A

Name:

MATH2800 Introduction to Discrete Structures

Final Exam, Wednesday, December 10, 2008.

You may use three sheets of handwritten notes, but no other sources. The exam lasts three hours. The exam consists of twelve questions. Answer **at least two of questions 10, 11, and 12**; your score will be the total of your **ten best** questions. Each question is worth ten points. You can work all twelve problems. You need to work at least ten problems, including at least two of problems 10, 11, and 12, to score over 90% on the exam. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am   2: Thursday 9am   3: Monday 2pm   4: Thursday 2pm

Q1		
Q2		
Q3		
Q4		
Q5		
Q6		
Q7		
Q8		
Q9		
Q10		
Q11		
Q12		
Total	100	

1. Solve the congruence $11x \equiv 1 \pmod{24}$.

$24 = 11 \times 2 + 2$

$11 = 5 \times 2 + 1$

so $1 = 11 - 5 \times 2 = 11 - 5 \times (24 - 11 \times 2) = -5 \times 24 + 11 \times 11$

so $11 \times 11 \equiv 1 \pmod{24}$.

Solns $x = 11 + 24k, \quad k = 0, \pm 1, \pm 2, \ldots$

2. A bag contains pink, yellow, and green balls. To be sure of withdrawing at least 2 pink balls you must take at least 12 balls. To be sure of withdrawing at least 2 yellow balls you must take at least 8 balls. To be sure of withdrawing at least 2 green balls you must take at least 14 balls. How many balls of each color are in the bag?

Let n = # balls in bag

Let n_p, n_y, n_g = # pink, yellow, green balls respectively.

By pigeonhole principle,

$$n_y + n_g = 10, \quad (1) \qquad n_p + n_g = 6, \quad (2)$$

$$n_p + n_y = 12 \quad (3)$$

$$(3) - (1) \Rightarrow n_p - n_g = 2 \quad (4)$$

$$(2) + (4) \Rightarrow 2n_p = 8$$

$$\Rightarrow \boxed{n_p = 4}, \xrightarrow{(2)} \boxed{n_g = 2} \xrightarrow{(1)} \boxed{n_y = 8}$$

3. Can there be a simple graph with n vertices all of different degrees? Explain.

Since the graph is simple, there are no loops or parallel edges.

Thus, the maximum degree of any vertex is $n-1$.

So the ~~p~~ degrees must be $0, 1, 2, \ldots, n-2$, and $n-1$.

But the vertex of degree 0 is not adjacent to the vertex of degree $n-1$,

So the vertex of degree $n-1$ is only adjacent to $n-2$ other vertices. Contradiction.

So, <u>no</u> there cannot be such a graph.

4. Dora has a coin that she suspects with probability $\frac{1}{5}$ might be biased. If the coin is biased it has a probability of $\frac{2}{3}$ of returning a head and only $\frac{1}{3}$ of returning a tail. Dora tosses the coin four times, obtaining three heads and one tail. How should Dora use this additional information to adjust her prior belief that the coin is biased with probability $\frac{1}{5}$?

Bayes Theorem:

~~$P(\text{head}) =$~~

Want $P(\text{coin is biased})$.

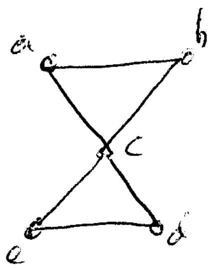
$$P\left(\text{coin is biased} \,\middle|\, \begin{array}{c}\text{three heads}\\ \text{out of 4}\end{array}\right) = \frac{P\left(\begin{array}{c}\text{3 heads}\\ \text{out of 4}\end{array} \,\middle|\, \text{coin is biased}\right) P\left(\begin{array}{c}\text{(prior)}\\ \text{coin is biased}\end{array}\right)}{P\left(\begin{array}{c}\text{3 heads out}\\ \text{of 4}\end{array} \,\middle|\, \begin{array}{c}\text{coin is}\\ \text{biased}\end{array}\right) P\left(\begin{array}{c}\text{coin is}\\ \text{biased}\end{array}\right) + P\left(\begin{array}{c}\text{3 heads}\\ \text{out of 4}\end{array} \,\middle|\, \begin{array}{c}\text{coin is}\\ \text{fair}\end{array}\right) P\left(\begin{array}{c}\text{coin is}\\ \text{fair}\end{array}\right)}$$

$$= \frac{4 \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right) \times \left(\frac{1}{5}\right)}{4 \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right) \times \frac{1}{5} + 4 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right) \times \frac{4}{5}}$$

$$= \frac{8/81}{8/81 + 1/4} = \frac{32}{32+81}$$

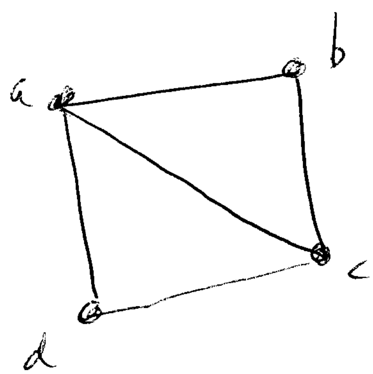
$$= \boxed{\frac{32}{113}}$$

5. Give an example of a graph that has an Euler circuit but not a Hamiltonian circuit, and give an example of a graph that has a Hamiltonian circuit but not an Euler circuit. Justify your answer.



Euler circuit (all vertices have even degree)

No Hamiltonian circuit:
need to include both edges adjacent
to each of a, b, e, d.
But then have 4 edges adjacent to c.



Hamiltonian circuit: a - b - c - d - a

No Euler circuit: a and c have
degree 3.

6. Let R be a symmetric and transitive relation on a set A. Assume for every $a \in A$ there exists $b \in A$ with aRb. Prove that R is an equivalence relation.

Need to show R is reflexive.

Given $a \in A$, $\exists b$ with aRb.

R is symmetric, so bRa also.

R is transitive and aRb and bRa so aRa.

Thus R is reflexive.

7. Jim is a minor league pitcher. His starts can be divided into four categories:

   (a) He gives up no runs.

   (b) He gives up one or two runs.

   (c) He gives up three or four runs.

   (d) He gives up at least five runs.

   Jim makes ten starts during the season. In how many ways can the number of runs be distributed? (Assume the order of the starts is irrelevant. Express your answer in factorials and/or powers of integers.)

   Assuming each outcome for the season is equally likely, what is probability that Jim pitches at least three shutouts? (A shutout is when Jim gives up no runs. Express your answer in factorials and/or powers of integers.)

Ten starts into 4 categories: $\binom{13}{3} = \boxed{\frac{13!}{3!\,10!}}$

No shutouts: Ten starts in 3 categories: $\binom{12}{2} = \frac{12!}{2!\,10!}$

One shutout: Nine starts in 3 categories: $\binom{11}{2} = \frac{11!}{2!\,9!}$

Multiply by $\binom{10}{1}$, because any start could be the shutout.

Two shutouts: 8 starts in 3 categories: $\binom{10}{2} = \frac{10!}{2!\,8!}$

Multiply by $\binom{10}{2}$, for the position of the two shutouts.

So $P(\text{at least 3 shutouts}) = 1 - P(0, 1, \text{or } 2 \text{ shutouts})$

$$= 1 - \left(\frac{\frac{12!}{2!\,10!} + \frac{11!}{2!\,9!}\,\frac{10!}{1!} + \frac{10!}{2!\,8!}\,\frac{10!}{2!\,8!}}{\frac{13!}{3!\,10!}} \right)$$

8. Answer these questions for the poset

$$(\{\{1\},\{2\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\},\subseteq).$$

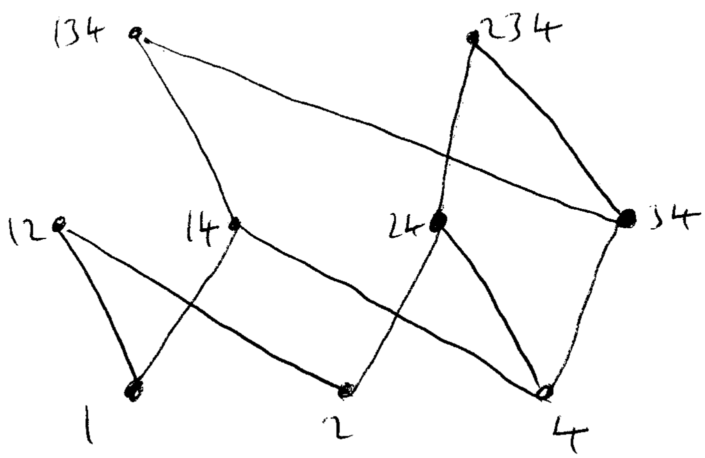
(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Find the least upper bound of $\{\{2\},\{4\}\}$.

(e) Find the greatest lower bound of $\{\{1,3,4\},\{2,3,4\}\}$.



(a) $\{1,2\}$, $\{1,3,4\}$, $\{2,3,4\}$

(b) $\{1\}$, $\{2\}$, $\{4\}$

(c) No

(d) $\{2,4\}$

(e) $\{3,4\}$.

9. (a) Show that if A and B are sets then $A - B = A \cap \bar{B}$.

   (b) Let A and B be independent events, with A having probability 0.6 and B having probability 0.3. What is the probability of $A - B$?

(a) $x \in A - B \Rightarrow x \in A,\ x \notin B \Rightarrow x \in A \cap \bar{B}$

$x \in A \cap \bar{B} \Rightarrow x \in A,\ x \notin B \Rightarrow x \in A - B$

(b) $p(A - B) = p(A \cap \bar{B}) = p(A)\,p(\bar{B})$ since independent

$= 0.6 \times 0.7$

$= 0.42$

10. Let G be a graph with n vertices and let u and v be distinct vertices of G. Prove that if there is a walk from u to v then there is a walk from u to v that has length less than or equal to $n-1$.

Assume we have a walk with $\geq n$ ~~vertices~~ edges.

So the walk visits $n-1$ vertices between u and v

So some vertex must be repeated.

Walk has form:

$$u - \cdots - \underbrace{a - \cdots - a} - \cdots - v$$ for some vertex a

This portion of the walk can be removed,
and we have a shorter walk from u to v.

So any walk of length $\geq n$ can be shrunk.

11. Use proof by contradiction to show that every integer greater than 11 is the sum of two composite numbers. (Hint: Let $p \geq 12$ and let $q = p - 6$. If q is prime, look at numbers near q. Recall that a number n is composite if $n > 1$ and n is not prime.)

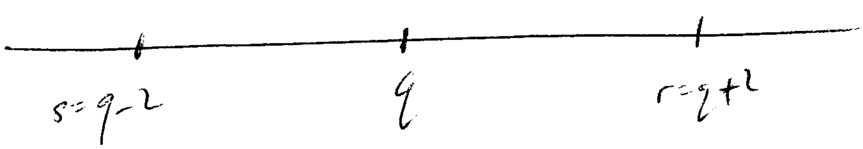
Let $p \geq 12$, let $q = p - 6$, so $q \geq 6$.

If q is composite, we are done, since 6 is composite also.

If q is prime:

Also have $p = 4 + r$, $p = 8 + s$ for two other numbers, $r \geq 8$, $s \geq 4$.

$r = q + 2$, $s = q - 2$.



~~If~~ Either $q-2$ or $q-1$ or q is divisible by 3.

If $q-2$ or q is divisible by 3, we are done.

Else, if $q-1$ is divisible by 3 then $q+2$ is also divisible by 3, so again we have p as a sum of two composite numbers.

12. Prove that $a_n = 2^n(2+n^2+n^3/6)$ is a solution to the recurrence relation $a_n = 4a_{n-1}-4a_{n-2}+(n+1)2^n$ for $n \geq 2$ when $a_0 = 2$ and $a_1 = 6\frac{1}{3}$.

Plug in the values and check:

$$a_0 = 2^0(2+0^2+0^3/6) = 2 \checkmark$$

$$a_1 = 2^1(2+1^2+1^3/6) = 6\tfrac{1}{3} \checkmark$$

$$a_{n-1} = 2^{n-1}(2+(n-1)^2+(n-1)^3/6)$$

$$a_{n-2} = 2^{n-2}(2+(n-2)^2+(n-2)^3/6)$$

$$4a_{n-1} - 4a_{n-2} + (n+1)2^n$$

$$= 2^{n+1}(2+(n-1)^2+(n-1)^3/6) - 2^n(2+(n-2)^2+(n-2)^3/6)$$
$$+ (n+1)2^n$$

$$= 2^n\left[4+2(n-1)^2+(n-1)^3/3 - 2 - (n-2)^2 - \frac{(n-2)^3}{6} + n + 1\right]$$

$$= 2^n\Big[4 + 2n^2 - 4n + 2 + \tfrac{1}{3}n^3 - n^2 + n - \tfrac{1}{3} - 2 - n^2 + 4n - 4$$
$$- \tfrac{1}{6}n^3 + n^2 - 2n + \tfrac{4}{3} + n + 1\Big]$$

$$= 2^n\left[2 + 0n + n^2 + \tfrac{1}{6}n^3\right] \checkmark$$