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ABSTRACT

In this dissertation we develop a method for synthetic-aperture radar (SAR) imaging through a dispersive medium and we provide a method to obtain the optimal waveform design for imaging.

We consider the case when the sensor and scatterers are embedded in a homogeneous dispersive material, and the scene to be imaged lies on a known surface. We use a linearized (Born) scalar scattering model, and allow the flight path of the radar antenna to be an arbitrary smooth curve.

We formulate our filtered back-projection imaging algorithm in a statistical framework where the measurements are polluted with thermal noise. We assume that we have prior knowledge about the power-spectral densities of the scene and the noise.

We test our algorithms when the scene consists of point-like scatterers located on the ground. The position of the targets is well resolved when the target-to-noise ration is relatively small. For relatively large noise levels, the position of the targets are still well resolved employing the optimal waveform as an input signal in the reconstruction algorithm.

We show the results of simulations in which the dispersive material is modeled with the Fung-Ulaby equations for leafy vegetation. However, the method is also applicable to other dielectric materials where the dispersion is considered relevant in a frequency range of the transmitted signals.
CHAPTER 1
WAVE PROPAGATION THROUGH A DISPERSIVE MEDIUM

If the speed of wave propagation in a medium depends on frequency, the medium is said to be dispersive. Most materials are dispersive to some extent [1]; however, most current work on radar imaging neglects the effect of dispersion. For most radar systems, this is a reasonable assumption, because a) the dispersion is very weak in dry air, and b) the frequency bands of most radar systems are not wide enough for dispersive effects to be important. However, for imaging radar systems, the range resolution is proportional to the bandwidth. As a consequence, high-resolution systems require a broadband pulses, and for such pulses, the issue of dispersion may become important. Therefore, it is of interest to develop synthetic-aperture radar (SAR) imaging algorithms that account for dispersive wave propagation [2].

SAR image formation is closely related to the solution of an inverse scattering problem. The time-domain direct and inverse scattering problems for dispersive media have been extensively studied by many authors [3, 4, 5, 6, 7, 9, 10]. However, most of this work is for the one-dimensional case. An optimization-based reconstruction of a parametrized multidimensional dispersive medium was demonstrated in [11]. A SAR image formation method for propagation through the ionosphere was recently developed in [8]. A SAR imaging method for imaging through a weak dispersive layer was developed in [2], where the wave propagation was modeled in terms of a Fourier integral operator (FIO).

This dissertation is based on SAR imaging methods developed in [29] and [23] for non-dispersive media. On the other hand, this work also complements the work developed in [22] and [39] in a statistical setting where the measurements are corrupted with noise and clutter.

In this dissertation we will develop an algorithm for forming SAR images of objects that are embedded in a known homogeneous dispersive background. Although we follow the general strategy of [2], we find that our linearized forward
model is not a FIO, and thus our imaging algorithm does not follow from the FIO theory.

The dissertation is divided into three chapters. In Chapter 1 we provide the mathematical model for scattering in a dispersive medium, and define the direct and inverse problem. In Chapter 2, we show how the inverse problem is solved approximately by means of a filtered back-projection reconstruction method. Numerical examples in Sec. 2.1 illustrate the implementation of the algorithm. Finally chapter 3 is dedicated to answering the question: Given a particular dispersive material model, what is the optimal waveform for imaging?

1.1 Electromagnetic Scattering Model

In this section we discuss the wave equations that govern the evolution of electromagnetic fields in the presence of scatterers in a dispersive medium.

1.1.1 Maxwell’s Equations and Constitutive Relations

Our starting point is the Maxwell equations, which are given in differential form by [12]

\[
\begin{align*}
\nabla \cdot \mathbf{D}(\mathbf{x}, t) &= \rho(\mathbf{x}, t) \\
\nabla \cdot \mathbf{B}(\mathbf{x}, t) &= 0 \\
\n\nabla \times \mathbf{E}(\mathbf{x}, t) &= -\partial_t \mathbf{B}(\mathbf{x}, t) \\
\n\nabla \times \mathbf{H}(\mathbf{x}, t) &= \partial_t \mathbf{D}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x}, t)
\end{align*}
\]

Here, \( \mathbf{D} \) is the electric displacement, \( \mathbf{B} \) is the magnetic induction, \( \mathbf{E} \) is the electric field, \( \mathbf{H} \) is the magnetic field, \( \rho \) is the charge density, \( \mathbf{J} \) is the current density, \( \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \) represents Cartesian coordinates, \( t \in \mathbb{R} \) is time, and \( \nabla \cdot \) and \( \nabla \times \) denote divergence and curl operators, respectively.

In order to complete this set of equations, we need to add some constitutive
relations. We will use the following relations [3, 13]:

\[
D(x,t) = \int_0^\infty \varepsilon(x,t') E(x, t - t') dt' \overset{\text{def}}{=} (\varepsilon \ast_t E)(x, t), \tag{1.2a}
\]

\[
B(x,t) = \mu_0 H(x,t). \tag{1.2b}
\]

Here \(\mu_0\) denotes the free-space magnetic permeability.

**Remark 1.** The fact that the lower limit of the convolution (1.2a) is 0 is due to causality: the dielectric response of the medium will be affected by the applied field \(E\) only at earlier times.

We can also write Maxwell’s equations and the constitutive laws in the frequency domain. For this, we write \(E\) in terms of its the Fourier transform \(\tilde{E}\):

\[
E(x,t) = \frac{1}{2\pi} \int e^{-i\omega t} \tilde{E}(x,\omega) d\omega, \tag{1.3}
\]

where \(\omega\) denotes the angular frequency. The frequency-domain counterpart of the dielectric constitutive relation in (1.2a) is given by

\[
\tilde{D}(x,\omega) = \varepsilon(x,\omega) \tilde{E}(x,\omega). \tag{1.4}
\]

Here \(\varepsilon\) denotes the Fourier transform of \(\varepsilon\).

The speed at which the phase of any one frequency component propagates through a (dispersive) medium is called the phase velocity \(v_p\), and is defined by

\[
v_p = \frac{\omega}{k}, \tag{1.5}
\]

\[
k = \omega \sqrt{\mu \varepsilon}, \tag{1.6}
\]

where \(k\) is called the wave number.

Conveniently, we rewrite the wave number \(k\) and the phase speed \(v_p\) in terms of the dimensionless relative-to-vacuum permittivity \(\varepsilon_r = \varepsilon/\varepsilon_0\) and permeability
\[ \mu_r = \mu / \mu_0: \]
\[ v_p = \frac{c_0}{\sqrt{\mu_r \epsilon_r}}, \]
\[ k = \frac{\omega}{c_0} \sqrt{\mu_r \epsilon_r}, \]

where \( c_0 = 1 / \sqrt{\mu_0 \epsilon_0} \) is the speed of light in vacuum. The quantity \( \sqrt{\mu_r \epsilon_r} \) is called the index of refraction, and is denoted by \( n \). In a dispersive medium \( n \) is a complex-valued function of the frequency:

\[ n(\omega) = n_R(\omega) + i n_I(\omega) = \sqrt{\epsilon_r(\omega)}, \]
\[ n_R(\omega) \overset{\text{def}}{=} \text{Re}\{\sqrt{\epsilon_r(\omega)}\}, \]
\[ n_I(\omega) \overset{\text{def}}{=} \text{Im}\{\sqrt{\epsilon_r(\omega)}\}, \]

where we have assumed, according to Eqn. (1.2b), that \( \mu_r \approx 1. \)

To determine the wave equation for the electric field in a dispersive medium, we use the equations (1.2a) and (1.2b) to eliminate \( D \) and \( B \) in (1.1c) and (1.1d). Then we substitute the curl of (1.1c) into (1.1d), and end up with

\[ \nabla \times \nabla \times E = -\partial_t \mu_0 J - \partial_t^2 (\mu_0 \epsilon_0 \ast_t E). \quad (1.9) \]

We use the identity

\[ \nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E \]

(1.10)

to write (1.9) as

\[ \nabla^2 E - \partial_t^2 (\mu_0 \epsilon_0 \ast_t E) - \nabla (\nabla \cdot E) = \mu_0 \partial_t J. \quad (1.11) \]

We assume that no charge densities are present, so we assume that \( \nabla \cdot E = 0 \). In this case (1.11) is reduced to the uncoupled form

\[ \nabla^2 E - \partial_t^2 (\mu_0 \epsilon_0 \ast_t E) = \mu_0 \partial_t J. \quad (1.12) \]

The assumption \( \nabla \cdot E = 0 \) does not hold in general. However, it does hold for wave propagation in any homogeneous dispersive medium with no charge densities, and
in particular, it holds within a homogeneous medium. At the interfaces between such homogeneous materials there will be a coupling between the components of the electric field. This coupling can result in a change in the polarization of the wave. Therefore, using (1.12) implies that we neglect polarization.

1.1.2 Scattering medium

Our aim is to image objects which are completely submerged in a dispersive background medium. These objects will be characterized as perturbations in the electromagnetic properties of the background. We make the following assumptions:

Figure 1.1: Background permittivity medium $\epsilon_r$ with a perturbation $\epsilon_T$.

1. The background medium is homogeneous with known permittivity $\varepsilon_0\varepsilon_r(t')$, where $\varepsilon_0$ denotes the electric permittivity of vacuum.

2. Electromagnetic scattering occurs at the surface given by $x' = \psi(x_1, x_2)$, where the function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is assumed known. This is a reasonable model for the situation in which electromagnetic waves are attenuated rapidly in the earth’s surface.

3. The scattering is caused by perturbations $\varepsilon_T$ in the instantaneous response of
the permittivity, so that the total permittivity in the medium is given by

\[ \varepsilon(t', x') = \varepsilon_0 \varepsilon_r(t') + \varepsilon_0 \varepsilon_T(x') \delta(x' - \psi(x_1, x_2)) \delta(t'). \] (1.13)

We denote by \( T(x') = \mu_0 \varepsilon_0 \varepsilon_T(x') \) the ground reflectivity function. Equivalently, we use the term target to refer to \( T \). The quantity \( T \) contains all the information that we want to reconstruct (from the received signals) about the inhomogeneities of the medium (see Fig. 1.1).

1.1.3 Scalar Wave Equation

Each component of (1.12) satisfies the scalar equation

\[ \nabla^2 E - \partial^2_t (c_0^{-2} \varepsilon_r \ast_t E) - (T \delta) \partial^2_t E = -j_s, \] (1.14)

where we use expression (1.13) for the permittivity, where \( j_s \) denotes one component of \( -\mu_0 \partial_t J \) in (1.12).

In order to solve (1.14), we will use the Green’s function \( g \) that solves, in the sense of distributions,

\[ \nabla^2 g(x, y, t) - \partial^2_t (c_0^{-2} \varepsilon_r \ast_t g)(x, y, t) = -\delta(x - y) \delta(t) \] (1.15)

together with the condition \( g(x, y, t) = 0 \) for \( t < 0 \). We take the Fourier transform of (1.15) in \( t \), which results in

\[ (\nabla^2 + k^2(\omega))G(x, y, \omega) = -\delta(x - y), \] (1.16)

where \( k(\omega) = \omega n(\omega)/c_0 = \omega \sqrt{\varepsilon_r(\omega)/c_0}, \) \( \varepsilon_r \) and \( G \) denote the Fourier transform of \( \varepsilon_r \) and \( g \), respectively. Equation (1.16) is solved by

\[ G(x, y, \omega) = \frac{e^{i k(\omega)|x-y|}}{4\pi |x - y|}. \] (1.17)
If we take the inverse Fourier transform of (1.17) we get a solution to (1.15):

\[ g(x, y, t) = \frac{1}{8\pi^2} \int \frac{e^{-i\omega(t-n(\omega)|x-y|/c_0)}}{|x - y|} d\omega. \]  

(1.18)

This Green’s function represents the wave-field at \((x, t)\) due to a point source at \(y\) at time \(t = 0\).

**Remark 2.** In (1.18) \(n(\omega) \overset{\text{def}}{=} \sqrt{\varepsilon_r(\omega)}\) is a complex-valued function of the frequency (see (1.8)). We choose the branch of the square root so that \(\omega n_I(\omega)\) is positive \(^2\). The sign of the imaginary part thus corresponds to attenuation and not amplification of the propagating waves.

We decompose the total field \(E\) of (1.14) as the sum of two fields: the incident field \(E^{\text{in}}\) which would have existed in the absence of the scatterers, and the scattered field \(E^{\text{sc}}\).

Let us now turn to the solution of the wave equation for the incident field \(E^{\text{in}}\). This field satisfies

\[ \nabla^2 E^{\text{in}}(x, t) - \frac{1}{c_0^2} \varepsilon_r(t) E^{\text{in}}(x, t) = -j_s(x, t) \]  

(1.19)

The source term \(j_s(x, t)\) is proportional to the current density over the antenna. For simplicity we consider an isotropic point-like antenna located at position \(y_c\): we replace \(j_s(x, t)\) by \(p(t)\delta(x - y_c)\). The function \(p(t)\) is a waveform sent to the antenna and it can be of almost any shape [21].

We can solve (1.19) by means of the Green’s function \(g\) as follows

\[ E^{\text{in}}(x, t) = (g * j_s)(x, t) \]
\[ = \int \frac{p(t - n(\omega)|x - y_c|/c_0)}{8\pi^2|x - y_c|} \delta(y - y_c) dy \]
\[ = \frac{p(t - n(\omega)|x - y_c|/c_0)}{8\pi^2|x - y_c|}. \]  

(1.20)

\(^2\)For a discussion on the analyticity of the index of refraction \(n(\omega)\) see §1.3 in [20]
If we write the waveform \( p \) in terms of its inverse Fourier transform \( P \)

\[
p(t) = \frac{1}{2\pi} \int e^{-i\omega t} P(\omega) d\omega, \tag{1.21}
\]
equation (1.20) becomes

\[
E^{\text{in}}(x, t) = \int \frac{e^{-i\omega(t-n(\omega)|x-y_c|/c_0)}}{16\pi^3|x-y_c|} P(\omega) d\omega. \tag{1.22}
\]

In practice, the waveform \( p \) is such that only the interval \([\omega_{\text{min}}, \omega_{\text{max}}]\) contributes significantly to (1.21); we call the set of these frequencies the effective support of \( P(\omega) \). We refer to the difference \((\omega_{\text{max}} - \omega_{\text{min}})\) as the (angular-frequency) bandwidth of the waveform.

Next we assume that the center of the antenna \( y_c \) is moving along the curve \( \gamma \). This introduces another parameters on which the expression (1.22) depends,

\[
E^{\text{in}}(x, t, s) = \int \frac{e^{-i\omega(t-n(\omega)|x-\gamma(s)|/c_0)}}{16\pi^3|x-\gamma(s)|} P(\omega) d\omega. \tag{1.23}
\]

We have replaced \( y_c \) by a parametric curve

\[
\gamma \overset{\text{def}}{=} \{ \gamma(s) \in \mathbb{R}^3 : s \in (s_{\text{min}}, s_{\text{max}}) \subset \mathbb{R} \}. \tag{1.24}
\]

The parameter \( s \), called the slow time, describes the position of the antenna along its flight path. On the other hand, the fast time \( t \) corresponds to the propagation of signals through the medium. The incident field \( E^{\text{in}} \) (1.23) describes the field emanating from the antenna located at \( \gamma(s) \). The field at position \( x \) is a superposition of fixed-frequency point sources over the effective support of \( P(\omega) \).

1.1.4 A Linearized Scattering Model

We substitute the expression \( E = E^{\text{in}} + E^{\text{sc}} \) in (1.14) and employ (1.19) to obtain

\[
\nabla^2 E^{\text{sc}} - \partial_t^2 (c_0^{-2} \varepsilon_r *_t E^{\text{sc}}) = T \delta \partial_t^2 E. \tag{1.25}
\]
We use the Green’s function (1.18) to write (1.25) as

\[ E_{\text{sc}}(x, t) = (g * T \delta(t^2 E))(x, t) \]

\[ = - \int g(x - x', t - \tau)T(x')\delta(x' - \psi(x_1, x_2))\partial_2 E(x', \tau)d\tau dx' \]  

(1.26)

This equation is often referred to as the Lippmann-Schwinger equation.

If we replace the full field \( E \) on the right-hand side of (1.25) and (1.26) by the incident field \( E_{\text{in}} \) we get the Born approximation or the single-scattering approximation to the scattered field

\[ E_{\text{sc}}(x, t, s) \approx E_{\text{sc}}^B(x, t, s), \]  

(1.27)

where

\[ E_{\text{sc}}^B \equiv - \int g(x - x', t - \tau)T(x')\delta(x' - \psi(x_1, x_2))\partial_2 E_{\text{in}}(x', \tau, s)d\tau dx'. \]  

(1.28)

We have used a subscript \( B \) to indicate that we are using the Born approximation. This approximation removes the non-linearity of the inverse problem: it replaces in (1.26) the product of two unknowns \((T \text{ and } E)\) by a single unknown \( (T) \) multiplied by the known incident field. The Born approximation is valid when the average intensity of the scattered field \( E_{\text{sc}} \) is small compared to the average intensity of the incident field \( E_{\text{in}} \) in a given volume \( |E_{\text{sc}}| \ll |E_{\text{in}}| \) \[28\]. If we substitute the Green’s function (1.18) and the incident field (1.23) into (1.28), we obtain the scattered wave-field

\[ E_{\text{sc}}^B = \int e^{-i\omega(t-n(\omega)|x-x_1, x_2|+|\psi(x_1, x_2)-\psi(s)|)/c_0)} \omega^2 P(\omega)T(x_1, x_2)d\omega dx_1 dx_2. \]  

(1.29)

1.1.5 Received Signal

We model reception of the field (1.29) by taking \( x \) to be \( \gamma(s) \). Thus we obtain

\[ E_{\text{sc}}^B(\gamma(s), t, s) = \int e^{-i\omega(t-n(\omega)|\psi(x_1, x_2)-\psi(s)|)/c_0)} \omega^2 P(\omega)T(x_1, x_2)d\omega dx_1 dx_2. \]  

(1.30)
In our model for the received signal, we also include additive noise \( n(t, s) \). Our model for the data is then

\[
d(t, s) = \int \frac{e^{-i\omega(t - 2n(\omega)\|r_{s,y}\|/c)}}{(16\pi^3\|r_{s,y}\|^2)}\omega^2 P(\omega)T(\mathbf{y})d\omega d\mathbf{y} + n(t, s).
\]  

(1.31)

Here we have introduced a bold italic font for 2D vector positions; i.e., \( \mathbf{y} \overset{\text{def}}{=} \{(y_1, y_2) \in \mathbb{R}^2\} \) and \( d\mathbf{y} \) is the 2D surface measure, and we have used the vector \( r_{s,y} \):

\[
r_{s,y} \overset{\text{def}}{=} \psi(\mathbf{y}) - \gamma(s).
\]  

(1.32)

### 1.2 The Inverse Problem.

The expression (1.31) defines a mapping \( F \) from \( T \) to \( d \).

**Definition 1.2.1** (forward problem). The problem of finding \( d(t, s) \) from the reflectivity function \( T \) and waveform \( p(t) \) is called the direct or forward problem.

The associated inverse problem is to compute the mapping \( F^{-1}[d] \) to obtain \( T \):

**Definition 1.2.2** (inverse problem). The inverse problem is to recover the function \( T \) from the data \( d(t, s) \) and the waveform \( p(t) \).
CHAPTER 2
IMAGE FORMATION

To form an image \( I(z) \) \( (i.e., \) an approximation to \( T(z) \)), we apply to the noisy data (1.31) an operator \( F_Q^{-1} \):

\[
I(z) = F_Q^{-1}[d](z), \tag{2.1}
\]

where \( F_Q^{-1} \) is defined by the integral operator:

\[
I(z) = \int e^{i\omega'(t-2n_R(\omega')|r_{s,z}|/c_0)}Q(\omega', z, s)\omega'd(t, s)dtds. \tag{2.2}
\]

The subscript \( Q \) on the operator \( F_Q^{-1} \) indicates that the operator depends on the filter \( Q \). The role of \( Q \) will be to remove blur, while controlling noise contributions to the image. In Sec. 2.0.1, we analyze the degree to which this operator is a good approximation to the inverse operator \( F^{-1} \).

In previous works on dispersionless media [21, 22, 23, 29] the mapping \( F \) was of the form of a Fourier Integral Operator (FIO) (see [24]). An approximate inverse of \( F \) was then constructed from the adjoint operator \( F^* \) followed by a filter to approximate \( (F^*F)^{-1} \). This resulted in a modified matched filtering operation for forming images. In the case of dispersion, the forward operator is not a FIO because of the non-linear dependence of the phase on the frequency \( \omega' \). Moreover, since \( \omega n_l(\omega) \) is positive, the adjoint \( F^* \) involves a decaying factor \( \exp(-\omega n_l(\omega)|r_{s,z}|/c_0) \).

Applying an inverse filter to approximate \( (F^*F)^{-1} \) would therefore exponentially amplify the noise. We will look for an inversion operator of the form \( F_Q \) in (2.1), whose explicit phase involves only the real part \( n_R(\omega) \). As the real part corresponds to the correct propagation speed, we will end up with a back-projection method which places the scatterers at the correct spatial location. We will subsequently determine the filter \( Q \) such that the variance in the image is minimized.
2.0.1 The relationship between the image \( I \) and the target \( T \)

At this point we assume that the filter \( Q \) plays the role of the amplitude of an FIO. To determine the fidelity of the image formed by (2.1), we substitute (1.31) into (2.2):

\[
I(z) = \int e^{-i2(\omega' n_R(\omega')|r_{s,z}|-\omega n_R(\omega)|r_{s,y}|)/c_0} Q(\omega', z, s) d\omega' \\
\times \frac{e^{-2\omega I(\omega)|r_{s,y}|/c_0}}{(4\pi|r_{s,y}|)^2} \omega^2 P(\omega) d\omega T(y) dy e^{i(\omega'-\omega)t} dt ds \\
+ \int e^{-i(2\omega' n_R(\omega')|r_{s,z}|/c_0-\omega t)} Q(\omega', z, s)n(t, s) dt d\omega' ds.
\]

(2.3)

In the first integral we perform integrations with respect to \( t \) and \( \omega \) and drop the prime on \( \omega' \) in the second integral to obtain

\[
I(z) = \int e^{-i2\omega n_R(\omega)(|r_{s,z}|-|r_{s,y}|)/c_0} Q(\omega, s, z) \\
\times \frac{e^{-2\omega I(\omega)|r_{s,y}|/c_0}}{(16\pi^3|r_{s,y}|)^2} \omega^2 P(\omega) d\omega T(y) dy ds \\
+ \int e^{-i(2\omega n_R(\omega)|r_{s,z}|/c_0-\omega t)} Q(\omega, s, z)n(t, s) dt d\omega ds.
\]

(2.4)

Now, in order to simplify notation we define the following functions:

\[
\vartheta_R(\omega) = 2\omega n_R(\omega)/c_0, \quad (2.5) \\
\vartheta_I(\omega) = 2\omega n_I(\omega)/c_0. \quad (2.6)
\]

The transmitted waveform \( P(\omega) \) and the geometrical spreading factor \( a(\omega, s, y) \) are included in \( A(\omega, s, y) \), which is defined by

\[
A(\omega, s, y) = a(\omega, s, y) P(\omega), \quad (2.7)
\]

where

\[
a(\omega, s, y) = \frac{\omega^2 e^{-\vartheta_I(\omega)|r_{s,y}|}}{(16\pi^3|r_{s,y}|)^2}. \quad (2.8)
\]
Hence, we rewrite $I(z)$ as follows
\begin{align}
I(z) &= \int e^{-i\vartheta_R(\omega)|r_{s,z}-r_{s,y}|}Q(\omega, s, z)A(\omega, s, y)T(y)d\omega dy ds \quad (2.9a) \\
&+ \int e^{-i[\vartheta_R(\omega)\omega t]}Q(\omega, s, z)n(t, s)dt d\omega ds \quad (2.9b)
\end{align}

Our next step is to simplify the phase of the exponent in (2.9a) using the integral form of the remainder for Taylor’s theorem. As in [23], we use the identity
\begin{equation}
h(z) - h(y) = (z - y) \cdot \int_{0}^{1} \nabla h(y - \lambda(z - y))d\lambda \quad (2.10)
\end{equation}
with $h(z) = \vartheta_R(\omega)|r_{s,z}|$ to write the exponent of (2.9a) as
\begin{equation}
\vartheta_R(\omega)|r_{s,z} - |r_{s,y}| = (z - y) \cdot \Xi(\omega, s, z, y), \quad (2.11)
\end{equation}
where
\begin{equation}
\Xi(\omega, s, z, y) = \vartheta_R(\omega) \int_{0}^{1} \nabla_{x}|r_{s,x}| \bigg|_{x=y+\lambda(z-y)} d\lambda. \quad (2.12)
\end{equation}

We make the Stolt change variables from $(\omega, s)$ to $\xi$, where
\begin{equation}
\xi \overset{\text{def}}{=} -\Xi(\omega, s, z, y) \quad (2.13)
\end{equation}
and the corresponding Jacobian determinant $J$ is defined by the expression (see Appendix C)
\begin{equation}
d\omega ds = \left| \frac{\partial(\omega, s)}{\partial \xi} \right| d\xi = J(\xi, z, y)d\xi. \quad (2.14)
\end{equation}
When $z = y$, $\xi$ becomes $-\Xi(\omega, s, z, z)$, where
\begin{equation}
\Xi(\omega, s, z, z) \overset{\text{def}}{=} \vartheta_R(\omega)\overline{r_{s,z}} \cdot D\psi(z), \quad (2.15)
\end{equation}
and $D\psi$ is the Jacobian matrix. We denote by $\Omega_z$ the set of values that $\xi$ can take on:
\begin{equation}
\Omega_z = \{ \xi = \vartheta_R(\omega)\overline{r_{s,z}} \cdot D\psi(z) \in \mathbb{R}^2 : \omega \in (\omega_{\text{min}}, \omega_{\text{max}}) \text{ and } s \in (s_{\text{min}}, s_{\text{max}}) \}. \quad (2.16)
\end{equation}
The set \( \Omega_z \) is called the data collection manifold, and is determined by the flight path and the frequency band obtained from the Stolt change of variables (2.13) (See [22] for further details).

The change of variables (2.13) transforms the integral (2.9) into

\[
I(z) = \int_{\Omega_z} e^{i(z-y) \cdot \xi} Q(\xi, z) A(\xi, y) T(y) J(\xi, z, y) d\xi d y
\]

(2.17)

\[
+ \int_{\Omega_z} \int e^{-i[\vartheta_R(\omega(\xi))]r_s(\xi, z)} Q(\xi, z) n(t, s(\xi)) d t d \xi.
\]

(2.18)

Given the noiseless data from the flight path and the frequency band of the radar system, the best \( L^2 \) approximation to \( T \) would be

\[
I_{\Omega_z}(z) = \int_{\Omega_z} e^{i(z-y) \cdot \xi} T(y) d y d \xi.
\]

(2.19)

In this case, the resolution of the imaging system is determined via (2.19) by the set \( \Omega_z \) of (2.16).

### 2.0.2 Determination of Filter \( Q \)

To determine the filter \( Q \) we examine the degree to which the image \( I \) is the best approximation to \( T \) in a statistical sense. In other words, we will determine \( Q_{opt} \) which is optimal in the sense that it yields an image with minimum variance.

#### 2.0.2.1 Image Variance, Bias, and Expected Mean-Square Error

The expected mean-square error (MSE) is

\[
\Delta(Q, P) = \int d z \langle |E(z)|^2 \rangle,
\]

(2.20)

where \( \langle \cdot \rangle \) denotes expectation operator, and \( E(z) \) denotes the error between the ideal image \( I_{\Omega_z}(z) \) and its estimate \( I(z) \) and is given by

\[
E(z) = I(z) - I_{\Omega_z}(z).
\]

(2.21)
The MSE comprises two components (for details see [22]):

$$\Delta(Q, P) = \mathcal{V}(Q, P) + \mathcal{B}(Q, P),$$

(2.22)

where $\mathcal{V}(Q, P)$ is the total variance of $\mathcal{E}$ defined as

$$\mathcal{V}(Q, P) = \int dz \langle |\mathcal{E}(z) - \langle \mathcal{E}(z) \rangle|^2 \rangle,$$

(2.23)

and $\mathcal{B}(Q, P)$ is the square of the $L^2$-norm of the bias

$$\mathcal{B}(Q, P) = \int dz |B(Q, P)(z)|^2,$$

(2.24)

where $B(Q, P)(z) = \langle I(z) \rangle - \langle I_{\Omega 0}(z) \rangle$.

It is well-known in statistical approaches to inverse problems that in general one cannot hope to both obtain an image with correct mean and at the same time minimize its variance [22]. Because it is desirable that the image not change appreciably for different realizations of noise, typically we minimize the variance of the image, but do not constrain the mean. We note, however, that minimizing the variance amounts to subtracting off the mean and then minimizing the expected mean-square error. In the rest of the thesis, for simplicity we address the case in which the target $T$ has zero mean and refer to the problem of minimizing the expected mean-square error, keeping in mind that this is equivalent to minimizing the variance.

We assume that the noise $n(t, s)$ is zero-mean second-order stochastic processes. Furthermore, we assume that the noise is stationary in the fast time variable $t$, statistically uncorrelated in the slow time variable $s$, and that its spectral density function $S_n$ is given by

$$\int e^{i\omega t_1} e^{-i\omega' t_2} \langle \bar{n}(t_1, s) n(t_2, s') \rangle dt_1 dt_2 = S_n(\omega, s) \delta(\omega - \omega') \delta(s - s'),$$

(2.25)

where $\bar{n}(t)$ is the complex conjugation of $n(t)$.

For simplicity, we assume here that the target $T$ is a zero-mean second-order
random field. Finally, we assume that the noise and the target are statistically independent.

The auto-correlation function $R_T$ of $T$ is defined as

$$R_T(y, y') \overset{\text{def}}{=} \langle T(y)T(y') \rangle. \quad (2.26)$$

Let now $S_T$ be defined through the following Fourier transform relation with $R_T$

$$R_T(y, y') = \int \int e^{-iy \cdot \xi e^{iy' \cdot \xi} S_T(\xi, \xi') d\xi d\xi'. \quad (2.27)$$

We will refer to $S_T$ as the target spectral density function.

The best image in the minimum MSE sense is obtained by minimizing the integral

$$\Delta(P, Q) = \int \langle |I(z) - I_{\Omega_z}(z)|^2 \rangle dz = \Delta_T(P, Q) + \Delta_N(P, Q), \quad (2.28)$$

where

$$\Delta_T(P, Q) = \int \left( \int_{\Omega_z} e^{i(z-y) \cdot \xi Q(\xi, z) A(\xi, y) J(\xi, z, y) d\xi d'y - I_{\Omega_z}(z) \right)^2 dz, \quad (2.29)$$

and

$$\Delta_N(P, Q) = \int_{\Omega_z} |Q(\xi, z)|^2 S_n(\xi) J(\xi, z) d\xi dz. \quad (2.30)$$

The cross terms vanish in (2.28) because the target and the noise are statistically independent and have zero mean: $\langle T(y)n(t, s) \rangle = \langle T(y) \rangle \langle n(t, s) \rangle = 0$. We expand and rewrite the expression (2.29) to get

$$\Delta_T(P, Q) = \int \int_{\Omega_z} e^{i[z-y] \cdot \xi - (z-y') \cdot \xi'} u(\xi, \xi', y, y', z) R_T(y, y') d\xi d\xi' dy dy' dz, \quad (2.31)$$

where

$$u(\xi, \xi', y, y', z) = [Q(\xi, z) A(\xi, y) J(\xi, z, y) - 1] [Q(\xi', z) A(\xi', y') J(\xi', z, y') - 1] \quad (2.32)$$
(bar denotes complex conjugate).

To simplify (2.31), we apply the stationary phase approximation [24] to the phase \((z - y) \cdot \xi - (z - y') \cdot \xi'\), where the critical set and the Hessian are (See Appendix B for more details)

\[
\text{critical set } \begin{cases} \frac{\partial \phi_1}{\partial z} = \xi - \xi' = 0 \\ \frac{\partial \phi_1}{\partial \xi'} = -(z - y') = 0 \end{cases} \\
\text{Hessian } \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial z \partial \xi'} & 0 \\ 0 & -I \end{pmatrix}
\]

Consequently, the main contribution to the integral (2.31) comes from the set \(\xi = \xi'\) and \(z = y'\):

\[
\Delta_T(P, Q) \sim \int_{\Omega_z} e^{i(y' - y) \cdot \xi} (Q(\xi, y') A(\xi, y) J(\xi, y', y) - 1) \\
\times \left( Q(\xi, y') A(\xi, y') J(\xi, y', y') - 1 \right) R_T(y, y') d\xi d y d y'
\]

\[
\sim \int_{\Omega_z} e^{i[(y' - y) \cdot \xi + y \cdot \zeta - y' \cdot \zeta']} (Q(\xi, y') A(\xi, y) J(\xi, y', y) - 1) \\
\times \left( Q(\xi, y') A(\xi, y') J(\xi, y', y') - 1 \right) S_T(\zeta, \zeta') d\zeta d \zeta' d\xi d y d y'
\]

where in the second line we have used (2.27). To (2.34) we again apply the method of stationary phase. In this case the phase is \(y' \cdot (\xi - \zeta') - y \cdot (\xi - \zeta)\), and the leading-order contribution comes from the points \(\xi = \zeta'\) and \(y' = y\). The leading-order term is

\[
\Delta_T(P, Q) \sim \int_{\Omega_z} e^{i(y' - y) \cdot (\xi - \zeta)} |Q(\xi, y) A(\xi, y) J(\xi, y) - 1|^2 S_T(\zeta, \xi) d\zeta d \zeta d\xi d y.
\]

We now make the assumption that the target \(T\) is a weak-sense stationary random field. In this case the correlation function \(R_T\) is a function only of the difference between its two variables. Thus \(S_T\) is then simplified to

\[
S_T(\zeta, \xi) = S_T(\zeta) \delta(\xi - \zeta).
\]
Therefore, from (2.28) we obtain

\[
\Delta(P, Q) = \int\int_{\Omega_y} |Q(\xi, y)A(\xi, y)J(\xi, y) - 1|^2 S_T(\xi) d\xi d\eta - 1 \int\int_{\Omega_y} |Q(\xi, y)|^2 S_n(\xi) J(\xi, y) d\xi d\eta \] 

(2.36a)

\[
+ \int\int_{\Omega_y} |Q(\xi, y)|^2 S_n(\xi) J(\xi, y) d\xi d\eta. \] 

(2.36b)

To obtain (2.36b) from (2.30), we have changed the dummy variable from \( z \) to \( y \).

We want to find the optimal filter \( Q = Q_{\text{opt}} \) for which (2.36) is minimized. To do this, we calculate the variation of \( \Delta(P, Q) \) with respect to \( Q \):

\[
0 = \frac{d}{d\rho} \bigg|_{\rho=0} \Delta(P, Q + \rho q)
\]

\[
= \frac{d}{d\rho} \bigg|_{\rho=0} \int\int_{\Omega_y} |(Q + \rho q)A - 1|^2 S_T d\xi d\eta + \frac{d}{d\rho} \bigg|_{\rho=0} \int\int_{\Omega_y} |Q + \rho q|^2 S_n J d\xi d\eta
\]

\[
= \int\int_{\Omega_y} 2\text{Re}\{qA\} |(Q + \rho q)A - 1|^2 S_T d\xi d\eta + \int\int_{\Omega_y} 2\text{Re}\{q\} |Q + \rho q|^2 S_n J d\xi d\eta
\]

\[
= \int\int_{\Omega_y} 2 \left( \text{Re}\{qA\} |AQ_{\text{opt}} - 1| \right) S_T + \text{Re}\{\overline{q}Q_{\text{opt}}\} S_n J ) d\xi d\eta. \]

(2.37)

As this should hold for all \( q \), we obtain

\[
(|A|^2 Q_{\text{opt}} |J|^2 - A J) S_T + Q_{\text{opt}} S_n J = 0. \] 

(2.38)

Solving for \( Q_{\text{opt}} \) we find that

\[
Q_{\text{opt}}(\xi, y) = \frac{A(\xi, y)}{J(\xi, y)|A(\xi, y)|^2 + S_n(\xi)/S_T(\xi)} \] 

(2.39)

Finally, we substitute back the variables \( (\omega, s) \), and recall from (2.7) that \( A = a(\xi, y) P(\xi) \). This gives \( Q_{\text{opt}} \) as

\[
Q_{\text{opt}}(\omega, s, y) = \frac{1}{a(\omega, s, y) P(\omega)} \left( \frac{|a(\omega, s, y) P(\omega)|^2}{|a(\omega, s, y) P(\omega)|^2 J + \sigma_{nT}(\omega, s)} \right). \] 

(2.40)

Here \( \sigma_{nT} = S_n(\omega, s)/S_T(\omega, s) \) is the (frequency-domain) noise-to-target ratio. A
filter of this form is commonly referred to as a *Wiener filter*. Four special cases of interest are:

1. **No noise**: \( \sigma_{nT} = 0 \), then the filter (2.40) reduces to the inverse filter \((aPJ)^{-1}\).
   This means that, if we substitute \(Q_{\text{opt}} = (aPJ)^{-1}\) into the first integral of (2.17), we obtain the best approximation to \(T\), Eqn. (2.19).

2. **No signal**: if the target does not scatter waves in some frequency range, \(i.e., S_T(\omega, s) = 0\) at some \(\omega\)’s, then formally \(\sigma_{nT}\) is infinite and \(Q_{\text{opt}} = 0\). This means that the filter does not pass frequencies at this range and there is no contribution to the image from these frequencies, see Eqns. (2.17) and (2.18).

3. **Non-zero signal and noise spectrum**: in the case of non-zero noise-to-target spectrum ratio, the filter is appropriately weighted [25, 26]. The quantity \(\sigma_{nT}\) is a frequency-dependent regularization term.

4. **White noise**: if the noise can be assumed to be spectrally white and independent of the slow-time parameter \(s\), Eqn. (2.40) reduces to a simple parametric filter
   \[Q_{\text{opt}}(\omega, s, y) = \frac{1}{aP} \left(\frac{|aP|^2 S_T}{|aP|^2 JS_T + \sigma^2}\right).\]  
   with a constant \(\sigma^2\) equal to the noise variance \(S_n\).

### 2.1 Numerical Simulations

In this section we present numerical simulations for the algorithm introduced in this chapter. The image formation simulations allow us to compare the effects of non-dispersive and dispersive intervening media. The effects of the additive noise on the image reconstruction are studied in the next chapter. To obtain the algorithm resolution; \(i.e., its ability to distinguish between separate scatterers\), we consider two point targets on the ground. We report results for a code written in MATLAB® [32] and performed on a desktop computer with 2 GHz Core Duo processor and 2 GB of memory. In the following two subsections, we describe the numerical implementation of Equations (1.31) and (2.2).
2.1.1 Forward Data $d(t, s)$

To obtain the simulated forward data $d(t, s)$ we proceed as follows. First, for numerical convenience, we work in the frequency domain. To this end,

(a) We rewrite Eqn. (1.31) as follows:

$$d(t, s) = \frac{1}{2\pi} \int e^{-i \omega t} D(\omega, s) d\omega,$$

(b) Square-wave-modulated sinusoidal waveform with smoothed edges.

(c) Waveform Power Spectrum Amplitude.

Figure 2.1: Radar scene and the transmitted waveform.
where
\[
D(\omega, s) = \frac{\omega^2 P(\omega)}{128\pi^5} \int T(y) \frac{e^{i2\omega n(\omega)|\gamma(s) - y|/c_0}}{|\gamma(s) - y|^2} \, dy.
\] (2.43)

(b) Then we discretize \(D\):
\[
D_{kl} \approx \frac{\omega^2 P(\omega_k)}{128\pi^5} \sum_m T_m e^{i2\omega_n(\omega_k)|\gamma(s_l) - y_m|/c_0} \frac{1}{|\gamma(s_l) - y_m|^2},
\] (2.44)

where \(D_{kl}\) is a matrix given by \(D_{kl} = D(\omega_k, s_l)\).

(c) To get the forward data \(d\) we apply an inverse fast Fourier transform to \(D_{kl}\) and we add normally distributed white noise:
\[
d(t_k, s_l) \approx \text{ifft} \, D(\omega_k, s_l)
\]
\[
d(t_k, s_l) \rightarrow d(t_k, s_l) + n(t_k, s_l).
\] (2.45)

To obtain a specific (frequency domain) target-to-noise ratio, we choose the spectral density noise amplitude \(S_n\) proportional to the amplitude of the spectral density target \(S_T\), i.e., \(S_n = \beta S_T\), where \(\beta\) is a constant.

(d) The scene consists of two weak \(^3\) scatterers located at different positions \(y_{m_1}\) and \(y_{m_2}\) with equal and constant reflectivity \(^4\) \(T_m = n(0)/c_0^2\). The antenna flight path is circular with radius \(r = 100 \text{ m}\) and altitude \(h = 10 \text{ m}\) (see Fig. 2.1(a))
\[
\gamma(s) = (r \cos(2\pi s), r \sin(2\pi s), h), \quad s \in [0, 1).
\] (2.46)

(e) The transmitted signal is an analytic\(^5\) square-wave-modulated sinusoidal waveform padded with zeros. Its carrier frequency is fixed at \(f_0 = 1.5 \text{ GHz}\) (see Fig. 2.1(b)):
\[
p(t) = \text{Re}\{\exp(i2\pi f_0 t - \phi)\},
\] (2.47)

where \(\phi\) is a phase. The signal has a total time duration of \(\tau_p = 5.33 \text{ ns}\) with a

\(^3\) The scatterer is known as weak or tenuous when its permittivity \(\epsilon_T\) is close to that of the background medium \(\epsilon_r\) (see Eqn. (1.13) and [16]).

\(^4\) The target \(T\) is considered delta-correlated: \(R_T = T_m \delta(y - y')\)

\(^5\) A signal which has no negative-frequency components is called an analytic signal.
maximum amplitude of 1 V·m\(^{-1}\). Because of the square-wave modulation, the waveform’s Fourier transform has non-zero values (commonly called leakage) at frequencies other than \(f_0\) [31] [32]. It is thus a wide-band pulse. In order to keep the effective support of our pulse well within the band of frequencies permitted by our temporal sampling rate, we applied a low-pass filter to the waveform (also after applying the second derivative term \(\omega_k^2\) in (1.28)). This low-pass filter corresponds to a smooth modulation of the sinusoid shown in Fig. 2.1(b) [33].

(f) In the dispersive case, the permittivity depends on frequency. Therefore, the phase velocity is given by \(v_p(\omega) = c_0/n_R(\omega)\) (See Eqn. (1.5)). We compare images for the non-dispersive case whose permittivity is taken to be \(\epsilon(\omega_0)\), where \(\omega_0 = 2\pi f_0\). This corresponds to a constant phase velocity \(v_p = c_0/n_R(\omega_0)\). Noiseless forward data \(d(t_k, s_l)\), for one single scatterer, is shown in the Figures 2.2(a) and 2.2(c). In the medical imaging community, such plots are called projections or sinograms [36].

To produce Figure 2.2(c), we have employed the Fung-Ulaby model for dispersion as described in (A.5) (see Appendix A.3.2 and Ref. [17]). This model involves three parameters, namely the concentration of leaves \(v_l\) in a given volume of air, the percentage of water \(v_w\) present on each leaf and the relaxation time \(\tau\) of water. We have taken \(v_w = 0.3\) (30 % of the weight of each leaf is water) and \(v_l = 0.1\) (10 % of the given volume of air is filled by leaves). In Fig. 2.5 we show the profile of the square-root of the Fung-Ulaby permittivity when \(\tau = 10.1\) ps. In Fig. 2.2(a) we show as well noiseless data but from a non-dispersive case, where the permittivity is taken to be the constant \(\epsilon_r(\omega_0)\).

We have made a slice over the dotted line on the plot in Fig. 2.2(c). This shows transient wave-fields in Fig. 2.2(d). When the scattered wave returns to the antenna after traveling through the dispersive medium, the windowed sinusoid has the form shown in this figure. The leading and trailing edge transients are called Brillouin precursors [1] [7]. These transient wavefields have been studied extensively and, because of its unique algebraic rather than exponential peak decay, the Brillouin precursor has been considered potentially useful for foliage- and ground-penetrating
radar as well for underwater communications (for details see [34] [35]).

2.1.2 Image Formation $I(z)$

The formula (2.2) is implemented in the following steps.

(a) We discretize $I(z)$ as the matrix $I_{jk} = I(z_j, z_k)$:

$$I_{jk} = \Delta s \sum_n \Delta \omega \exp \left( -i \omega_n 2n_R(\omega_n) |\gamma(s_m) - z_{jk} |/c_0 \right) Q_{mn} \delta_{mn},$$  

(2.48)

In the non-dispersive case, we replace $n_R(\omega_n)$ in (2.48) by the constant $n_R(\omega_0)$ (see step (f) in 2.1.1).

(b) We discretize the filter (2.39), $Q_{mn}^{\text{opt}}$, as follows

$$Q_{mn}^{\text{opt}} = \frac{a_{nm}(z_{jk}) P_n}{|a_{nm}(z_{jk})|^2 |P_n|^2 J_{nm}(z_{jk}) + S_n/S_T}.$$  

(2.49)

where $a_{nm}(z_{jk}) = a(\omega_n, s_m, z_j, z_k)$, $P_n = P(\omega_n)$ and $J_{nm}(z_{jk}) = J(\omega_n, s_m, z_j, z_k)$. The Jacobian $J$ is shown in Appendix C for flat topography and a circular flight path.

(c) We use normally-distributed white noise with constant spectrum $S_n$, and a constant reflectivity spectrum $S_T$ for the scatterers.

(d) Finally, we form the image of the scatterer positions on a square region in $z$-space where $(z_j, z_k) = z$.

Figure 2.3 displays surface plots of the image reconstructions of two scatterers embedded in a homogeneous non-dispersive and a dispersive medium, respectively. The scatterers are separated by a distance of 40 cm and their locations are well resolved by the algorithm. The wiggles on the base of the peaks in Fig. 2.3(d) are typical “star pattern” artifacts associated with back-projection reconstructions [30].

In Fig. 2.4, we show the effect of decreasing the distance between scatterers. We display only transversal cuts of the (not shown) full reconstructed images. In Fig. 2.4(a), we show a slice of Fig. 2.3(c); in Fig. 2.4(b) we display a slice of figure 2.3(d). In Figs. 2.4(c) and 2.4(d) the distance between the scatterers is 14 cm.
Figure 2.2: In Figs. 2.2(a) and 2.2(c) we show the noiseless data projections generated by one single scatterer. The dotted lines over these plots indicate slices on the projections shown in Figs. 2.2(b) and 2.2(d), respectively. In the dispersive case 2.2(c) the projections have been smoothed out by attenuation and Fig. 2.2(d) shows clearly the presence of the edge transients.
However, we can still distinguish two peaks in both cases. We see that the resolution in the dispersive case is approximately 14 cm. From Fig. 2.4(e) we see that the maximum resolution for the non-dispersive case is reached around 1 cm. In the dispersive case, Fig. 2.4(f), we see only one point. Notice that in both cases noise is not present in the data, but the dispersion smoothes the images. Observe the relative decrement in amplitude for the dispersive case relative to the non-dispersive one.

Figure 2.3: 3D surface plots of the image reconstructions from noiseless data for the same two scatterers separated by $\Delta x = 20$ cm. Note that the scale on the vertical axis is different in the two plots. We have normalized the amplitudes with respect to the non-dispersive case image. In 2.3(d), we note the smoothness and the attenuation of the image due to dispersion.
(a) A cross-sectional view of 2.3(c).
(b) A cross-sectional view of 2.3(d).

(c) $\Delta x = 14$ cm. Non-dispersive medium.
(d) $\Delta x = 14$ cm. Dispersive medium.

(e) $\Delta x = 1$ cm. Non-dispersive medium.
(f) $\Delta x = 1$ cm. Dispersive medium.

Figure 2.4: Cross-sectional comparisons of the reconstructed positions of two different scatterer’s in different media. In the dispersive medium, 2.4(d) and 2.4(f), the two scatterers positions have merged into a single one. In 2.4(e) we see that the minimum resolvable distance between the scatterers is $\lesssim 1$ cm. In the last two plots, 2.4(e) and 2.4(f), we have increased the number of sampled points.
2.2 Concluding Remarks

In this chapter, we have developed a formula for forming SAR images through a temporally dispersive medium. For a given waveform, we derived a filtered-back-projection reconstruction method, the filter of which was optimally designed to minimize the reconstruction error in the mean-square sense.

We showed reconstructions from this method for the case of a transmitted waveform consisting of a square-wave-modulated sine wave.

In the next chapter, we address the issue of optimal waveform design for imaging in dispersive media.
A problem of particular interest is the determination of the structural form of the input pulse that is the best for imaging in a dispersive medium [34]. In this chapter, in a statistical framework, we provide an optimal waveform design theory for synthetic-aperture imaging through dispersive media. From previous work [22] and [39], we follow a variational approach to obtain an optimal waveform in terms of minimising the mean square error in the resulting image.

The strategy is as follows. After the optimal filter $Q_{\text{opt}}$ is substituted into the functional (2.36), this functional depends on the signal power spectrum amplitude only. From the extrema of the functional we get the optimal power spectrum. We impose the constraint that the total power of the input signal is constant along the entire antenna flight path trajectory.

The final objective in this chapter is to find an optimal waveform and not only its power spectrum. To this end, we perform a spectral decomposition getting the minimum-phase and causal signal with the initial optimal power spectrum. This spectral decomposition method is known as the Kolmogorov spectral decomposition.

In the previous chapter, we suppressed the possible dependence of the waveform on the slow-time parameter $s$. Here, however, we explicitly include the possibility that different waveforms could be transmitted from different positions on the flight path, and consequently the power spectrum amplitude depends on both $\omega$ and $s$. 
3.1 Determination of the Power Spectrum Amplitude of the Waveform

The departure point is the functional given by Eqn. (2.36). If we substitute the optimal filter $Q^\text{opt}$ into (2.36), we get a new functional

$$\Delta(P) = \int_{\Omega_y} d\omega ds \int_{\Gamma} dy \frac{S_T(\omega, s) / J(\omega, s, y)}{S_n(\omega, s) / J(\omega, s, y) |A(\omega, s, y)|^2 |P(\omega, s)|^2 + 1},$$

(3.1)

where $|P|^2$ is the power spectrum that we want to determine, $J$ is the Jacobian that comes from the Stolt change of variables (2.13), $a$ is the geometrical spreading factor (2.8), which is a real-valued and non-negative function (we will drop the absolute value sign), $S_T$ and $S_n$ are the target and noise power spectrum respectively (see Sec. 2.0.2), the domain $\Omega_y$ is defined in (2.16) and $\Gamma \subset \mathbb{R}^2$ is the surface where we form images.

In terms of the spectral density functions, we define the (frequency-dependent) target-to-noise ratio as follows:

$$\sigma_{Tn} \overset{\text{def}}{=} \frac{S_T}{S_n} \quad \text{(target-to-noise ratio).}$$

(3.2)

The noise-to-target ratio is defined by $\sigma_{nT} = S_n / S_T$.

We perform a variational derivative of Eqn. (3.1) with respect to $|P|^2$, subject to the constraint that the total transmitted energy along the flight path is a constant $M_P$; i.e.,

$$\int_{\Omega_y} d\omega ds |P(\omega, s)|^2 = M_P.$$

(3.3)

Thus, the cost functional $E(\lambda, P)$ is defined as follows:

$$E(\lambda, P) \overset{\text{def}}{=} \Delta(P) + \lambda \left( \int_{\Omega_y} d\omega ds |P(\omega, s)|^2 - M_P \right).$$

(3.4)

The parameter $\lambda$ is the Lagrange multiplier and is constant with respect to the variables $(\omega, s)$. Note that $E$ depends explicitly on $|P|^2$. Then, we simplify the calculations writing $W = |P|^2$, where $W = W(\omega, s)$. Therefore, we minimize $E(\lambda, W)$
with respect to $W$:

$$0 = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} E(\lambda, W + \epsilon W),$$

(3.5)

where $\epsilon$ is a parameter and $W_\epsilon(\omega, s)$ is an admissible function. From Eqn. (3.5) we obtain the necessary condition for the existence of the optimal $W$:

$$0 = -\int_{\Omega_y} d\omega ds \left( W_\epsilon S_T \sigma T_n \int_{\Gamma} dy \frac{a^2}{(\sigma T_n J a^2 W + 1)^2} \right) + \lambda \int_{\Omega_y} d\omega ds W_\epsilon.$$  (3.6)

Eqn. (3.6) must be valid for any arbitrary continuous function $W_\epsilon(\omega, s)$. Note that the two terms on the right-hand side of Eqn. (3.6) are constant. We rewrite this expression as follows:

$$0 = \int_{\Omega_y} d\omega ds W_\epsilon \Lambda,$$  (3.7)

where

$$\Lambda(\omega, s) = \lambda - S_T \sigma T_n \int_{\Gamma} dy \frac{a^2}{(\sigma T_n J a^2 W + 1)^2}.$$  (3.8)

From (3.7) we conclude that, if $\Lambda$ is a continuous function of $(\omega, s) \in \Omega_y$, it must be identically zero \(^6\). Therefore, from Eqn. (3.8) we have that

$$\lambda = S_T \sigma T_n \int_{\Gamma} dy \frac{a^2}{(\sigma T_n J a^2 W + 1)^2}.$$  (3.9)

Note that $\lambda$ must be positive.

Next we develop an approach to obtain a formula for $W$ in the case when the surface $\Gamma$, on which we form images, is considered small in the sense that the integrand does not vary too much on $\Gamma$. We observe that the integrand of the right-hand side of (3.9) is a function of $(\omega, s)$ and $y$; i.e.,

$$\frac{a^2(\omega, s, y)}{(\sigma T_n(\omega, s) J(\omega, s, y)a^2(\omega, s, y)W(\omega, s) + 1)^2} \equiv L(\omega, s, y).$$  (3.10)

Because $L(\omega, s, y)$ is everywhere non-negative and continuous in the region $\Gamma$, we

---

\(^6\)See [37] Ch. IV, §3.4.
can approximate the integral in Eqn. (3.9) by the Mean Value Theorem. Thus,

$$\int_{\Gamma} d\mathbf{y} L(\omega, s, \mathbf{y}) = L_0(\omega, s, \mathbf{y}_0) U, \tag{3.11}$$

where $U = \int_{\Gamma} d\mathbf{y}$ is the area of the region $\Gamma \subset \mathbb{R}^2$, $L_0 = L(\omega, s, \mathbf{y})|_{\mathbf{y} = \mathbf{y}_0}$ and $\mathbf{y}_0 \in \Gamma$. Then Eqn. (3.9) becomes

$$\lambda = \frac{S_T \sigma_T a_0^2 U}{(\sigma_T J_0 a_0^2 W + 1)^2}, \tag{3.12}$$

where $a_0 = a(\omega, \mathbf{s}, \mathbf{y}_0)$ and $J_0 = J(\omega, \mathbf{s}, \mathbf{y}_0)$.

From (3.12), we solve for $W$ in terms of $\lambda$:

$$W_\lambda = \frac{1}{\sigma_T J_0 a_0} \left( \sigma_T \sqrt{\frac{S_n U}{\lambda}} - \frac{1}{a_0} \right). \tag{3.13}$$

The function $W_\lambda$ depends on the functions $a_0$ and $J_0$ (see (2.8) and (C.2)), which both depend on the complex-valued index of refraction $n(\omega)$. Explicitly,

$$a_0 = \frac{\omega^2 e^{-\vartheta_I(\omega)|\mathbf{r}_{s,\mathbf{y}_0}|}}{(16\pi^3 |\mathbf{r}_{s,\mathbf{y}_0}|^2)}, \quad \text{and} \quad J_0 = \frac{|\mathbf{v}_p(\omega)\mathbf{v}_g(\omega)|}{\omega} \frac{|\mathbf{r}_{s,\mathbf{y}_0}|}{4|\mathbf{\psi}(\mathbf{y}_0) \cdot \gamma(s) - r^2|}, \tag{3.14}$$

where $\mathbf{\psi}$ is the ground surface defined in Subsection (1.1.2), $\gamma(s)$ is the antenna flight trajectory introduced in Sec. 1.1.3, $|\mathbf{r}_{s,\mathbf{y}_0}|$ is the distance between the ground point $\mathbf{y}_0$ and the antenna position, $r$ is the radius of the antenna circular flight path trajectory, $v_g$ and $v_p$ are the group and the phase velocity respectively [See (C.3)], $\vartheta_I(\omega) = 2\omega n_I(\omega)/c_0$ and $n_I(\omega)$ is the imaginary part of the complex-valued index of refraction (see Fig. 3.2).

Observe that $a_0$ is a well-behaved function of $s$ if the distance $|\mathbf{r}_{s,\mathbf{y}_0}|$ is a smooth and bounded function of $s$. On the other hand, when the frequency $\omega \to 0$ and $\omega \to \infty$, the term $-1/a_0$ in (3.13) diverges to $-\infty$ (see below Fig. (3.1)). However, $W_\lambda$ must be a positive quantity to be considered a meaningful power

---

^7See [38] Ch. IV, §5.
spectrum amplitude $|P|^2$. Thus, from (3.13) we have that

$$
\sigma_{Tn} \sqrt{\frac{S_n U}{\lambda}} > \frac{1}{a_0},
$$

(3.15)

which implies that the Lagrange multiplier $\lambda$ must satisfy the inequality

$$
0 < \lambda < U \sigma^2_{Tn}(\omega, s) S_n(\omega, s) a_0^2(\omega, s, y_0).
$$

(3.16)

This expression (3.16) is equivalent to the expression (68) in [39]. The positiveness condition of $W_\lambda$ restricts the values of $(\omega, s)$ to some finite sub-region of $^8$

$$
\Omega_0 = \{ (\omega, s) : \omega \in (\omega_{\text{min}}, \omega_{\text{max}}) \subset \mathbb{R}^+ \text{ and } s \in (0, 2\pi] \subset \mathbb{R} \},
$$

(3.17)

where we use the same notation as in (2.16). In practice, outside of the interval $(\omega_{\text{min}}, \omega_{\text{max}})$, $W_\lambda$ can be approximated by zero (see the numerical simulation section below).

### 3.1.1 Determination of the Lagrange multiplier with the help of the total energy

Now, we determine the Lagrange multiplier $\lambda$ from the expression (3.13) and with the help of the constraint (3.3). We integrate both sides of (3.13) with respect to $\omega$ and $s$ to get that

$$
M_P = \left[ \sqrt{\frac{U}{\lambda}} \int_{\Omega_0} d\omega ds W_\lambda = \sqrt{\frac{U}{\lambda}} \int_{\Omega_0} d\omega ds \frac{S_n}{J_0 a_0} - \int_{\Omega_0} d\omega ds \frac{1}{\sigma_{Tn} J_0 a_0^2} \right].
$$

(3.18)

We solve (3.18) for $\sqrt{U/\lambda}$ to obtain

$$
\sqrt{\frac{U}{\lambda}} = \frac{M_P + \int_{\Omega_0} d\omega ds \frac{1}{\sigma_{Tn} J_0 a_0^2}}{\int_{\Omega_0} d\omega ds \frac{S_n}{J_0 a_0^2}},
$$

(3.19)

$$
\lambda = U \left( \frac{\int_{\Omega_0} d\omega ds \frac{S_n}{J_0 a_0^2}}{M_P + \int_{\Omega_0} d\omega ds \frac{1}{\sigma_{Tn} J_0 a_0^2}} \right)^2.
$$

---

$^8$Notice that we change $\Omega_y$ by $\Omega_0$ in (3.3), because the expressions are evaluated at a particular point $y = 0$. 
For fixed values of $S_n$ and $S_T$, we observe that the relation between $\lambda$ and the total energy $M_P$ is $M_P \propto 1/\sqrt{\lambda}$. Thus, if we decrease $\lambda$ we increase the total energy and vice versa. From (3.16) we see that $\lambda$ is bounded above, which implies that the total energy $M_P$ is bounded below:

$$\frac{1}{\sqrt{\lambda_{\text{max}}}} \lesssim M_P,$$

(3.20)

where $\lambda_{\text{max}}$ is provided in (3.16). This means that there exists a minimum total energy $M_P$ that guarantees that the function $W_{\lambda}$ is positive in the region $\Omega_0$. With the help of the Lagrange multiplier, we can adjust the power constraint (3.18) to a desired value. Actually, we use this approach in the numerical simulations, where we look for a $\lambda$ that allows to $W_{\lambda}$ satisfies the power constraint.

Consider the case in which $W_{\lambda}$ is positive. We rewrite (3.13) as follows

$$\frac{W_{\lambda}}{S_n} = \frac{1}{J_0 a_0} \left( \sqrt{\frac{U}{S_n \lambda}} - \frac{1}{S_T a_0} \right).$$

(3.21)

When $\lambda \to 0$, the value of the ratio $W_{\lambda}/S_n$ increases without bound. This solution for $W_{\lambda}$ means that we could transmit high power to compensate for the noise level. When $\lambda \to \lambda_{\text{max}}$, the ratio $W_{\lambda}/S_n$ decreases its value monotonically to zero.

### 3.2 Getting the Full Waveform via Spectral Factorization

We have obtained a formula for the optimal power spectrum $W(\omega)$; now we need to find a causal waveform $p(t)$ whose Fourier transform $P(\omega)$ satisfies

$$W(\omega) = |P(\omega)|^2.$$  

(3.22)

The problem of factoring the power spectrum $W(\omega)$ in this manner is the same as the problem of finding the transfer function of a desired minimum-delay filter, i.e., the transfer function and its inverse are causal and stable [40].

Spectral decomposition is used to recover the minimum-phase complex-valued (causal) signal given its power spectrum amplitude only. Thus, the transfer function
$P(\omega)$ of a desired filter may be expressed as

$$P(\omega) = \sqrt{W(\omega)}e^{i\Theta(\omega)},$$

where $\Theta(\omega)$ is the desired phase spectrum. The method is due to Kolmogorov (see [41]).

We use of the z-transform where we define $z = e^{i\omega}$. Under this transform $P(\omega)$ becomes $\tilde{P}(z)$. Since $\tilde{P}(z)$ has no singularities or zeros within the unit circle, then $\ln \tilde{P}(z)$ is analytic within this region. On the unit circle, the spectrum is specified as follows

$$\tilde{P}(z)\overline{\tilde{P}(1/z)} = \tilde{W}(z) = e^{\tilde{K}(z)} = e^{\overline{\tilde{C}(1/z)} + \tilde{C}(z)} = e^{\overline{\tilde{C}(1/z)}}e^{\tilde{C}(z)}. \quad (3.24)$$

Given the spectrum $\tilde{W}(z)$ for each value on the unit circle, we could deduce the log spectrum $\tilde{K}(z) = \ln \tilde{W}(z)$ at each point on the unit circle:

$$\tilde{K}(z) = \ln[\overline{\tilde{S}(z)}] = \overline{\tilde{C}(1/z)} + \tilde{C}(z). \quad (3.25)$$

This is the answer we have been looking for. Given $\tilde{K}(z)$ for all real values of $\omega$, we could inverse transform to the time domain, obtaining the two-sided function $u(t) = c(-t) + c(t)$. Setting to zero the coefficients at negative times eliminates $\overline{c(-t)}$, leaving just $c(t)$; whose Fourier transform is $\tilde{C}(z)$. And we know that the exponential of $\tilde{C}(z)$ gives $\tilde{P}(z)$ with a causal $p(t)$. This method is implemented below in the next section.

### 3.3 Numerical Simulations

We calculate numerically the optimal waveform and its power spectrum following the procedure above. Then, we use that waveform for imaging. At the end of this section we compare the variance, Eqn. (2.28), of the image reconstructions from the optimal, the Brillouin precursor and the square-wave-modulated sinusoidal

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9Since this spectrum can be zero at some frequencies, and since the logarithm of zero is infinite, there is a pitfall. When the logarithm of zero arises during the computation, it is replaced by the log of a small number (see [41]).
In our simulations, we consider a simplified scenario in which the amplitude of the spectral density functions of noise $S_n$ and the target $S_T$ are constants. The spectral density function of the target is proportional to the target reflectivity $T$ and we assume that the noise spectral density is proportional to $S_T$, i.e., $S_n = \beta S_T$, where $\beta$ is a constant (see Sec. 2.1).

We choose the ground point of reference that appears in the Eqn. (3.11) to be $y_0 = (0, 0)$. With this value of $y_0$, the formulas (3.14) no longer depend on the slow-time parameter $s$ because the distances from the antenna flight trajectory to the point $y_0$ are the same:

$$a_0 = \frac{\omega^2 e^{-\vartheta_i(\omega)\sqrt{r^2 + h^2}}}{(16\pi^3)^{\frac{3}{2}}(r^2 + h^2)}, \quad \text{and}$$

$$J_0 = \left| \frac{v_p(\omega)v_g(\omega)}{\omega} \right| \frac{\sqrt{r^2 + h^2}}{4r^2}. \quad (3.26)$$

The point $y_0$ is located exactly beneath the center of the circular flight trajectory of radius $r = 100$ m and an altitude of $h = 10$ m. The area of reconstruction is $U = 100$ m$^2$. The two point-like scatterers are located on the ground at $(-1.5, 0)$ and $(1.5, 0)$, respectively. In Fig. 3.1 we show the profile of the function $a_0(\omega, y_0)$.

When we uniformly discretize the slow-time parameter $s$, the total energy $M_P$ is then equally distributed in $N_s$ firings ($N_s$ is the number of points on the antenna flight path trajectory). Consequently, on the plots, for a fixed $\lambda$, the (positive) area under the graph of $|P(\omega, s)|^2$ for every $s$ is proportional to a fraction $(M_P/N_s)$ of the total energy.

The complex-valued refractive index $n(\omega)$ used in the simulations is provided by the Fung-Ulaby model dielectric (see Appendix A.3.2). We show the profile of this refractive index in Fig. 3.8.

In Fig. 3.2 we show the behavior of $W_\lambda$ when we change the Lagrange multiplier value $\lambda$. There are values of $\lambda$ that give us $W_\lambda < 0$; this is the case for $\lambda_3$. On the other hand, there are $\lambda$ for which $W_\lambda$ has both positive and negative values; this is the case for $\lambda_1$. The intermediate case is $\lambda_2$, when we get that $W_\lambda \leq 0$. The relation between the $\lambda$ values is the following: $\lambda_1 < \lambda_2 < \lambda_3$.  

waveforms.
In Fig. 3.3 we show how the function $W_\lambda$ changes as $\lambda \to 0$. Note that the peak of $W_\lambda$ moves to higher frequencies when $\lambda$ decreases.

In Fig. 3.4 we show how the function $W_\lambda$ changes when we change the target-to-noise ratio $\sigma_{Tn}$. We keep $S_T$ constant and change the noise amplitude $S_n$ only. When we decrease $\sigma_{Tn}$, i.e., when we increase the noise amplitude, it is interesting to note that the normalized amplitude of $W_\lambda$ decreases slightly but its peak moves notably toward lower frequencies.

In Fig. 3.5 we show the procedure to obtain a causal waveform from the amplitude of its power spectrum. Firstly, given a value for the Lagrange multiplier $\lambda$, we obtain the function $W_\lambda(\omega)$, shown in Fig. 3.5(a). We replace the negative values by zeros (zero-padding) to get the power spectrum prototype $|P(\omega)|^2$, shown in Fig. 3.5(b). We take the logarithm $\ln(|P|^2)$ and, by means of the inverse Fourier transform, we obtain a time domain representation of this quantity. We set to zero the non-causal part in the time domain. Then, we apply the Fourier transform to get an expression in the frequency domain and exponentiate to invert the logarithm. Finally, we apply the inverse Fourier transform to return to the time-domain, obtaining a causal waveform, shown in Fig. 3.5(c). In Fig. 3.5(d) is shown the power spectrum from Fig. 3.5(c), which is indeed an approximation of the power spectrum shown in Fig. 3.5(b).

In Fig. 3.6 we show a detail of the optimal waveform 3.5(c). Note that the wavelength of this wave-field is shortened and its amplitude increases at the beginning of the plot.

In Fig. 3.7 we show the three different input signals that we employ for imaging. In Fig. 3.8 we show the profile of the real and imaginary parts of the complex-valued index of refraction when the relaxation time parameter $\tau = 8$ ns. The noticeable change of the index of refraction takes place in the vicinity of 0.1 GHz, around this value are located the carrier frequencies of the input signals.

In Figs. 3.9, 3.10 and 3.11 we show ten reconstructions of the three different input signals at different realization of noise with the same target-to-noise ratio $\sigma_{Tn} = 1$.

In Fig. 3.12 we show the image reconstruction averages that come from each
input signal. When we employ the square-wave-modulated sinusoidal and the optimal waveforms, we can distinguish the two point-like scatterers located on the ground. In Figs. 3.13, 3.14 and 3.15 we show surfaces of this reconstructions. In Fig. 3.16 we show a slice of each average image at $y = 0$ position, where we observe the difference between reconstructions when we employ different input signals.

In Chapter 2 we determined the optimal filter $Q$, and in this chapter we determine the optimal waveform so that the MSE $\Delta(Q,P)$ is minimized. From Eqn. (2.23) and (2.21) we can see that

$$V(Q,P) = \int dz \langle |E(z) - \langle E(z) \rangle|^2 \rangle$$

$$= \int dz \langle |I(z) - \langle I(z) \rangle|^2 \rangle. \quad (3.27)$$

We show the normalized variance $V$ in Table 3.3 for the three different input signals. The variance calculated with the optimal waveform input signal is the smallest one.

<table>
<thead>
<tr>
<th>Input Waveform</th>
<th>$\frac{V}{|I(z)|_{L^2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Windowed Sinusoid</td>
<td>26.85</td>
</tr>
<tr>
<td>Brillouin Precursor</td>
<td>10.46</td>
</tr>
<tr>
<td>Optimal Waveform</td>
<td>7.93</td>
</tr>
</tbody>
</table>

Table 3.1: Comparisons of the image reconstruction variances by the different input signals.

### 3.4 Concluding Remarks

For a large amount of power (small $\lambda$), the optimal waveform spectrum $W_\lambda$ shifts to higher frequencies. If the target-to-noise ratio is larger in a particular frequency band, then this is also where the waveform power spectrum is large.

For a given noise level on the forward data and when we employ the windowed sinusoidal and the optimal waveforms, we can resolve the position and reflectivity amplitude of the two point-like scatterers on the ground (see Fig. 3.16). However, from the variances reported in Table 3.3, we observe that imaging with the optimal
Figure 3.1: Profile of the function $a_0$ (Eqn. (3.26)) when $y_0 = 0$, $r = 100$ m, $h = 10$ m, and $n_I$ is the imaginary part of the square-root of the Fung-Ulaby permittivity.

We define the total energy constraint $M_P$ in (3.3) on a region given in (3.17). We found that the Lagrange multiplier $\lambda$ is related to this total energy in (3.19). This expression (3.19) comes from the fact that the function $W_\lambda$ is non-negative only on the region $\Omega_0$ (see Fig. 2.1). Outside of $\Omega_0$, $W_\lambda$ is negative. This means that if $W_\lambda$ is an approximation of a power spectrum (PS) $|P(\omega, s)|^2$, then the corresponding real-valued and causal signal $p(t)$ must be determined via the spectral factorization using the frequencies in $\Omega_0$. From this point of view, the zero-padding of $W_\lambda$ should
Figure 3.2: Profile of $W_\lambda$ at different values of $\lambda: \lambda_1 < \lambda_2 < \lambda_3$. Other parameters are fixed as follows: $\sigma_{Tn} = 1$, $y_0 = 0$, $r = 100$ m, $h = 10$ m, and $n_I$ is the imaginary part of the square-root of the Fung-Ulaby permittivity.

be considered another approximation to $|P(\omega, s)|^2$ (see Fig. 3.5).

On the other hand, we do not restrict the frequency domain of the precursor and the windowed sinusoid waveforms in the same way as we do with the optimal waveform. For the comparisons in this work to hold, we design the other waveforms in a way that their carrier frequencies remain in the same region $\Omega_0$. Thus, the comparison between different signals, with the same $M_P$, is valid.
Figure 3.3: Function $W_\lambda$ vs. $\lambda$. Observe how the amplitude of $W_\lambda$ increases considerably and how its peak moves to higher frequencies when we decrease the value of $\lambda$. 
Figure 3.4: Function $W_\lambda$ vs. the target-to-noise ratio $\sigma_{Tn} = S_T/S_n$ for a fixed value of $\lambda$. The amplitude of $W_\lambda$ is reduced slightly and its peak moves to lower frequencies when we decrease the value of $\sigma_{Tn}$. We have normalized the $W_\lambda$ amplitude.
Figure 3.5: Minimum-phase optimal waveform via spectral factorization for Fung-Ulaby model dielectric.
Figure 3.6: Minimum-phase optimal waveform detail.
Figure 3.7: Input signals used in the numerical simulations.
Figure 3.8: Profile of the real and imaginary parts of the complex-valued refractive index with a relaxation time parameter $\tau = 8$ ns.
Figure 3.9: Ten reconstructions from the Brillouin precursor waveform input signal for different realizations of noise when the target-to-noise ratio = 1.
Figure 3.10: Ten reconstructions from the windowed sinusoidal waveform input signal for different realizations of noise when the target-to-noise ratio = 1.
Figure 3.11: Ten reconstructions from the optimal waveform input signal for different realizations of noise when the target-to-noise ratio = 1.
Figure 3.12: Average of the ten reconstructions from the three input signals for different realizations of noise.
Figure 3.13: Surface of the average of the reconstruction from the precursor input signal.
Figure 3.14: Surface of the average of the reconstruction from the windowed sinusoidal input signal.
Figure 3.15: Surface of the average of the reconstruction from the optimal input signal.
Figure 3.16: Slice on the average of the reconstructions at position $y = 0$. The scatterers are located at $x = -1.5$ and $x = 1.5$. 
LITERATURE CITED


[8] Tsynkov S V 2007 *On SAR Imaging through the Earth Ionosphere*, to be published


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[36] Epstein C L 2003 Introduction to the Mathematics of Medical imaging Pearson, NJ USA


APPENDIX A

SOME DISPERSIVE MATERIALS

A.1 Debye model.

Water is an example of a ubiquitous dispersive medium [13]. Between 0 and 20°C, water’s relative permittivity ranges from 88 to 80.1 in contrast to the value of 1 for vacuum and approximately 1.00054 for air [14]. Water molecules absorb microwaves and other radio wave frequencies and attenuate radar signals. Different frequencies attenuate at different rates, such that some components of air are opaque to some frequencies and transparent to others. Therefore, in its various forms, depending on whether it is vapor or liquid, water affects strongly the permittivity of the medium [15]. The Debye model is generally good for polar molecules such as water in microwave regime [13, 16]. The Debye model is

\[
\epsilon_r(\omega) = \epsilon_\infty + \frac{\epsilon_s - \epsilon_\infty}{1 - i\omega\tau},
\]

where for water, typical values are as follows. The zero-frequency relative permittivity is \(\epsilon_s = 80.35\), the infinite-frequency relative permittivity is \(\epsilon_\infty = 1.00\) and the relaxation time is \(\tau = 8.13\) ps.

We assume that (A.1) satisfies

\[
\partial_\omega^\beta f(\omega) \leq C_\beta (1 + \omega^2)^{-1+\beta/2},
\]

where \(f(\omega) = (\epsilon_s - \epsilon_\infty)/(1 - i\omega\tau)\) for every non-negative integer \(\beta\). The large-\(\omega\) decay of \(f(\omega)\) implies that \(f(\omega)\) appears only in the remainder terms of large-\(\omega\) asymptotic calculations [13].
A.2 Lorentz model.

The Lorentz model, which is a harmonic-oscillator model, is generally good for solid materials. The permittivity for a Lorentz medium is given by

$$\epsilon_r(\omega) = \epsilon_\infty - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + 2i\omega \nu},$$  \hspace{1cm} (A.3)

where again $\epsilon_\infty$ is the high-frequency permittivity of the material. The other parameters are the plasma frequency $\omega_p$, the resonance frequency $\omega_0$, and the damping amplitude $\nu$ [13].

A.3 Models for vegetation

Vegetation is a dispersive medium, because water is dispersive and is a major constituent of leaves and living wood [13][16].

A.3.1 Brown-Curry-Ding model (100 MHz - 10 GHz).

This is a model for a sparse random medium such as the branches and tree trunks of a forest. A modification due to Ding of the Brown-Curry model is

$$\epsilon_r(\omega) = \left[ \epsilon_\infty + \frac{b}{1 + (\omega/\omega_c)^2} \right] + i \left[ \frac{(\omega/\omega_c)b}{1 + (\omega/\omega_c)^2} + \frac{a}{(2\pi\omega)^{0.96}} \right].$$  \hspace{1cm} (A.4)

Here $\epsilon_\infty \approx 4.5$, $b$ is a constant that depends on the density and fractional volume of wood, $a = 1.5 \times 10^9$ (respectively $4 \times 10^9$) when the electric field is parallel (perpendicular) to the wood grain and $2\pi\omega_c$ is the temperature-dependent frequency that is roughly 20 GHz at 25°C [2].

For time-domain continuity, the permittivity $\epsilon(\omega)$ must decay at high frequencies faster than $\omega^{-1}$ [12]. The Brown-Curry-Ding model does not obey this condition.

A.3.2 Fung-Ulaby model (8 - 17 GHz).

Roughly speaking, vegetation could be considered a simple mixture of dry matter, air and water. In our numerical examples, we considered the model for complex-valued permittivity developed by Fung and Ulaby [17]. In this model, the
permittivity of the vegetation (taken to be a combination of water and some solid material) was estimated by a mixing formula for two-phase mixtures [18] [19]. The effective relative Debye-type permittivity is given by

$$\epsilon_{\text{r,eff}}(\omega) = v_l \epsilon_l + (1 - v_l),$$

$$\epsilon_l = \epsilon_R(\omega) + i \epsilon_I(\omega),$$

$$\epsilon_R(\omega) = 5.5 + \frac{e_m - 5.5}{1 + \tau^2 \omega^2},$$

$$\epsilon_I(\omega) = \frac{(e_m - 5.5) \tau \omega}{1 + \tau^2 \omega^2},$$

$$e_m = 5 + 51.56 v_m.$$  \hspace{1cm} (A.5)

where $v_l$ fractional volume occupied by leaves, $v_w$ water volume fraction within a “typical leaf”, and $\tau$ empirical relaxation time of water molecules.

The relaxation time $\tau$ is governed by the interaction of the water molecules with its environment and by the temperature $T$. For pure water at $20^\circ\text{C}$, $\tau \approx 10.1 \times 10^{-12} \text{ s}$. This is the value that we employ in Sec. 2.1. However, when water is mixed with other materials, that is the case of bulk vegetation and water, its response to the electric field changes. If the water molecules are under the influence of non-electrical forces, such as mechanical or chemical forces, its response to an applied wave-field is impeded by these forces, which has the equivalent effect of increasing the relaxation time $\tau$ (see for details [18], [19] and [13]).
In this appendix we show how to use the method of stationary phase to approximate the multiple integral (2.31). This integral is of the type:

\[ \Delta_T(P,Q) = \int e^{i\phi(x)}u(x)dx. \]  

The stationary phase method deals with integrals of the form (B.1) in which the integration is over a compact set \( K \). We consider only the case in which the phase has only non-degenerate critical points in \( K \). We say that \( x_l \in \mathbb{R}^n \) is a non-degenerate critical point of \( \phi \) if \( \phi'(x_l) = 0 \) and \( \det(\phi''(x_l)) \neq 0 \), where \( \phi''(x) = \left( \frac{\partial^2 \phi(x)}{\partial x_j \partial x_k} \right)_{1 \leq j,k \leq n} \) is the Hessian of \( \phi \) at \( x \). The stationary phase theorem states that if \( u \in C^\infty_0 \subset \mathbb{R}^n \), and \( \phi \) has only non-degenerate critical points, then as \( \lambda \to \infty \) the leading-order approximation to (B.1) is obtained from the estimate (See Proposition 2.3 in [24]):

\[ \left| \int e^{i\lambda \phi(x)}u(x)dx - \sum_l R_l u(x_l)\lambda^{-n/2}e^{i\lambda \phi(x_l)} \right| \leq C\lambda^{-n/2} \sum_{|\kappa| \leq n+3} \left( \sup_K |\partial^{\kappa} u(x)| \right), \]

where \( \lambda \geq 1 \), \( \kappa \) is a multi-index, where

\[ R_l = \frac{(2\pi)^{n/2}e^{i\pi\text{sgn}(\phi''(x_l))/4}}{|\det \phi''(x_l)|^{1/2}}, \]

and where \( x_l \) denote the critical points in \( K \), and where \( \text{sgn}(\cdot) \) denotes the signature of the Hessian, i.e., the number of positive eigenvalues minus the number of negative ones.

To put (2.31) into the form (B.1), we first make the change of variables \( \xi = \lambda \xi', \xi' = \lambda \tilde{\xi}' \), where \( \lambda = |\xi| \). This change of variables converts the phase to \( \phi = \dots \).
\( \lambda[(z - y) \cdot \tilde{\xi} - (z - y') \cdot \tilde{\xi}'] \). The integrand \( u \) of (B.2) (with \( x \leftrightarrow (z, \tilde{\xi}') \)) is then

\[
\begin{align*}
\lambda(\tilde{\xi}) 
\times \left[ \frac{1}{\lambda^3} \left[ Q(\lambda\tilde{\xi}, z) A(\lambda\tilde{\xi}, y) J(\lambda\tilde{\xi}, z, y) - 1 \right] \right. \\
\left. - Q(\lambda\tilde{\xi}', z) A(\lambda\tilde{\xi}', y') J(\lambda\tilde{\xi}', z, y') - 1 \right] 
\end{align*}
\]

(B.4)

We apply the stationary phase analysis to the \( z \) and \( \tilde{\xi}' \) integrals, so that \( n \) in (B.2) is 4; the leading-order term is (2.34).

The right side of (B.2) decays more rapidly than \( \lambda^{-2} \) provided \( u \) satisfies a symbol estimate as in [24]. This is the case because a) the flight path is smooth; b) for the operating frequencies of the radar, the permittivity \( \epsilon_r \) is smooth; and c) \( P \) is assumed to satisfy symbol estimates. Although derivatives with respect to \( \tilde{\xi}' \) give rise to factors of \( \lambda \), because \( u \) involves \( P \), whose derivatives decay increasingly rapidly in \( \omega \) (\( \omega \) being proportional to \( |\xi| = \lambda \)), these extra factors of \( \lambda \) do not lead to an overall worsening of the decay rate in \( \lambda \). For the calculation of (2.34), we assume that \( Q \) is smooth; then smoothness of (2.40) can be checked, so that the assumption is consistent with the result.

The more rapid large-\( \lambda \) decay of the error term corresponds to its being smoother than the leading-order term.
APPENDIX C
BEYLKIN DETERMINANT

The Stolt change of variables (2.13) introduce the Jacobian

\[
J(\omega, s, y) = \left| \frac{\partial(s, \omega)}{\partial \xi} \right|. \tag{C.1}
\]

This Jacobian is calculated when the topography is flat and the antenna flight path trajectory is circular of radius \(r\) and height \(h\) (the center of this trajectory is given by \((0, 0, h)\)). We obtain the following expression

\[
J(\omega, s, y) = \left| \frac{v_p(\omega)v_g(\omega)}{\omega} \right| \frac{\left| \psi(y) - \gamma(s) \right|}{4|\psi(y) \cdot \gamma(s) - r^2|}. \tag{C.2}
\]

where

\[
v_g(\omega) = \frac{c_0}{n_R(\omega) + \omega \frac{dn_R(\omega)}{d\omega}},
\]

\[
v_p(\omega) = \frac{c_0}{n_R(\omega)},
\]

\[
n_R(\omega) = \text{Re}\{\sqrt{\epsilon_r(\omega)}\}, \tag{C.3}
\]

\[
\psi(y) = (y_1, y_2, 0),
\]

\[
\gamma(s) = (r \cos(s), r \sin(s), h).
\]

If we do not take into account a dispersive medium, \(i.e., n_R = \sqrt{\epsilon_r}\) is constant, then the Jacobian becomes

\[
J(\omega, s, y) = \left| \frac{c_0^2}{\omega \epsilon_r} \right| \frac{\left| \psi(y) - \gamma(s) \right|}{4|\psi(y) \cdot \gamma(s) - r^2|}. \tag{C.4}
\]

In the numerical simulations we take into account the dispersive medium and that the ground position of reference is the center of the scene \(y_0 = (0, 0)\), the Jacobian is then reduced to the expression

\[
J(\omega, s, y) = \left| \frac{v_p(\omega)v_g(\omega)}{\omega} \right| \frac{\sqrt{r^2 + h^2}}{4r^2}. \tag{C.5}
\]