) \setminus M(a)$, *so that the potential in* (2) *satisfies*

$$\left| q(x') - ⨍ q - [\lambda_{\alpha 0} - \lambda_{1,0}(\Omega'_\alpha)] \right| < \frac{1}{2} |\alpha|^{-7/4}$$

Here $\lambda_{1,0}(\Omega'_\alpha)$ *is the smallest eigenvalue for*

$$-\Delta u = \lambda u \,, \; x \in \Omega'_\alpha, \qquad u = 0 \,, \; x \in \partial\Omega'_\alpha.$$

Remark: We show that the set $L(a) \setminus M(a)$ has density one in $L(a)$. This means

$$\lim_{r \to \infty} \frac{\#\{\alpha \in L(a) \backslash M(a) \;\;|\;\; |\alpha| < r\}}{\{\alpha \in \text{Ł}(a) \;\;|\;\; |\alpha| < r\}} = 1.$$

Further, for α to be in $L(a) \setminus M(a)$ it is necessary, though not sufficient, for the three conditions described above to be satisfied. Finally, for any $N > 0$ we can choose our dense set $\{x'\}$ so that the formula for approximating q satisfies the above bound for some $|\alpha| > N$.

To establish Theorem 5, and hence Theorem 4, we proved the following perturbation result whose demonstration requires the solution of a 'small divisor' problem, see [9].

<u>Theorem 6</u>: Let $a \in V$. Let $q \in C_0^\infty(R)$. Then there exists a set $L(a) \setminus M(a)$ of density one such that for $\alpha \in L(a) \setminus M(a)$ there is a unique eigenvalue, normalized eigenfunction pair $\lambda_{\alpha q}, u_{\alpha q}$ of (2) satisfying

$$\begin{aligned} |\lambda_{\alpha q} - ⨍ q - \lambda_{\alpha 0}| &\leq |\alpha|^{-15/8}, \\ \left\| u_{\alpha q} - u_{\alpha 0} - \sum_{\beta \neq \alpha} \frac{(q u_{\alpha 0}, u_{\beta 0})}{\lambda_{\alpha 0} - \lambda_{\beta 0}} \right\|_\infty &\leq \sqrt{a}\, |\alpha|^{-15/8}. \end{aligned}$$

Remark: Our proof of this result was influenced by the perturbation results in [1], [2], [3]. There L^2 bounds for the eigenfunctions were obtained. We require L^∞ bounds for the eigenfunctions for two reasons. One is so that we can get good estimates for the positions of the nodal lines for $u_{\alpha q}$. The second reason is to obtain a sharp estimate for $u_{\alpha q}$ near the points of intersection of the nodal lines of $u_{\alpha 0}$.

Having established the perturbation result we encounter one more difficulty: the nodal domains for $u_{\alpha q}$ are, in general, not small, see [15],[16]. The following figure gives a typical case

nodal domains for $u_{\alpha 0}$

nodal domains for $u_{\alpha q}$

To illustrate our result we consider the shaded region which we call Ω_α. We cut the nodal domain Ω_α with the straight diagonal lines, D_1, D_2, illustrated below. The upper cut

domain of Ω_α is called Ω'_α, see the figure on the right above. We establish a bound for $u_{\alpha q}$ near $\Omega_\alpha \cap D_i$, $i = 1, 2$, and hence show that there exists an $x' \in \Omega'_\alpha$ satisfying

$$|\, q(x') - \fint q - [\lambda_{\alpha 0} - \lambda_{1,0}(\Omega'_\alpha)]\,| \leq \frac{1}{2|\alpha|^{7/4}}.$$

The choice of the particular cut domain Ω'_α was rather arbitrary. Each nodal domain of $u_{\alpha q}$, for $\alpha \in L(a) \setminus M(a)$ can be cut in similar ways to obtain approximate nodal domains where a similar approximation to $q - \fint q$ can be obtained. Because of the conditions satisfied by $L(a) \setminus M(a)$ we can establish Theorem 6 by selecting all the approximate domains for $u_{\alpha q}$ for any infinite sequence of $\alpha \in L(a) \setminus M(a)$.

The above results are for the two dimensional inverse nodal problem. The three dimensional inverse nodal problem has also been considered. In that case the nodal sets are surfaces instead of lines. There are substantially more eigenvalues on the real line. Nonetheless, results analogous to the two dimensional case have been obtained in [10]. That is, conditions have been established so that the eigenvalues are well-separated; perturbation results have been established for a set of density one eigenvalues and eigenfunctions; a formula for an approximation to the potential has been obtained. Bounds for the difference between the potential and its approximation have been established. A uniqueness theorem has been proved.

A Related Result

We briefly discuss one other result. This is a specific case study to examine the effect of structural damping on nodal position measurements. It is important to show that structural damping has a small effect on nodal position measurement. The reason for this is that the mathematical models used to develop the the formulas for material parameters do not contain terms which model damping. Our study was not for elastic systems for which second order mathematical models are used but rather for stiff vibrating systems where structural damping may have a larger effect. Hence we chose the Euler-Bernoulli beam which models transverse vibration; we also chose Kelvin-Voigt damping.

For this specific case, the beam is free at each end and driven with an oscillating force at the center. The natural frequencies are defined as those frequencies where the response of the beam is maximized. When the beam is driven at a natural frequency, the nodal positions of the damped, driven beam are defined as the positions where the amplitude of the displacement is minimized. We compare those positions with the nodal positions of the mode shapes of the undamped beam and establish a bound on the difference. The bound shows that the difference is small when the beam is driven at low frequencies. This result is contained in [4], [5].

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References

[1] L. Friedlander, "On Certain Spectral Properties of Very Weak Nonselfadjoint Perturbations of Selfadjoint Operators", Trans. Moscow Math. (1982), Is. 1, pp. 185-218.

[2] L. Friedlander, "On the Spectrum of the Periodic Problem for the Schrödinger Operator", Comm. P.D.E., Vol. 15 (1990), pp. 1631-1647.

[3] J.Feldman, H. Knörrer, and E. Trubowitz, ”The Perturbatively Stable Spectrum of a Periodic Schrödinger Operator”, Inventiones Matematicae, Vol. 100 (1990), pp. 250-300.

[4] B. Geist and J.R. McLaughlin, ”The effect of structural damping on the nodes of a uniform, pinned, Euler-Bernoulli beam: A specific case study”, Rensselaer Polytechnic Institute Mathematics Technical Report Number 199, (1991).

[5] B. Geist and J.R. McLaughlin, ”The effect of structural damping on nodes for the Euler-Bernoulli beam: A specific case study”, App. Math. Lett., Vol. 7 (1994), pp. 51-55.

[6] O.H. Hald and J.R. McLaughlin, ”Solutions of Inverse Nodal Problems,” Inverse Problems, Vol. 5 (1989), pp. 307-347.

[7] O.H. Hald and J.R. McLaughlin, ”Inverse Problems Using Nodal Position Data: Uniqueness Results, Algorithms and Bounds,” Special Program on Inverse Problems Proceedings of the Centre for Math. Analysis, A.N.U., Vol. 17 (1988), pp. 32-59.

[8] O.H. Hald and J.R. McLaughlin, ”Inverse Problems: Recovery of BV Coefficients from Nodes”, (to appear).

[9] O.H. Hald and J.R. McLaughlin, ”Inverse Problems: Finding the potential from nodal lines”, (accepted for publication in the Memoirs of the AMS).

[10] Yu. Karpeshina and J.R. McLaughlin, ”Inverse Problems: Finding the potential from nodal surfaces”, (to appear)

[11] J.R. McLaughlin, ”Inverse Spectral Theory Using Nodal Points as Data - Uniqueness Result,” J. Diff. Eq., Vol. 73 (1988), pp. 354-363.

[12] J.R. McLaughlin and O.H. Hald, ”A Formula for Finding the Potential from Nodal Lines”, (accepted for publication in Bull. Amer. Math. Soc.,

[13] J. Moser, ”Lectures on Hamiltonian Systems”, Memoirs A.M.S. Vol. 81, 1968.

[14] K.F. Roth, ”Rational approximations to algebraic numbers,” Mathematika, Vol.2 (1955), pp.1-20, p.168.

[15] K. Uhlenbeck, "Eigenfunctions of Laplace operators", Bull. Amer. Math. Soc., vol. 78 (1972), pp. 1073-1076.

[16] K. Uhlenbeck, "Generic Properties of Eigenfunctions", Amer. J. Math., vol. 98 (1976) pp. 1059-1078.

[17] M.D. Waller, Chladni Figures: A study in Symmetry, Bell, London.

Appendix

In order to establish the perturbation results, we require that $\alpha \in L(a)$ satisfy five conditions. We give three of these conditions first. The description is somewhat technical. We require that α is in each of the following three sets:

$$L(a)\setminus M_{10}(a) = \left\{\alpha \in L \mid \forall\beta \in L \quad \text{with} \quad \beta \neq \alpha, \quad ||\alpha|^2 - |\beta|^2| > \frac{4}{|\alpha|^{\frac{1}{4}}}\right\};$$

let $C_1, C_2 \geq 1$ and label

$$L(a)\setminus M_{11}(a) = \left\{\alpha \in L \mid \forall\beta \in L \text{ with } \beta \neq \alpha,\ |\alpha - \beta| < C_1|\alpha|^{\frac{1}{20}},\ ||\alpha|^2 - |\beta|^2| > C_2|\alpha|^{\frac{15}{16}}\right\};$$

let $\zeta = 5^{\frac{1}{2}}2^{-\frac{1}{8}}$ and label

$$L(a)\setminus M_{12}(a) = \left\{\alpha \in L \mid m > (\zeta C_1)^2(an)^{\frac{1}{2}} \quad \text{and} \quad an > (\zeta C_1)^2 m^{\frac{1}{2}}\right\}.$$

In words, $L(a)\setminus M_{10}(a)$ contains those $\alpha \in L(a)$ for which the corresponding eigenvalue $|\alpha|^2$ has no other eigenvalue $|\beta|^2$ in a small interval about $|\alpha|^2$. Note that the size of this interval decreases as $|\alpha|^2$ increases. For the $\alpha \in L(a)\setminus M_{11}(a)$ the distance between $|\alpha|^2$ and $|\beta|^2$ is large when the corresponding $u_{\alpha 0}$, $u_{\beta 0}$ have nearly the same oscillations in both the x and y directions. Note that, in this case, the distance between $|\alpha|^2$ and $|\beta|^2$ increases as $|\alpha|^2$ increases. Finally, the elements in $L(a)\setminus M_{12}(a)$ are such that an and m are always comparable.

The three subsets, $M_{10}(a)$, $M_{11}(a)$, $M_{12}(a)$ are combined with two others. We define

$$M_1(a) = M_{10} \cup M_{11} \cup M_{12} \cup \{\alpha \in L \mid |\alpha| < 20\} \cup \left\{\alpha \in L \mid |\alpha| < \left(\frac{4}{C_1}\right)^{20}\right\}.$$

We show that $L(a)\backslash M_1(a)$ has density zero in $L(a)$. The full set of density one, $L(a)\backslash M(a)$, for which we establish perturbation results, is contained in $L(a)\backslash M_1(a)$. The elements of $M(a)\backslash M_1(a)$ are determined without specific description of their lattice properties.