INTRODUCTION TO DIFFERENTIAL EQUATIONS, FINAL EXAM

Sections 5-8, Spring 2008

Section ________________ Name ____

Instructions. You are allowed to use two 8 1/2 × 11 inch sheets of notes. No books or electronic equipment (including calculators, PDAs, computers, cell phones) are allowed. Do not collaborate in any way. In order to receive credit, your answers must be clear, legible, and coherent. In case of an error in a test question, simply write in the correct answer. All questions are 5 points each.

_____1. For the differential equation $2ty' + y + \sqrt{t} = 0$, the integrating factor is
A) e^t B) $e^{t/2}$ C) e^{t^2} D) $\sqrt{t}$

_____2. To solve $y' + xy^2 = 0$, which method should we use?
A) the method of integrating factors B) the method of separation of variables
C) either A) or B) will work D) neither A) nor B) will work

_____3. The general solution to a certain first-order differential equation is $y(x) = e^{-x^2}(x + c)$. The solution satisfying $y(0) = 10$ is $y(x) =$
A) $e^{-x^2}(x + 10)$ B) 10 C) $10e^{-x^2}(x + c)$ D) $e^{-x^2}(10x + C)$ E) $e^{-10x^2}(x + C)$

For problems 4, 5, and 6, choose the best description of the system from the following:
A) simple harmonic motion B) overdamped C) underdamped
D) critically damped E) resonant F) steady-state plus transient G) beating

_____4. $y'' + y = \cos t$

_____5. $y'' + y = \cos 2t$

_____6. $y'' + y' + y = \cos t$

_____7. The eigenvalues and eigenfunctions for the two-point boundary-value problem
$X'' + \lambda X = 0, \quad X'(0) = 0 = X(\pi)$ are

A) $\{\lambda = n^2,\ X_n(x) = \cos[(n + 1/2)x],\ n = 0, 1, 2, \ldots\}$
B) $\{\lambda = (n + 1/2)^2,\ X_n(x) = \cos[(n + 1/2)x],\ n = 0, 1, 2, \ldots\}$
C) $\{\lambda = n^2,\ X_n(x) = \sin[(n + 1/2)x],\ n = 0, 1, 2, \ldots\}$
D) $\{\lambda = (n + 1/2)^2,\ X_n(x) = \sin[(n + 1/2)x],\ n = 0, 1, 2, \ldots\}$
E) $\{\lambda = n^2,\ X_n(x) = \cos(nx),\ n = 0, 1, 2, \ldots\}$
F) $\{\lambda = n^2,\ X_n(x) = \sin(nx),\ n = 1, 2, \ldots\}$

______8. The plot that best represents

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \; dx \qquad$ is

A)

B)

C)

D)

E)

F)

Questions 9 and 10 deal with the problem

$$u_t = u_x + u_{xx}, \quad u(0,t) = 0 = u(\pi,t), \quad u(x,0) = f(x) \qquad (*)$$

______9. When we apply the method of separation of variables to equation (*), the X problem is

A) $X'' + \lambda X = 0,\ X(0) = 0 = X(\pi)$ B) $X'' + X' + \lambda X = 0,\ X(0) = 0 = X(\pi)$
C) $X'' + X + \lambda X = 0,\ X(0) = 0 = X(\pi)$ D) $X'' + \lambda X' = 0,\ X(0) = 0 = X(\pi)$
E) $X'' + X' = \lambda,\ X(0) = 0 = X(\pi)$

______10. and the T problem for equation (*) is

A) $T' + \lambda T = 0$ B) $T' = 0$ C) $T' + \lambda T = 0,\ T(0) = f(x)$ D) $T' = 0,\ T(0) = f(x)$

For problems 11 and 12, you may use the fact that the problem $u_t = u_{xx},\ u(0,t) = 0 = u(\pi,t)$ has solutions of the form $\exp(-n^2 t)\sin(nx),\ n = 1, 2, \ldots$.

_____11. The solution to $u_t = u_{xx},\ u(0,t) = 0 = u(\pi,t),\ u(x,0) = \sin 3x + 2\sin 5x$ is $u(x,t) =$

A) $\sum_{n=1}^{\infty} e^{-n^2 t}\sin(nx)$ B) $\sin 3x + 2\sin 5x$ C) $e^{-3t}\sin(3x) + 2e^{-5t}\sin(5x)$

D) $e^{-9t}\sin(3x) + 2e^{-25t}\sin(5x)$ E) $e^{-n^2 t}\sin(nx)$

_____12. The solution to $u_t = u_{xx},\ u(0,t) = 0 = u(\pi,t),\ u(x,0) = \begin{cases} 0 & 0 < x < \pi/2 \\ 1 & \pi/2 \le x < \pi \end{cases}$ is

A) $\sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$ where $b_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)dx$

B) $\sum_{n=1}^{\infty} b_n \sin(nx)$ where $b_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)dx$

C) $\sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$ where $b_n = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)dx$

D) $\sum_{n=1}^{\infty} b_n \sin(nx)$ where $b_n = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)dx$

_____13. In the process of solving

$$u_t + u = u_{xx}, \qquad u(0,t) = 0 = u(\pi,t), \qquad u(x,0) = f(x) \qquad (\#)$$

we find that the nontrivial solutions of $X'' + \lambda X = 0, X(0) = 0 = X(\pi)$ are $\lambda = n^2,\ X_n(x) = \sin(nx),\ n = 1, 2, \ldots$, and the solution of $T' + (n^2+1)T = 0$ is $T(t) = \exp[-(n^2+1)t]$. Then the solution of (#) is $u(x,t) =$

A) $\sum_{n=1}^{\infty} b_n \sin(nx)$ where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$

B) $\sum_{n=1}^{\infty} b_n e^{-n^2 t}\sin(nx)$ where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$

C) $\sum_{n=1}^{\infty} b_n e^{-(n^2+1)t}\sin(nx)$ where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx$

D) $\sum_{n=1}^{\infty} e^{-n^2 t}\sin(nx)$

E) $\sum_{n=1}^{\infty} e^{-(n^2+1)t}\sin(nx)$

F) $e^{-(n^2+1)t}f(x)$

_____14. The Laplace transform of the initial value problem $2y'' + y' = 5,\ y'(0) = 1,\ y(0) = 3$ is

A) $(2s^2 + s)\mathcal{L}[y] = 5$ B) $(2s^2 + s)\mathcal{L}[y] - 2s - 5 = 5$ C) $(2s^2 + s)\mathcal{L}[y] - 6s - 5 = 5/s$
D) $(2s^2 + 1)\mathcal{L}[y] - 6s - 5 = 5/s$ E) $(2s^2 + 1)\mathcal{L}[y] - 2s - 9 = 5/s$

_____15. The solution to the initial value problem $y'' + y = g(t), \quad y(0) = 1, \quad y'(0) = 0$, can be written $y(t) =$
A) $\dfrac{\mathcal{L}[g] + s}{s^2 + 1}$ B) $A\cos t + B\sin t + g(t)$ C) $\int_0^t \sin(t - t')g(t')dt'$ D) $\int_0^t \cos(t - t')g(t')dt$
E) $\sin t g(t)$

_____16. In solving an initial value problem by the Laplace transform method, we obtain $\mathcal{L}[y] = e^{-2s}\left(\dfrac{1}{s} - \dfrac{1}{s+1}\right)$. The solution to the initial value problem is $y(t) =$
A) $\begin{cases} 1 - e^{-(t-2)} & t > 2 \\ 0 & t < 2 \end{cases}$ B) $\begin{cases} 1 - e^{-t} & t > 2 \\ 0 & t < 2 \end{cases}$ C) $1 - e^{-t}$ D) $1 - e^{2-t}$

_____17. An eigenvector of the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ that corresponds to the eigenvalue $\lambda = 3$ is any multiple of
A) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ B) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ C) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ D) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ E) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ F) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

_____18. The real-valued matrix A has eigenvalue $-i$ and eigenvector $\begin{pmatrix} 2+i \\ 3i \end{pmatrix}$. The system $\boldsymbol{x}' = A\boldsymbol{x}$ has general solution

A) $c_1 \begin{pmatrix} 2\cos t + \sin t \\ 3\sin t \end{pmatrix} + c_2 \begin{pmatrix} \cos t - 2\sin t \\ 3\cos t \end{pmatrix}$ B) $c_1 \begin{pmatrix} 2\cos t + \sin t \\ 3\cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ 3\sin t \end{pmatrix}$

C) $c_1 \begin{pmatrix} 2\cos t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t \\ 3\sin t \end{pmatrix}$ D) $c_1 \begin{pmatrix} 2\sin t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ 3\cos t \end{pmatrix}$

E) $c_1 \begin{pmatrix} 2\cos t - \sin t \\ \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \cos 3t \end{pmatrix}$ F) $c_1 \begin{pmatrix} 2\cos 3t - \sin 3t \\ \sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin 3t + \cos t \\ \cos t \end{pmatrix}$

For the following two problems, classify the phase-plane diagram for the system.
A) asymptotically stable node B) unstable node C) saddle point (unstable) D) asymptotically stable spiral E) unstable spiral F) center (stable)

_____19. $\begin{cases} x' = 3x + y \\ y' = y \end{cases}$

_____20. $\begin{cases} x' = x + y \\ y' = -x \end{cases}$