Scaling, self-similarity, and intermediate asymptotics

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0.1 Dimensional analysis and physical similarity

The starting point of this book is dimensional analysis and it is used throughout. Like unhappy families, every unfortunate scientific idea is unfortunate in its own way. Many of those who have taught dimensional analysis (or have merely thought about how it should be taught) have realized that it has suffered an unfortunate fate.

In fact, the idea on which dimensional analysis is based is very simple, and can be understood by everybody: physical laws do not depend on arbitrarily chosen basic units of measurement. An important conclusion can be drawn from this simple idea, using a simple argument: the functions that express physical laws must possess a certain fundamental property, which in mathematics is called generalized homogeneity or symmetry. This property allows the number of arguments in these functions to be reduced, thereby making it simpler to obtain them (by calculating them or determining them experimentally). This is, in fact, the entire content of dimensional analysis – there is nothing more to it.

Nevertheless, using dimensional analysis, researchers have been able to obtain remarkably deep results that have sometimes changed entire branches of science. The mathematical techniques required to derive these results turn out to be simple and accessible to all. The list of great names involved runs from Newton and Fourier to Maxwell, Rayleigh and Kolmogorov. Among recent developments, it is sufficient to recall the triumph of the Kolmogorov–Obukhov theory in turbulence.
Everyone would like to score a classical triumph. Many people therefore attacked what one would think were almost identical problems using the same simple dimensional analysis approach. Alas, they almost always failed. Dimensional analysis was cursed and reproached for being untrustworthy and unfounded, even mystical. Paradoxically, the reason for this lack of success was that only a few people understood the content and real abilities of dimensional analysis.

It was like the old Deanna Durbin film: a girl with a small suitcase arrives in New York and, in no time, charms the son of a millionaire. Films like this are pleasant to watch. However, if they are treated as a guide to what provincial girls should do, disillusionment is inevitable.

Let us describe here what dimensional analysis is using several simple examples.

From elementary physics, the reader knows that the period $\theta$ for small oscillations of a simple pendulum of length $l$ (Figure 0.1) is

$$\theta = 2\pi \sqrt{\frac{l}{g}} \approx 6.28 \sqrt{\frac{l}{g}},$$  \hspace{1cm} (0.1)

where $g$ is the gravitational acceleration.

![Figure 0.1. A pendulum performs small oscillations. Experiment shows that the period of the small oscillations is independent of the maximum deviation of the pendulum.](image)

Equation (0.1) is usually obtained by deriving and solving a differential equation for the oscillations of the pendulum. We shall now obtain it from completely different considerations without any use of calculus. First of all, we ask ourselves; on what can the period of oscillation of the pendulum depend? It is clear that in principle, it can depend only on (a) the length of the pendulum, (b) the mass of the bob, and (c) the gravitational acceleration – if there were no gravitational force (i.e., under weightless conditions), the pendulum would not oscillate. The length of the pendulum, mass of the bob, period of oscillation, and gravitational acceleration can be written in terms of the numbers $l$, $m$, $\theta$, and $g$, which are obtained in the following way. Definite objects representing units of length, mass, and time are chosen; these are agreed standards,
which are either carefully preserved or reproducible. Then, the number \( l \) is obtained by measuring the length of the pendulum, i.e., comparing the length of the pendulum with the unit of length. The number \( m \) is obtained by comparing the mass of the bob with the unit of mass, and the number \( \theta \) is obtained by comparing the period of oscillation with the unit of time. The situation is slightly more complicated for the gravitational acceleration:

First, we recall that velocity by its very definition is the ratio of the distance travelled in an infinitesimal time interval to the magnitude of that time interval. We therefore adopt the velocity of uniform motion in which one unit of length is traversed per unit of time as the unit for velocity. Analogously, the acceleration is the variation in velocity over an infinitesimal time interval divided by the magnitude of that time interval. We therefore adopt the acceleration of uniformly accelerated motion in which the velocity increases by one velocity unit per unit time as the unit for acceleration.

Let us now decrease the unit of length by a factor \( L \), the unit of mass by a factor \( M \), and the unit of time by a factor \( T \). We are justified in doing so, and in selecting the abstract positive numbers \( L, M, \) and \( T \) as we like: the choice of units for mass, length, and time – the fundamental units – is arbitrary. In doing so, since the units have been decreased in magnitude, the numerical values of all lengths increase by a factor \( L \), all masses increase by a factor \( M \), and all times increase by a factor \( T \). The velocity increases by a factor \( LT^{-1} \) with respect to its original magnitude under this transformation. Indeed, in uniform motion at a velocity assumed to be equal to the new unit of velocity, the new unit of length (which is a factor \( L \) smaller than the original unit) is now traversed in one new unit of time (which is a factor \( T \) smaller than the original unit). Because of this, the numerical values of all velocities increase by a factor \( LT^{-1} \). Analogously, the unit of acceleration decreases by a factor \( LT^{-2} \) under this transformation of fundamental units. Thus, the numerical values of all accelerations (and, in particular, the gravitational acceleration) increase by a factor \( LT^{-2} \).

Therefore, in general, when the magnitudes of the fundamental units – those in which length, mass, and time are measured – are changed, the numerical value of a physical quantity also changes. The factor which gives the magnitude of this change is determined by the dimension of the quantity in question. For example, if the unit of length is decreased by a factor \( L \), the numerical values of all lengths are increased by a factor \( L \). We say that length has dimension \( L \). Analogously, mass has
dimension $M$, time has dimension $T$, velocity has dimension $LT^{-1}$, and acceleration has dimension $LT^{-2}$. We emphasize once again: $L$, $M$, and $T$ are nothing more than abstract positive numbers. As is evident from these examples, the dimensions of velocity and acceleration are functions of these numbers, in fact very special functions of these numbers, power-law monomials. (In chapter 1 the reason why is explained in detail.)

Consider the quantity $l/g$. This quantity is a ratio of two numbers that depend on the units defined for length and acceleration. If the unit of length is decreased by a factor $L$, and the unit of time by a factor $T$, the numerical value of the length in the numerator increases by a factor $L$, and the numerical value of the acceleration in the denominator increases by a factor $LT^{-2}$. Consequently, the ratio $l/g$ increases by a factor $T^2$, and the quantity $(l/g)^{1/2}$ increases by a factor $T$, i.e., exactly the same factor by which the period of oscillation $\theta$ increases in this case. Therefore, the ratio

$$\Pi = \frac{\theta}{\sqrt{l/g}}$$

is invariant under a change of fundamental units. Quantities like $\Pi$, which do not change when the fundamental units of measurement are changed, are called dimensionless: their dimensions are equal to unity. All other physical quantities are called dimensional. A correspondence is set up between each dimensional physical quantity and its dimension, which differs from unity and indicates the factor by which the numerical value of this quantity increases for a given decrease in the magnitude of the fundamental units of measurement.

We shall now proceed from the fact that, just like the period of oscillation $\theta$, the quantity $\Pi$ may, in principle, depend on these same quantities $l$, $m$ and $g$; thus, $\Pi = \Pi(l, m, g)$. Once again we recall that $l$, $m$, and $g$ are numbers that hold for one particular system of fundamental units of measurement. We now decrease the unit of mass by some factor $M$, and leave all the other units unchanged. In this case, the number $m$ increases by an arbitrary factor $M$, but the numbers $\Pi$, $l$, and $g$ remain unchanged. But this means that the function $\Pi(l, m, g)$ remains unchanged for any change in the argument $m$ while the other two arguments $l$ and $g$ remain unchanged, i.e., that this function is independent of $m$. Next, we decrease the unit of time by some factor $T$, leaving the unit of length unchanged. Then, in accordance with the above, the numerical value of $g$ increases by a factor $T^{-2}$. (The dimension of acceleration is actually $LT^{-2}$, but $L$ is equal to unity in this case.) The quantities $\Pi$ and $l$ remain unchanged (recall that $\Pi$ is dimensionless). But this
means that the function $\Pi(l, m, g)$ is also independent of $g$. Finally, we
decrease the unit of length by some factor $L$. The numerical value of the
last remaining argument $l$ then increases by a factor $L$, and the dimen-
sionless quantity $\Pi$ once again remains unchanged. This means that $\Pi$
is also independent of $l$, and is therefore completely independent of all
the parameters. Therefore, it is in fact a constant:

$$\Pi = \frac{\theta}{\sqrt{l/g}} = \text{const},$$  \hspace{1cm} (0.2)

from which we obtain

$$\theta = \text{const} \sqrt{\frac{l}{g}},$$  \hspace{1cm} (0.3)

which is the same as (0.1) up to a constant. The constant in (0.3) can
be determined fairly accurately from a single experiment that the reader
may carry out him- or herself by measuring the period of oscillation
of a weight hung on a thread. With this step, the derivation of the
relation (0.1) for the period of oscillation of a pendulum is complete. The
example just presented (which is due to the French mathematician P.
Appell) is instructive. It would seem that we have succeeded in obtaining
an answer to an interesting problem from nothing – or, more precisely,
only from a list of the quantities on which the period of oscillation of the
pendulum is expected to depend, and a comparison (analysis) of their
dimensions.

The following example concerns the steady uniform motion of a body
in a gas at high velocity. To be specific, we shall discuss the simplest
such case: the motion of a sphere (Figure 0.2(a)). At high velocities, it
intuitively seems possible to neglect the internal friction in the gas (the
viscosity), since the resistance to the motion of the body is mainly due
to the inertia of the gas as it is pushed apart by the body. Therefore,
the drag force that the gas exerts against the motion of the sphere in
it depends on the static gas density $\rho$, the static gas pressure $p$, the
velocity with which the body moves, $U$, and the diameter of the sphere,$D$. Let us now determine the dimensions of density, force, and pressure,
since we already know the dimensions of the remaining quantities.

Density is by definition the ratio of a mass to the volume occupied by
that mass. Consequently, the density of a homogeneous body in which a
unit mass occupies a unit volume can be adopted as the unit for density.
In decreasing the unit of mass by a factor $M$ and the unit of length by a
factor $L$, we decrease the unit of density by a factor $ML^{-3}$. Therefore,
all the density values increase by this same factor. This means that
density has dimension $ML^{-3}$. 
Force is related to mass and acceleration via Newton's second law, force equals mass times acceleration. Now it must be true that the dimensions of both sides of any equation having physical sense must be identical. Otherwise, the equation would no longer hold under a change of fundamental units of measurement. Thus, the dimension of force must be identical to the dimension of the product of mass and acceleration. When the unit of mass is decreased by a factor $M$, the unit of length is decreased by a factor $L$, and the unit of time is decreased by a factor $T$, the value of the product of mass and acceleration increases by a factor of $MLT^{-2}$. Force therefore has dimension $MLT^{-2}$. Pressure is normal force per unit area; area has dimension $L^2$, so pressure has dimension $ML^{-1}T^{-2}$.

We shall now turn to the quantity $p/\rho$. From the above discussion, when the fundamental units of measurement are decreased by factors $M$, $L$, and $T$, respectively, the numerator of this quantity increases by a factor $ML^{-1}T^{-2}$, and the denominator increases by a factor $ML^{-3}$, so that the quantity $p/\rho$ increases by a factor $L^2T^{-2}$, i.e., as the velocity squared. The quantity $(p/\rho)^{1/2}$ therefore has the same dimension as velocity. Indeed, this is not surprising, since the quantity $c = (\gamma p/\rho)^{1/2}$, where $\gamma$ is a dimensionless constant that is a characteristic property of a given gas, is the speed of sound in the gas. Thus, it may be assumed that the drag force against the motion of the sphere, $f$, depends on the density of the gas $\rho$, the velocity of the sphere $U$, the diameter of the

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Such equations which hold only in one system of fundamental units do exist and may be very useful. For instance, my colleague Professor A. Yu. Ishlinsky proposed a formula for the time taken to drive a given distance in Moscow: the time (in minutes) is equal to the distance (in kilometres) plus the number of traffic lights. Of course, the formula

\[ \text{time} = \text{distance} + \text{number of traffic lights} \]

does not work in other units.

$\gamma$ is the ratio of the specific heat at constant pressure to the specific heat at constant volume ($\gamma = 1.4$ for air at room temperature).
sphere \( D \), and the speed of sound in the gas at rest, \( c \), which may be conveniently introduced in place of the static gas pressure:

\[
f = f(\rho, U, D, c). \tag{0.4}
\]

We now form the combination \( f/\rho U^2 D^2 \). The reader may easily verify that it is dimensionless. Equation (0.4) may obviously be rewritten in the form

\[
\Pi = \frac{f}{\rho U^2 D^2} = \Pi(\rho, U, D, \Pi_1), \tag{0.5}
\]

where \( \Pi_1 = U/c \) is also a dimensionless quantity – the ratio of the velocity of the body to the sound speed. This quantity is called the *Mach number*, in honour of the Austrian scientist who performed pioneering experiments with shock waves in a gas. We now decrease the unit of mass by an arbitrary factor \( M \), and leave all other units unchanged. The numerical value of the density then increases by a factor \( M \), while the numerical values of the dimensionless quantities \( U \) and \( D \) and the dimensionless quantities \( \Pi \) and \( \Pi_1 \) remain unchanged. But this means that the quantity \( \Pi \) is independent of the density \( \rho \). Further, upon changing the unit of time alone by an arbitrary factor, we find that the numerical value of the velocity \( U \) changes by the same arbitrary factor, while the quantities \( D \), \( \Pi \), and \( \Pi_1 \) remain unchanged. This means that \( \Pi \) is also independent of the velocity. Finally, upon changing the unit of length, we find that \( \Pi \) is independent of \( D \) as well, so that

\[
\Pi = \Pi(\Pi_1). \tag{0.6}
\]

However, further simplification is now impossible! The quantity \( \Pi_1 \) is dimensionless, and does not change when the fundamental units are changed. But even without further simplification, the result is impressive: restoring dimensional variables in (0.6), we find that

\[
f = f(\rho, U, D, c) = \rho U^2 D^2 \Pi \left(\frac{U}{c}\right); \tag{0.7}
\]

thus, the problem has been reduced to determining a function of one (!) variable instead of a function of four variables.

In order to complete the analysis, the function of one variable, \( \Pi(U/c) \), must be determined either experimentally or by calculation. For example, it can be obtained from experiments on a small model in a wind tunnel (Figure 0.3). A graph of the function \( \Pi(U/c) \) obtained in this way is shown in Figure 0.4.

The above examples illustrate a complete recipe for applying dimensional analysis. Outwardly, it appears very simple.
Figure 0.3. A nylon sphere moves in air at Mach number 7.6. A detached shock wave is visible ahead of the sphere. (From van Dyke (1982)).

\[ c_x = \frac{2}{\pi} \Pi \]

Figure 0.4. The dimensionless drag on a sphere, \( c_x = (2/\pi)\Pi \), as a function of the dimensionless governing parameter \( \Pi_1 = U/c \) (the Mach number) (Chernyi, 1961). The quantity \( \Pi \) approaches a constant for large values of \( \Pi_1 \).
The parameters on which the quantity to be determined depends are ascertained;

Those parameters whose dimensions are independent (so that their numerical values can change independently of one another when the fundamental units of measurement are changed) are selected; and then

The relations being studied are transformed into relations between dimensionless quantities.

The advantage of using dimensional analysis is that the number of dimensionless quantities is smaller than the total number of dimensional quantities between which we are searching for a relationship. Once again, we find that the difference between the total number of dimensional parameters and the number of dimensionless parameters is equal to the number of dimensional parameters with independent dimensions.

0.2 Assumptions underlying dimensional analysis

However, the apparent simplicity of the above procedure is illusory. In fact, it is most effective when the problem is in the end reduced to determining a constant or a function of one dimensionless variable. This is why it is important to restrict oneself to the minimum necessary number of parameters when finding out on which of them the quantity to be determined depends. Moreover, none of the essential parameters may be left out. How should we proceed here, especially in those cases where we do not have a mathematical formulation for the problem for instance, in turbulence?

We shall now illustrate the real conceptual difficulties that arise in doing this, using the following apparently very similar example. Namely, consider the steady uniform motion of a thin plate in a fluid (Figure 0.2(b)). In this case the effects of compressibility can be neglected for small enough values of the Mach number whereas viscosity effects are essential. Then the drag force \( f \) (per unit width) will depend on the velocity \( U \) with which the plate moves, the plate length \( l \), and the fluid’s properties, its density \( \rho \) and its kinematic viscosity \( \nu \); so

\[
  f = f(\rho, U, l, \nu).
\]

Repeating the discussion from the preceding example, this relation can also be reduced to the form (0.6), where now

\[
  \Pi = \frac{f}{\rho U^2 l}, \quad \Pi_1 = \frac{U l}{\nu}.
\]
Thus, the problem has again been reduced to one of determining the function of one variable $\Pi(\Pi_1)$. If the dependence on kinematic viscosity were not essential, the problem would be reduced to one of determining a single constant, as in the first example: a clear gain. How can one determine that a certain governing parameter is not essential? The reasoning very often (it could even be said, usually) goes like this. If the dimensionless parameter $\Pi_1$ corresponding to some dimensional parameter is either very small or very large compared to unity, it may be assumed to be not essential, and the function $\Pi(\Pi_1)$ can be assumed to be a constant (or, in general, when there are several dimensionless parameters $\Pi_1, \Pi_2, \ldots$, a function of one fewer arguments).

A very strong assumption, which is usually not mentioned, and which is satisfied in some cases and not in others, is in fact being made when reasoning along these lines. Namely, the function $\Pi(\Pi_1)$ is assumed to approach a finite, non-zero limit at either large or small $\Pi_1$ (i.e., as $\Pi_1$ goes to either zero or infinity): $\Pi(0) = C$ or $\Pi(\infty) = C$. If it is in fact the case and the parameter $\Pi_1$ is sufficiently large or sufficiently small, the equation $\Pi = \Pi(\Pi_1)$ can, to the required accuracy, be written as a simpler relation similar to (0.2):

$$\Pi = C.$$  \hspace{1cm} (0.10)

This is precisely the situation that obtains in the motion of a sphere at high velocity (see Figure 0.4): to sufficient accuracy, the function $\Pi(U/c)$ is constant for ratios $U/c$ greater than four, and so for large velocities $f = C \rho \ U^2 \ D^2$.

However, it is obvious that this is far from always the case. If, for example, for a certain process

$$\Pi(\Pi_1) = \ln \Pi_1 \quad \text{or} \quad \Pi(\Pi_1) = \sin \Pi_1$$  \hspace{1cm} (0.11)

(which obviously cannot be excluded in a general discussion), it is not permissible to replace the function by a constant, no matter how large or small the parameter $\Pi_1$ is. That is, no matter how large or small the parameter $\Pi_1$, the assumption that a particular dimensional parameter may be neglected is a strong hypothesis that must be supported by experiment, numerical calculation, or (at least) the intuition of the investigator. However, since dimensional analysis is normally used only when we cannot obtain a more complete solution to the problem, this means that we can rarely answer in advance the subtle question of whether the function $\Pi(\Pi_1)$ has a non-zero limit as $\Pi_1$ goes to zero or infinity. Moreover, there is yet another, rather insidious situation that may arise here. It is illustrated by the example of plate motion considered above.
Let us make the same assumption of a finite non-zero limit at large $\Pi_1$, i.e. at large velocities or small viscosities. We then obtain a limiting relation in the form (0.10), i.e.

$$f = \text{const} \, \rho \, U^2 l.$$  
(0.12)

In fact, however (see chapter 7), the function $\Pi(\Pi_1)$ in this case approaches zero at large $\Pi_1$, according to

$$\Pi = \text{const}/\sqrt{\Pi_1},$$  
(0.13)

so, for large $\Pi_1$, the relation (0.8) can be represented in the form

$$\Pi^* = C,$$  
where $\Pi^* = f/\rho U^{3/2}(l\nu)^{1/2}$  
(0.14)

whence

$$f = \text{const} \, \rho \, U^{3/2}(l\nu)^{1/2}.$$  
(0.15)

So, we have obtained scaling laws identical in form to (0.2) and (0.10). However, although the relation for $\Pi^*$ has in principle the same monomial form as $\Pi$, it differs from $\Pi$ in two important respects. First, the powers in (0.14) cannot be determined from simple considerations, that is, an analysis of the dimensions of the quantities in the problem. Second, in contrast with the sound velocity $c$ in the first example, the parameter $\nu$ remains in (0.14) and (0.15). Thus the simplification that has occurred here is no longer due to dimensional analysis alone but to a special property of the problem being studied: the existence of the power-law representation (0.13) of the function $\Pi$ for large $\Pi_1$.

We shall now give one more example (this time, geometric) of this type of situation. Consider two continuous curves. One of them (Figure 0.5) is a normal circle. We inscribe a regular $n$-gon with side length $\eta$ in it. The perimeter of this inscribed polygon, $L_\eta$ obviously depends only on the diameter of the circle $d$ and side length $\eta$:

$$L_\eta = f(d, \eta).$$  
(0.16)

Proceeding in much the same way as in the previous examples, we transform this relation, using dimensional analysis, to the form (0.6),

$$\Pi = \Pi(\Pi_1),$$  
(0.17)

where this time $\Pi = L_\eta/d$, $\Pi_1 = \eta/d$, whence

$$L_\eta = d\Pi(\eta/d).$$

Let the number of sides of the polygon, $n$, approach infinity, i.e., let the side length $\eta$ approach zero. From elementary geometry, it is known that the perimeter of the inscribed polygon approaches the finite limit $L_0 = \pi d$ (which is, in fact, adopted as the circumference of a circle). Thus, as $\eta/d \to 0$, the function $\Pi(\eta/d)$ approaches a finite limit equal
Figure 0.5. A circle with inscribed regular polygons. As the number of sides in the polygon approaches infinity, and the side length approaches zero, the perimeter of the polygon approaches a finite limit.

Figure 0.6. A fractal curve – the Koch triad. (a) The original triangle, (b) the elementary operation, and (c) the broken line that approximates the fractal curve for a large number of sides. As the number of sides increases, the perimeter of the broken line approaches infinity according to a power law.

to $\pi$. Therefore, for sufficiently small $\eta/d$, it is possible to neglect the influence of the parameter $\eta$ and to assume that the following relation is satisfied to the required accuracy for polygons with a large number of sides:

$$\Pi = \text{const} = \pi,$$

(0.18)
i.e., $L_\eta = \pi d$. 
The second curve is obtained in the following way (Figure 0.6). An equilateral triangle of side $d$ is taken, and each of its three sides is subjected to the following elementary operation: the side is divided into three sections, and the middle section is replaced by two sides of an equilateral triangle constructed using it as a base. The sides of the polygon obtained are once again subjected to the same elementary operation, and so on to infinity. Obviously, the side length of this polygon at the $n$th stage, $\eta$, is equal to $d/3^n$, and the perimeter of the entire polygon, $L_\eta$, is equal to $3d(4/3)^n$. Equations (0.16) and (0.17) clearly also hold in this case. However, it can easily be shown that (since it is obvious that $n = \log(d/\eta)/\log 3$)

$$L_\eta = 3d[10^n(\log 4 - \log 3)] = 3d[10^\alpha \log(d/\eta)] = 3d(d/\eta)^\alpha,$$  

(0.19)

where

$$\alpha = (\log 4 - \log 3)/\log 3 \approx 0.26\ldots.$$  

Comparing (0.19) and (0.17), we find that

$$\Pi(\eta/d) = 3(\eta/d)^{-\alpha}$$  

(0.20)

(i.e., the length of the curve $L_0$ is infinite in this case, so that only the empty relation $\Pi = \infty$ is obtained in going to the limit $\eta/d \to 0$). Thus, if one is interested in the perimeter of the polygon for large $n$, it is not possible to pass to the limit and use a limiting relationship such as (0.18). At the same time, equation (0.19) can be rewritten in the form $\Pi^* = C$, setting

$$\Pi^* = \frac{L_\eta}{d^{1+\alpha} \eta^{-\alpha}}, \quad C = 3.$$  

(0.21)

The parameter $\Pi^*$ is (like $\Pi$) a power-law combination of the parameters that determine it. However, the structure of (0.21) is not determined by dimensional considerations alone; we do not know the number $\alpha$ beforehand as we did in the case of a circle. Furthermore, unlike the case of a circle (equation (0.18)), the parameter $\eta$ remains in the resulting equation no matter how small $\eta/d$ is. Therefore, the length of the inscribed broken line, $L_\eta = 3d^{1+\alpha}/\eta^\alpha$, turns out to be proportional to

\footnote{Curves and, in general, geometric objects of this type are called fractals, as suggested by Mandelbrot (1975). They were intensively studied by mathematicians at the beginning of this century. Mandelbrot in his papers, and especially in his illuminating collections of essays (Mandelbrot, 1975, 1977, 1982) has revived interest in such geometric objects by showing that they provide adequate descriptions of important objects in nature. The curve shown in Figure 0.6 was constructed by von Koch (1904) and is called the Koch triad. We will discuss such objects and the whole concept at some length in chapter 12.}
$d^{1+\alpha}$ rather than $d$, and the length of a segment of this broken line, $\eta$ remains in the constant of proportionality.

The discussion presented above shows that, for a given quantity of interest, correctly choosing the parameters on which it depends and correctly evaluating the nature of this dependence are what really come first, rather than the formal procedure behind dimensional analysis. We should not assume that too many of the parameters are essential, since dimensional analysis then becomes ineffective. However, only with extreme caution should one discard particular parameters as nonessential merely because corresponding dimensionless parameters are either large or small.

Dimensional analysis and the general concepts of dynamical similarity are presented in chapter 1. Our exposition is essentially different from those available in the literature, although it follows in its general ideas the excellent book of P.W. Bridgman (1931), undeservedly forgotten in recent years.

0.3 Self-similar phenomena

A time-developing phenomenon is called self-similar if the spatial distributions of its properties at various different moments of time can be obtained from one another by a similarity transformation.† Establishing self-similarity has always represented progress for a researcher: self-similarity has simplified computations and the representation of the properties of phenomena under investigation. In handling experimental data, self-similarity has reduced what would seem to be a random cloud of empirical points so as to lie on a single curve or surface, constructed using self-similar variables chosen in some special way. The self-similarity of the solutions of partial differential equations has allowed their reduction to ordinary differential equations, which often simplifies the investigation. Therefore, with the help of self-similar solutions researchers have attempted to envisage the characteristic properties of new phenomena. Self-similar solutions have also served as standards in evaluating approximate methods for solving more complicated problems.

The appearance of computers changed the general attitude toward self-similar solutions but did not decrease the interest in them. Previously it had been considered that the reduction of partial to ordinary

† The fact that we identify one of the independent variables with time is of no significance.
differential equations simplified matters, and hence self-similar solutions had attracted attention, first of all, because of the simplicity of obtaining and analyzing them. Gradually, however, the situation grew more complicated, and in many cases it turned out that the simplest method of numerically solving boundary-value problems for the systems of ordinary equations that resulted from the construction of self-similar solutions was computation by the method of stabilization of the solutions of the original partial differential equations. Nevertheless, self-similarity continued as before to attract attention as a profound physical fact indicating the presence of a certain type of stabilization of the processes under investigation, valid for a rather wide range of conditions. Moreover, self-similar solutions were used as a first step in starting numerical calculations on computers. For all these reasons the search for self-similarity was undertaken at the outset, as soon as a new domain of investigation was opened up.

Instructive examples of self-similarities are given by several highly idealized problems in the mathematical theory of filtration – slow ground-water motion in porous media.

Suppose that, in a porous stratum over an underlying horizontal impermeable bed, at an initial instant \( t = 0 \) a finite volume of water \( V \) is supplied instantaneously by a well of very small, let us say infinitesimally small, radius (Figure 0.7). Then at time \( t \) the local height \( h \) of the ground water mound formed in such a way will be given (Barenblatt, 1952, see the details below in chapter 12) by

\[
h = \frac{Q^{1/2}}{(\kappa t)^{1/2}} \left[ 8 - \frac{r^2}{(Q\kappa t)^{1/2}} \right] \quad (0.22)
\]

for \( r < r_f = \sqrt[4]{8(Q\kappa t)^{1/4}} \), and by \( h = 0 \) for \( r \geq r_f \). Here

\[
\kappa = \frac{k \rho g}{2m \mu}, \quad Q = \frac{V}{2\pi m} \quad (0.23)
\]

where \( k, m \) are the permeability and porosity (the relative volume occupied by the pores) of the stratum which are statistical geometric properties of the porous medium; \( \rho \) and \( \mu \) are the water density and dynamic viscosity, \( g \) is the gravitational acceleration, and \( r \) is the distance from the well of the point at which the observation is made.

The form of the relation (0.22) is instructive: there exist a mound height scale \( h_0(t) \) and a linear scale \( r_0(t) \), both depending on time,

\[
h_0(t) = \frac{Q^{1/2}}{(\kappa t)^{1/2}}, \quad r_0 = (Q\kappa t)^{1/4} \quad (0.24)
\]

such that the spatial distribution of the ground-water mound height,