

MATH 1020 WORKSHEET 10.1
Parametric Equations

If f and g are continuous functions on an interval I , then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations. The parametric equations along with the graph of (x,y) for all t in the interval make up the plane curve.

Sketch the curve represented by the parametric equations $x = 3t - 1$, $y = 2t + 1$ making sure to indicate the direction of motion. Write the corresponding rectangular equation by eliminating the parameter.

Solution. Note that a sketch is not provided, but direction of motion is indicated in the description below. Solving the x equation for t one has that $t = \frac{x+1}{3}$. Substituting this into the y equation the resulting rectangular equation is

$$y = \frac{2}{3}(x+1) + 1 = \frac{2}{3}x + \frac{5}{3}.$$

This is the equation of a line with slope $\frac{2}{3}$ with y -intercept of $\frac{5}{3}$ and x -intercept of $-\frac{5}{2}$. There are no restrictions given on t and there are no natural restrictions on t imposed by $f(t)$ or $g(t)$ so the sketch would include the entire line. The direction of motion in the rectangular plane is from bottom left to top right. This was determined by noting that a particle moves along the line from the point $(-1, 1)$ when $t = 0$ to the point $(2, 3)$ when $t = 1$.

Eliminate the parameter and write the corresponding rectangular equation for $x = 4 + 2 \cos \theta$, $y = -1 + 4 \sin \theta$.

Solution. To solve this type of parametric equations problem, one needs to solve for $\cos \theta$ and $\sin \theta$ in terms of x and y respectively and then substitute into the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$.

$$\begin{aligned} x &= 4 + 2 \cos \theta & y &= -1 + 4 \sin \theta \\ \frac{x-4}{2} &= \cos \theta & \frac{y+1}{4} &= \sin \theta \end{aligned}$$

Substituting into the trigonometric identity results in

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= 1 \\ \left(\frac{x-4}{2}\right)^2 + \left(\frac{y+1}{4}\right)^2 &= 1 \\ \frac{(x-4)^2}{4} + \frac{(y+1)^2}{16} &= 1 \end{aligned}$$

This equation is the equation of an ellipse with center at $(4, -1)$ and vertices at $(2, -1)$, $(6, -1)$, $(4, -5)$ and $(4, 3)$. Since there is no restriction given on θ and none

exists from the definitions of $f(\theta)$ and $g(\theta)$, the entire ellipse is traced out.

The last step in our solution is to determine the direction of motion the curve is traced out. The chart on the right helps us to determine that the direction of motion is counterclockwise.

θ	x	y
0	6	-1
$\frac{\pi}{2}$	4	3
π	2	-1

Find two different sets of parametric equations for the line $y = 3x - 2$.

Solution. The simplest set of parametric equations for $y = 3x - 2$ is to set $x = t$ and then substitute that into the equation of the line to determine the resulting y equation. This gives

$$x = t \quad y = 3t - 2$$

A second set of parametric equations can be formed by using some other function of t for x . However, care must be used to not introduce a restriction on the t -interval that would result in part of the line not being represented by the new set of parametric equations. One such second set of parametric equations that would work is

$$x = 5t \quad y = 15t - 2$$

Another is the following

$$x = t + 3 \quad y = 3t + 7$$

One that only gives you half of the line and therefore is not correct is

$$x = t^2 \quad y = 3t^2 - 2$$

MATH 1020 WORKSHEET 10.2
Calculus with Parametric Curves

This sections included formulas for slope and concavity of parametric curves and for arc length when given parametric curves.

Find dy/dx and d^2y/dx^2 for the parametric equations $x = \sqrt{t}$, $y = \sqrt{t-1}$ and evaluate each at $t = 2$.

Solution. For both the slope and concavity, one first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \quad \frac{dy}{dt} = \frac{1}{2\sqrt{t-1}}$$

Next one applies the formula for the slope in parametric form

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\frac{1}{2\sqrt{t-1}}}{\frac{1}{2\sqrt{t}}} \\ &= \frac{\sqrt{t}}{\sqrt{t-1}} \\ &= \sqrt{\frac{t}{t-1}} \end{aligned}$$

Thus

$$\boxed{\left. \frac{dy}{dx} \right|_{t=2} = \sqrt{2}}.$$

To determine the concavity, we apply to parametric formula

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt} \left(\sqrt{\frac{t}{t-1}} \right)}{\frac{dx}{dt}} \\ &= \frac{\frac{1}{2} \left(\frac{t}{t-1} \right)^{-1/2} \left(\frac{(t-1)-t}{(t-1)^2} \right)}{\frac{1}{2\sqrt{t}}} \\ &= 2\sqrt{t} \cdot \left(\frac{1}{2} \left(\frac{t}{t-1} \right)^{-1/2} \left(\frac{-1}{(t-1)^2} \right) \right) \end{aligned}$$

Evaluating at $t = 2$ one gets

$$\begin{aligned} \left. \frac{d^2y}{dx^2} \right|_{t=2} &= 2\sqrt{2} \cdot \left(\frac{1}{2} \left(\frac{2}{1} \right)^{-1/2} \left(\frac{-1}{1^2} \right) \right) \\ &= 2\sqrt{2} \cdot \left(\frac{1}{2} \frac{1}{\sqrt{2}} (-1) \right) \\ &= -1 \end{aligned}$$

Thus

$$\boxed{\left. \frac{d^2y}{dx^2} \right|_{t=2} = -1}.$$

Find the arc length of the curve given by $x = t$, $y = \frac{t^5}{10} + \frac{1}{6t^3}$ on the interval $1 \leq t \leq 2$.

Solution. One first needs to determine dx/dt and dy/dt before proceeding.

$$\frac{dx}{dt} = 1 \quad \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$$

Next one applies the formula for the arc length in parametric form

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2} dt \\ &= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - 2 \cdot \frac{t^4}{2} \cdot \frac{1}{2t^4} + \frac{1}{4t^8}\right)} dt \\ &= \int_1^2 \sqrt{1 + \left(\frac{t^8}{4} - \frac{1}{2} + \frac{1}{4t^8}\right)} dt \\ &= \int_1^2 \sqrt{\left(\frac{t^8}{4} + \frac{1}{2} + \frac{1}{4t^8}\right)} dt \\ &= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2} dt \\ &= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right) dt \\ &= \left(\frac{t^5}{10} - \frac{1}{6t^3}\right) \Big|_1^2 \\ &= \left(\frac{32}{10} - \frac{1}{10}\right) - \left(\frac{1}{48} - \frac{1}{6}\right) \\ &= \frac{31}{10} - \frac{-7}{8} \\ &= \frac{744 + 35}{240} \end{aligned}$$

Thus

$$\boxed{\int_1^2 \sqrt{(1)^2 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2} dt = \frac{779}{24}}.$$

MATH 1020 WORKSHEET 10.3
Polar Coordinates and Polar Graphs

Polar coordinates require the basic transformation equations

$$\begin{array}{ll} x = r \cos \theta & y = r \sin \theta \\ r^2 = x^2 + y^2 & \tan \theta = y/x \end{array}$$

Given the polar point $(-1, 5\pi/4)$, find the corresponding Cartesian coordinates for the point.

Solution. We apply the transformation equations to determine the x and y coordinates of the point.

$$\begin{array}{ll} x = -1 \cdot \cos\left(\frac{5\pi}{4}\right) & y = -1 \cdot \sin\left(\frac{5\pi}{4}\right) \\ x = -1 \left(\frac{-1}{\sqrt{2}}\right) & y = -1 \left(\frac{-1}{\sqrt{2}}\right) \\ x = \frac{1}{\sqrt{2}} & y = \frac{1}{\sqrt{2}} \end{array}$$

Thus, writing the answer as a point we have

$$\boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}.$$

Convert the polar equation $r = 4 \sin \theta$ to a Cartesian form and sketch its graph.

Solution. Transforming to a polar equation, it is best to start by replacing any trigonometric functions in the equation before regrouping and then replacing any r terms.

$$\begin{aligned} r &= 4 \sin \theta \\ &= 4 \left(\frac{y}{r}\right) \\ r^2 &= 4y \\ x^2 + y^2 &= 4y \end{aligned}$$

From here, the easiest way to sketch the equation is to bring the $4y$ over to the LHS and complete the square so that the Cartesian equation is in a familiar form.

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + (y^2 - 4y + _) &= 0 + _ \\ x^2 + (y^2 - 4y + \underline{4}) &= 0 + \underline{4} \\ x^2 + (y - 2)^2 &= 4 \end{aligned}$$

One recognizes that we have the equation of a circle. Thus

$$\boxed{x^2 + y^2 = 4y \text{ is a circle with center at } (0, 2) \text{ and radius } r = 2.}$$

Convert the Cartesian equation $xy = 4$ to a polar form.

Solution. Converting a Cartesian equation into polar form is done by substituting the transformation equations for $x = r \cos \theta$ and $y = r \sin \theta$ into the given equation.

$$\boxed{r^2 \cos \theta \sin \theta = 4.}$$

Find the slope of the tangent line to the polar curve $r = 2 - \sin \theta$ at $\theta = \frac{\pi}{3}$.

Solution. To determine slope of the tangent line to the polar curve, we must first convert the polar equations into parametric form using the transformation equations. For parametric form we have

$$\begin{array}{ll} x = r \cos \theta & y = r \sin \theta \\ x = (2 - \sin \theta) \cos \theta & y = (2 - \sin \theta) \sin \theta \\ x = 2 \cos \theta - \cos \theta \sin \theta & y = 2 \sin \theta - \sin^2 \theta \end{array}$$

Now we need to determine $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$.

$$\frac{dx}{d\theta} = -2 \sin \theta - \cos^2 \theta + \sin^2 \theta \quad \frac{dy}{d\theta} = 2 \cos \theta - 2 \sin \theta \cos \theta$$

Substituting into the parametric formula for slope we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{2 \cos \theta - 2 \sin \theta \cos \theta}{-2 \sin \theta - \cos^2 \theta + \sin^2 \theta} \end{aligned}$$

Now we can evaluate at $\theta = \pi/3$

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\pi/3} &= \left. \frac{2 \cos \theta - 2 \sin \theta \cos \theta}{-2 \sin \theta - \cos^2 \theta + \sin^2 \theta} \right|_{\theta=\pi/3} \\ &= \frac{2 \cos(\pi/3) - 2 \sin(\pi/3) \cos(\pi/3)}{-2 \sin(\pi/3) - \cos^2(\pi/3) + \sin^2(\pi/3)} \\ &= \frac{2 \left(\frac{1}{2}\right) - 2 \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right)}{-2 \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{1 - \frac{\sqrt{3}}{2}}{-\sqrt{3} - \frac{1}{4} + \frac{3}{4}} \\ &= \frac{1 - \frac{\sqrt{3}}{2}}{-\sqrt{3} + \frac{1}{2}} \end{aligned}$$

Thus the slope of the tangent line to the polar curve $r = 2 - \sin \theta$ at $\theta = \frac{\pi}{3}$ is

$$\boxed{\left. \frac{dy}{dx} \right|_{\theta=\pi/3} = \frac{2 - \sqrt{3}}{-2\sqrt{3} + 1}.}$$

MATH 1020 WORKSHEET 10.4
Area and Arc Length in Polar Coordinates

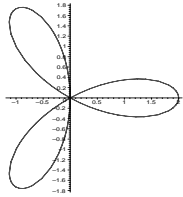
Area in polar coordinates is found with the formula $A = 1/2 \int_a^b r^2 d\theta$

Arc Length has the formula $s = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta$

Graphs of polar curves will be given to you on quizzes and exams

Find the area of one petal of the region $r = 2 \cos(3\theta)$.

Solution. One must first determine values of θ where $r = 0$.



Setting $r = 0$ we find that

$$0 = 2 \cos(3\theta)$$

$$= \cos(3\theta)$$

$$\text{Thus } 3\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}$$

Thus we have that the petal that lies on the x -axis is traced out on the interval $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Note that one can also use symmetry and find the area by calculating $2 \times$ the area on the interval $0 \leq \theta \leq \frac{\pi}{6}$. Our area integral becomes

$$\begin{aligned} \frac{1}{2} \int_{-\pi/6}^{\pi/6} (2 \cos(3\theta))^2 d\theta &= 2 \int_{-\pi/6}^{\pi/6} \cos^2(3\theta) d\theta \\ &= 2 \int_{-\pi/6}^{\pi/6} \frac{1}{2} (1 + \cos(6\theta)) d\theta \\ &= \int_{-\pi/6}^{\pi/6} (1 + \cos(6\theta)) d\theta \\ &= \left(\theta + \frac{\sin(6\theta)}{6} \right) \Big|_{-\pi/6}^{\pi/6} \\ &= \frac{\pi}{6} - \frac{-\pi}{6} + \frac{\sin(6(\pi/6))}{6} - \frac{\sin(6(-\pi/6))}{6} \\ &= \frac{2\pi}{6} + 0 - 0 = \underline{\underline{\frac{\pi}{3}}} \end{aligned}$$

Find the θ values for the points of intersection for the graphs of the equations $r = 1 + \cos\theta$ and $r = 3 \cos\theta$.

Solution.

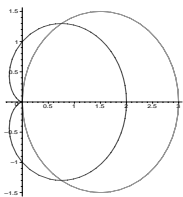
Setting the two equations equal one finds

$$3 \cos\theta = 1 + \cos\theta$$

$$2 \cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

$$\text{Thus } \theta = \underline{\underline{\frac{\pi}{3}}} \quad \text{and} \quad \theta = \underline{\underline{-\frac{\pi}{3}}}$$



Using the results from the previous problem, find the area inside $r = 3 \cos \theta$ and outside $r = 1 + \cos \theta$.

Solution. The area enclosed between the two curves on the interval $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$ can be found by calculating $2 \times$ the area between the two curves on the interval $0 \leq \theta \leq \frac{\pi}{3}$. The calculation follows

$$\begin{aligned}
 2 \cdot \left[\frac{1}{2} \int_0^{\frac{\pi}{3}} (3 \cos \theta)^2 d\theta - \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 + \cos \theta)^2 d\theta \right] &= 2 \cdot \left(\frac{1}{2} \right) \left[\int_0^{\frac{\pi}{3}} 9 \cos^2 \theta d\theta - \int_0^{\frac{\pi}{3}} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \right] \\
 &= \int_0^{\frac{\pi}{3}} (9 \cos^2 \theta - 1 - 2 \cos \theta - \cos^2 \theta) d\theta \\
 &= \int_0^{\frac{\pi}{3}} (8 \cos^2 \theta - 1 - 2 \cos \theta) d\theta \\
 &= \int_0^{\frac{\pi}{3}} 8 \frac{1}{2} (1 + \cos(2\theta)) - 1 - 2 \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{3}} 4 + 4 \cos(2\theta) - 1 - 2 \cos \theta d\theta \\
 &= \int_0^{\frac{\pi}{3}} 3 + 4 \cos(2\theta) - 2 \cos \theta d\theta \\
 &= (3\theta + 2 \sin(2\theta) - 2 \sin \theta) \Big|_0^{\frac{\pi}{3}} \\
 &= 3 \left(\frac{\pi}{3} \right) + 2 \sin \frac{2\pi}{3} - 2 \sin \frac{\pi}{3} - 3 \cdot 0 + 2 \cdot 0 - 2 \cdot 0 \\
 &= \pi + 2 \frac{\sqrt{3}}{2} - 2 \frac{\sqrt{3}}{2} = \underline{\underline{\pi}}
 \end{aligned}$$

Find the length of $r = 2a \cos \theta$ on the interval $-\pi/2 \leq \theta \leq \pi/2$.

Solution. First we find $\frac{dr}{d\theta}$

$$\frac{dr}{d\theta} = -2a \sin \theta$$

We can now use this result in the polar arc length formula.

$$\begin{aligned}
 L &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(2a \cos \theta)^2 + (-2a \sin \theta)^2} d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2 \cos^2 \theta + 4a^2 \sin^2 \theta} d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2 (\cos^2 \theta + \sin^2 \theta)} d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4a^2 (1)} d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2a d\theta \\
 &= 2a\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
 &= 2a \left(\frac{\pi}{2} - \frac{-\pi}{2} \right) = 2a \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \underline{\underline{2a\pi}}.
 \end{aligned}$$

MATH 1020 WORKSHEET 10.5 & App C
Conic Sections & Graphing

The conic sections are circles, parabolas, ellipses, and hyperbolas. Standard Equations for parabolas, ellipses and hyperbolas will be given to you on a quiz or exam if needed.

Find the vertex, focus, and directrix of the parabola and sketch its graph.

$$y^2 - 4y - 4x = 0$$

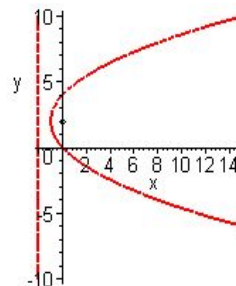
Solution. This is most easily done if the equation for the parabola is put in standard form.

$$\begin{aligned} y^2 - 4y &= 4x \\ (y^2 - 4y + \underline{\quad}) &= 4x + \underline{\quad} \\ (y^2 - 4y + \underline{4}) &= 4x + \underline{4} \\ (y - 2)^2 &= 4(x + 1) \end{aligned}$$

Solving for p : $4 = 4p$ so $p = 1$

Thus we have

vertex :	$(-1, 2)$
focus :	$(0, 2)$
directrix :	$x = -2$

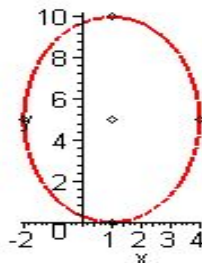


Find the center, foci and vertices of the ellipse and sketch the its graph.

$$\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1$$

Solution. From the equation, we can read off the following information

Center :	$(1, 5)$
vertices :	$(4, 5), (-2, 5)$
	$(1, 10), (1, 0)$



Write the equation in standard form **and** identify the conic section.

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

Solution. To write the equation in standard form we must complete the square in both x and y .

$$\begin{aligned}9x^2 - 36x + 4y^2 + 24y &= -36 \\9(x^2 - 4x + \underline{\quad}) + 4(y^2 + 6y + \underline{\quad}) &= -36 + 9(\underline{\quad}) + 4(\underline{\quad}) \\9(x^2 - 4x + \underline{4}) + 4(y^2 + 6y + \underline{9}) &= -36 + 9(\underline{4}) + 4(\underline{9}) \\9(x - 2)^2 + 4(y + 3)^2 &= 36\end{aligned}$$

Thus

$$\boxed{\frac{(x - 2)^2}{4} + \frac{(y + 3)^2}{9} = 1 \text{ is an ellipse.}}$$

Write the equation in standard form **and** identify the conic section.

$$6y^2 + x - 36y + 55 = 0$$

Solution. To write the equation in standard form we must complete the square in y .

$$\begin{aligned}6y^2 - 36y &= -x - 55 \\6(y^2 - 6y + \underline{\quad}) &= -x - 55 + 6(\underline{\quad}) \\6(y^2 - 6y + \underline{9}) &= -x - 55 + 6(\underline{9}) \\6(y - 3)^2 &= -x - 1\end{aligned}$$

Thus

$$\boxed{(y - 3)^2 = \frac{-1}{6}(x + 1) \text{ is a parabola.}}$$