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Name:

MATH2800 Introduction to Discrete Structures

Final Exam, Thursday, December 13, 2007.

You may use three sheets of handwritten notes, but no other sources. The exam lasts three hours. The exam consists of twelve questions. Answer **at least two of questions 10, 11, and 12**; your score will be the total of your **ten best** questions. Each question is worth ten points. You can work all twelve problems. You need to work at least ten problems, including at least two of problems 10, 11, and 12, to score over 90% on the exam. Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit. No calculators are allowed. Please ring your section below:

1: Monday 9am 2: Thursday 9am 3: Monday 2pm 4: Thursday 2pm

SOLUTIONS.

Q1		
Q2		
Q3		
Q4		
Q5		
Q6		
Q7		
Q8		
Q9		
Q10		
Q11		
Q12		
Total	100	

1. What is the probability that a five card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?

Total number of hands $= \binom{52}{5}$

\# two pair hands $= \binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}$

choose the pairs

choose cards from pairs

$$= \frac{13!\,4!\,4!\,44}{11!\,2!\,2!\,2!\,2!\,2!}$$

Probability $= \dfrac{13!\,4!\,4!\,44 \cdot 5!\,47!}{11!\,2!\,2!\,2!\,2!\,2!\ \ 52!}$

2. A directed graph $G = (V, E)$ has 8 vertices and 12 directed edges.

   (a) We are interested in the out-degrees of the vertices. How many different out-degree sequences are possible? (Assume the order of the vertices is important, so a sequence (2,1,3,1,0,0,3,2) is different from a sequence (3,0,1,0,2,2,1,3). Express your answer in terms of factorials and/or powers of integers.)

   (b) Assume each arrangement is equally likely. What is the probability that every vertex has at least one outgoing edge? (Express your answer in terms of factorials and/or powers of integers.)

(a) $v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8$

$\ast \ast \mid \quad \mid \ast \ast \ast \mid \quad \mid \ast \mid \ast \mid \ast \ast \ast \mid \ast \ast$

Distribute 12 arcs between eight vertices

Get $\binom{19}{12} = \frac{19!}{7!\,12!}$

(b) Distribute eight arcs first, then distribute remaining 4 in $\binom{11}{4}$ ways

Probability $= \frac{11!\,7!\,12!}{4!\,7!\,19!}$

3. Assume 6 distinct integers are chosen from the first 10 positive integers. Show that there must be a pair of these integers with sum equal to 11.

Use pigeonhole principle.

Have five pairs that add up to 11.

$1 + 10$
$2 + 9$
$3 + 8$
$4 + 7$
$5 + 6$

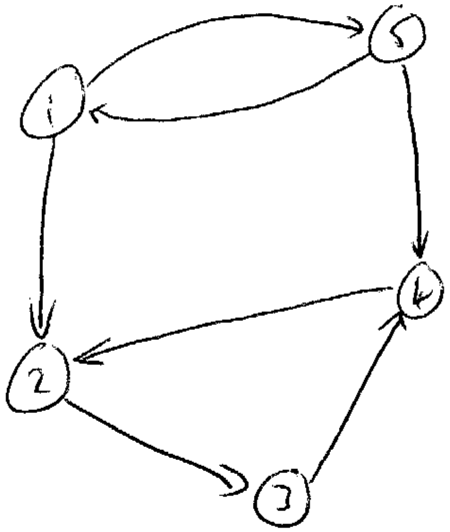
Six integers are picked, so both of at least one pair must be picked.

4. A relation R on a set with five elements has zero-one matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

What is the transitive closure of this relation?

Closure is $R^* = \bigcup_{n=1}^{\infty} M_{R^n}$, the connectivity relation.



1, 5 can reach any vertex.

2, 3, 4 can reach each other

So $M_{R^*} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

5. Let x, y, and z be variables with domains equal to $\{0, 1\}$. Let p be the statement $x = 1$. Let q be the statement $y = 1$. Let r be the statement $z = 1$. Assume the statements $p \vee \neg q$ and $q \vee r$ are true. Show that we must have $x + z \geq 1$.

Resolution: $p \vee \neg q$ and $q \vee r \Rightarrow p \vee r$.

$\Rightarrow$ either $x = 1$ or $z = 1$

$\Rightarrow$ $x + z \geq 1$.

6. A *matching* on a graph $G = (V, E)$ is a subset of the edges that do not share any end vertices. Let a_n equal the number of matchings on K_n. Show that $a_n = a_{n-1} + (n-1)a_{n-2}$. (For an example of a matching, see question 8.)

   Assume $a_1 = 1$ and $a_2 = 2$. Show that if $n \geq 3$ is odd then

$$a_n \geq 2^{\frac{n-1}{2}}(\frac{n-1}{2})!$$

Consider vertex n.

Have two cases (i) vertex n is not matched.

Number of such matchings is a_{n-1}

(ii) vertex n is matched to some vertex i.

Number of matchings on remaining $n-2$ vertices is a_{n-2}

Have $(n-1)$ choices for i

So $a_n = a_{n-1} + (n-1)\, a_{n-2}$, as required.

Second part. Use induction

For $n=1$, $2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)! = 2^0\, 0! \leq 1 = a_1$ ✓

~~For $n=2$, $2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)! \leq 2^{\frac{1}{2}} \frac{1}{2}!$~~

Assume true for $n \leq k$, k odd

Have $a_{k+2} \geq (k+1)\, a_k \geq (k+1)\, 2^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)!$

$$= \frac{k+1}{2}\, 2\, 2^{\frac{k-1}{2}}\left(\frac{k-1}{2}\right)! = 2^{\frac{k+1}{2}}\left(\frac{k+1}{2}\right)!,$$

as required.

7. Let $\boldsymbol{Z}$ denote the set of integers. Which of these relations on the set of all functions from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

   (a) $\{(f, g) \mid f(1) = g(1)\}$.

   (b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$.

   (c) $\{(f, g) \mid f(x) - g(x) = 1 \ \forall x \in \boldsymbol{Z}\}$.

   (d) $\{(f, g) \mid f(0) = g(0) \text{ and } f(1) = g(1)\}$.

(a) Reflexive, symmetric, and transitive ✓ So equiv rel

(b) Not transitive Eg $f(x)=0 \ \forall x$, $g(x)=x \ \forall x$, $h(x)=1 \ \forall x$.

Then $f(0)=g(0)$ and $g(\cancel{0}1)=h(1)$, but f is not related to h.

| Is reflexive & symmetric

(c) Not reflexive. Not symmetric $\left(g(x)-f(x)=-(f(x)-g(x)) \atop =-1\right)$ Not transitive.

$f(x)-g(x)=1 \ \forall x$

$g(x)-h(x)=1 \ \forall x$

$\Rightarrow f(x)-h(x)=2 \ \forall x$

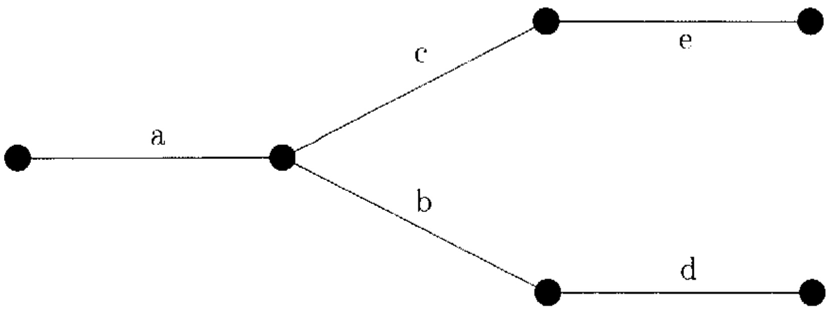
(d) Reflexive ✓

Symmetric ✓

Transitive ✓ $f(0)=g(0)$, $f(1)=g(1)$, $\cancel{f}g(0)=h(0)$, $g(1)=h(1)$

$\Rightarrow f(0)=h(0)$, $g(1)=h(1)$ ✓

8. A *matching* on a graph $G = (V, E)$ is a subset of the edges that do not share any end vertices. Consider the graph:

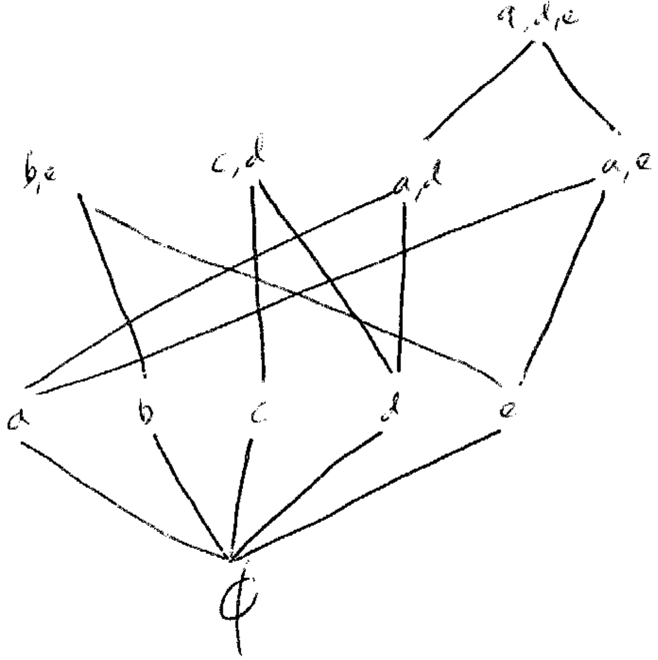


For example, the set $\{a, e\}$ is a matching, but the set $\{b, c\}$ is not. Note that the empty set and the singleton sets consisting of just a single edge, for example $\{b\}$, are also matchings. Let S_1 and S_2 be two subsets of the edges that are matchings. The relation $\preceq$ is defined as set inclusion, so $S_1 \preceq S_2$ if and only if $S_1 \subseteq S_2$. This defines a partially ordered set.

(a) Draw the Hasse diagram for this poset.

(b) Give a maximal element of the poset with two edges and another maximal element with three edges.

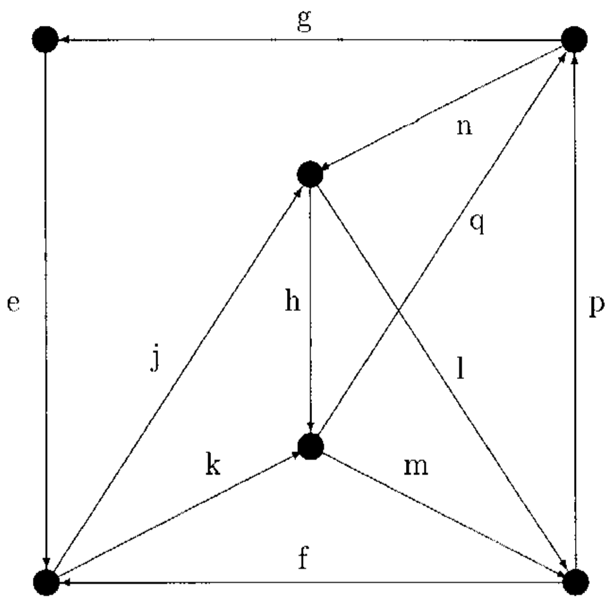
(a)



(b) $\{a, d, e\}$ maximal with 3 edges

$\{c, d\}$ maximal with 2 edges

9. A directed Euler circuit on a directed graph is an Euler circuit that traverses each edge in the correct direction. Let $G = (V, E)$ be the following directed graph.



Show how the edges can be partitioned into a collection of directed circuits, so each edge appears in exactly one of the circuits. Hence find a directed Euler circuit on the graph.

Directed circuit (i) g, e, j, h, m, p

(ii) n, l, f, k, q

Directed Euler circuit

g, e, j, h, m, p, n, l, f, k, q

10. Let $G = (V, E)$ be a connected directed graph. For each vertex $v \in V$, the *in-degree* is the number of edges entering vertex v and the *out-degree* is the number of edges leaving vertex v. A directed Euler circuit on a directed graph is an Euler circuit that traverses each edge in the correct direction. Prove that G contains a directed Euler circuit if and only if the in-degree of each vertex is equal to its out-degree.

Pick a vertex, v, with out-degree ≥ 1

Construct a directed cycle

Walk along edges until return to original vertex, v

(This is possible because in-degree = out-degree, so when a vertex is entered it is always possible to leave.)

Delete the cycle

If no edges left, stop

Else, loop

These cycles can then be hooked together to give an Euler directed circuit.

11. Use strong induction to show that every number greater than or equal to 18 can be written as a sum of a nonnegative integer multiple of 4 and a nonnegative integer multiple of 7. (For example, $29 = 2 \times 4 + 3 \times 7$.)

Base case:

$$18 = 4 + 2\times 7$$

$$19 = 3\times 4 + 7$$

$$20 = 5\times 4$$

$$21 = 3\times 7$$

Inductive step:

Assume true for $n \leq k$, with $k \geq 21$

Let $n = k+1$.

Then $n = 4 + (k-3)$

$= 4 + 4a + 7b$ for some nonnegative integers a, b

$= 4(a+1) + 7b$ ∎

12. Show that a positive integer is divisible by 3 if and only if the sum of its decimal digits is divisible by 3. (Hint: a number n is divisible by 3 if and only if $n \equiv 0 \pmod 3$. You may assume $10^i \equiv 1 \pmod 3$ for any nonnegative integer i.)

Let $x = \sum_{i=0}^{k} a_i 10^i$ be the decimal expansion of x,

so $a_i \geq 0 \ \forall i$

and $a_k > 0$

Then $x \equiv \left(\sum_{i=0}^{k} a_i 10^i \right) \pmod 3$

$\equiv \sum_{i=0}^{k} \left(a_i 10^i \pmod 3 \right)$

$\equiv \sum_{i=0}^{k} a_i \pmod 3$ since $10^i \equiv 1 \pmod 3$

It follows that $x \equiv 0 \pmod 3$

if and only if $\sum_{i=0}^{k} a_i \equiv 0 \pmod 3$