1 Introduction

The motivation for this inquiry is to study prevention of decompression sickness in scuba diving. Decompression sickness, commonly called “the bends”, which is believed to be caused by nitrogen bubbles in tissue which form when a diver returns too quickly to the surface after a dive. The specific inquiry made here is to discover how slowly must a diver ascend to the surface so that bubbles of a size that cause internal damage do not occur.

In order to understand the physiological process which occurs during decompression sickness we began by considering a bottle of water pressurized by CO$_2$. Carbonated water in a bottle is a saturated system under pressure higher than the surrounding atmosphere. When the cap is removed, the pressure decreases immediately, causing the dissolved gas to leave the water. This reduction in pressure, which results in a decrease in gas concentration is analogous to what occurs in the body of a diver as he ascends to the surface. The diver experiences increased pressure during a dive, causing an increase in the concentration of a gas in his body. When the diver ascends, pressure is decreased and in order to equilibrate, the gas must leave their system similar to the gas leaving the cup of soda.

2 Bubble Growth Due to The Change of Pressure

2.1 Pressure Changes Alone

We observed a two liter bottle of carbonated water (water pressurized with CO$_2$) and noticed that small bubbles appeared toward the bottom and grew significantly as they ascended to the top. We first questioned whether the increase in radius was simply
correlated to the decrease in pressure as the bubble ascended to the surface. A simple expression relating pressure to the size of a bubble is Boyle’s law (also known as the ideal gas law), $PV = nRT$. Here $P$ is the pressure in atm, $V$ is the volume in L, $n$ is the number of moles, $R$ is the universal gas constant $0.082$ in atm·L/mol·K, and $T$ is our absolute temperature in K. Assuming the mass of the bubble remains constant, $nRT$ = constant, it follows that $P_0V_0 = P_1V_1$, where the latter is the pressure and volume of a bubble near the top of the bottle, and the prior is the pressure and volume of the bubble at the bottom.

Using $V$ as the volume of a spherical bubble we set up the following relationship:

$$P_0\frac{4}{3}\pi r_0^3 = P_1\frac{4}{3}\pi r_1^3$$

For this described experiment $P_0$, $P_1$ are known.

$$P_0 = 2\cdot\text{atm} = 2\cdot10^5\text{Pa}$$

$$P_1 = P_0 + \rho_w\cdot g\cdot h \approx 2\cdot10^5 + .01\cdot10^5 \approx 2.01\cdot10^5\text{Pa}$$

where, $\rho_w = 10^3\frac{\text{kg}}{\text{m}^3}$, $g = 9.81\frac{\text{m}}{\text{s}}$, and $h = .1\text{ m}$. Then we have $r_1 = (\frac{2.01}{2})r_0 = 1.0017\cdot r_0$. Since we were able to observe noticeable changes in the radius of a bubble we came to the conclusion that the change in bubble size was not due to pressure change alone and thus the mass of gas in the bubble is not constant.

### 2.2 Mass of Gas Also Changes

Next, we consider that $n$ is not a constant (i.e. more $CO_2$ is collected as the bubble ascends). We rewrite the ideal gas law by letting $n = \frac{m}{M}$ where $m$ is mass, and $M$ is the molecular weight. By rearranging the ideal gas law we have $\frac{PV}{m} = \frac{RT}{M}$, where the
right hand side is a given constant. In the same manner as in the previous section we can now set \( \frac{P_0 V_0}{m_0} = \frac{P_1 V_1}{m_1} \). We can then consider \( r_0 \) and \( r_1 \) to be known and assume that, from our observations, the bubble increases in radius by a factor of two, i.e. \( r_1 = 2 \cdot r_0 \). By solving this equation for the ratio of masses we find that \( \frac{m_1}{m_0} = 8 \). For this example, if the radius of the bubble doubles then the mass will be 8 times as large.

### 3 Diffusion in Soda Cup

#### 3.1 Henry’s Law

We first considered how much \( CO_2 \) is dissolved into the liquid under 2 atm of pressure and 1 atm of pressure in addition to how much \( CO_2 \) we would expect to come out of the water when it is changed from 2 atm to 1 atm of pressure.

Statement of Henry’s Law:

\[
C = k_H P_g
\]  

(3)

where \( C \) is the concentration of the gas in the solvent, \( k_H \) is the proportionality constant (Henry’s constant) depending on nature of both gas and solvent, and \( P_g \) is the partial pressure of the gas.

Now, if we know the concentration of gas in a solvent, we can calculate the mass of the gas by multiplying by the volume of the liquid. Given \( k_H \) of \( CO_2 \) at 298K, which is equal to 0.034 \( \text{mol} \cdot \text{atm}^{-1} \cdot \text{L} \), and pressure, \( P_g = 2 \text{ atm} \), the concentration of \( CO_2 \) in water is

\[
c = k_H P_g = 0.068 \text{ mol} / \text{L} \approx 3 \text{ g} / \text{L}.
\]

What is the volume of \( CO_2 \) that is released when the pressure is decreased from 2 atm to 1 atm? To answer this question we revisit the ideal gas law of form \( PV = \frac{m}{M} RT \) where \( R \) is 0.82 \( \frac{\text{L} \cdot \text{atm}}{\text{K} \cdot \text{mol}} \), and \( T = 298 \text{ K} \) \( \Rightarrow V = 1.6 \text{L} \) of \( CO_2 \) is released.

#### 3.2 Time For Diffusion

How long will it take for all of the \( CO_2 \) to diffuse out of the top of the cup?

\( c(z, t) = \) concentration of \( CO_2 \) at a depth \( z \) and time \( t \). \( z \in [0, h] \), \( t \in [0, \infty) \), where \( h \) is the height of the water. From conservation of mass \( \frac{\partial c(z, t)}{\partial t} = -\frac{\partial q}{\partial z} \), and the flux of the
concentration, $q(z, t)$ is taken from Fick’s Law: $q = -D \frac{\partial c}{\partial z}$. Thus we have,

$$\frac{\partial}{\partial t} c(z, t) = D \frac{\partial^2}{\partial z^2} c(z, t)$$
$$\frac{\partial}{\partial z} c(z, t)|_{z=0} = 0$$
$$c(h, t) = g_1$$
$$c(z, 0) = c_0$$

(4)

Where $D = 1.6 \times 10^{-5}\text{cm}^2/\text{s}$ is the diffusion coefficient and $h$ is the height of soda water. In our case, we use $h = 2\text{in}$, $c(h, t) = 1.5\frac{\text{g}}{\text{L}}$, the saturated concentration at 1 atm, and $c_0 = 3\frac{\text{g}}{\text{L}}$ the saturated concentration at 2 atm.

Using separation of variables, the series solution for the system is

$$c(z, t) = c(h, t) + \sum_{n=1}^{\infty} \frac{c_0}{\lambda_n h} \sin(\lambda_n h) \cos(\lambda_n z) \exp(-D\lambda_n^2 t)$$

(5)

where $\lambda_n = \frac{2^{n-1} \pi}{2h}$. The system was also solved using the method of lines where space was discretized using Chebyshev points and the resulting system was marched forward in time using the Matlab solver ode15s. Fig.1 shows the time evolution of the concentration as it diffuses from the bottom to the top. The results for the diffusion PDE determined the time for the concentration to reach below $2\frac{\text{g}}{\text{L}}$ everywhere, which we called ‘flat’, to be $\approx 10$ days. This amount of time is physically unreasonable for $\text{CO}_2$ to diffuse out of the cup given our observations. Thus, there must be some other means by which the $\text{CO}_2$ leaves the water.

![Figure 1: Evolution of concentration of $\text{CO}_2$ in cup. The bottom curve corresponds to $t = 250$ hours at which point the computation was stopped as the concentration fell below our critical value, $2\frac{\text{g}}{\text{L}}$.](image-url)
4 Growth of an isolated gas bubble in a fluid

4.1 Derivation of the equations

Consider a gas bubble surrounded by water saturated with the gas. Concentration of the gas in water is denoted by $c$. The flux of the gas at any point in the fluid is given by

$$\mathbf{q} = -D \nabla c + \mathbf{uc}.$$  

(6)

where $u$ is the velocity of the fluid and $D$ is constant of diffusion. Conservation of mass results in the equation

$$\frac{\partial c}{\partial t} = -\nabla \cdot \mathbf{q} = D \frac{\partial}{\partial r} \left( r^2 \frac{\partial c}{\partial r} \right) - u \frac{\partial c}{\partial r}.$$  

(7)

By the ideal gas law

$$\frac{P(t)V(t)}{m(t)} = \frac{P_0 V_0}{m_0} = \frac{P_0}{\rho_0} = \alpha_0$$  

(8)

or

$$P(t)V(t) = \alpha_0 m(t).$$  

(9)

Differentiate both sides of the equation with respect to $t$ to get

$$\frac{dP(t)}{dt} V(t) + P(t) \frac{dV(t)}{dt} = \alpha_0 \frac{dm(t)}{dt}.$$  

(10)

For the spherical bubble the volume is $V(t) = \frac{4\pi}{3} R^3(t)$, after differentiation we get

$$\frac{dV(t)}{dt} = 4\pi R^2(t) \frac{dR(t)}{dt}.$$  

(11)

The total mass can be written as

$$m(t) = m_0 + \int_0^t (SA) D \frac{\partial c}{\partial r} \bigg|_{R(t)} \, dt$$  

(12)

where $SA$ is the surface area of the bubble, $SA = 4\pi r^2$. The integral part defines the mass influx. Differentiating above equation we get

$$\frac{dm(t)}{dt} = 4\pi R^2(t) D \frac{\partial c}{\partial r} \bigg|_{R(t)}.$$  

(13)
Substituting into (10) we get
\[
\frac{dP(t)}{dt} \frac{4\pi}{3} R^2(t) + 4\pi R^3(t) P(t) \frac{dR(t)}{dt} = \alpha_0 \frac{dm(t)}{dt}.
\] (14)

The problem of the bubble growth can be modeled by
\[
\frac{\partial c}{\partial t} = D \frac{r^2 \partial}{\partial r} \left( r^2 \frac{\partial c}{\partial r} \right) - u \frac{\partial c}{\partial r},
\] (15)
with the initial condition
\[
c(r, 0) = c_0 \quad r > R(t = 0) = r_0,
\] (16)
and boundary conditions
\[
c(R(t), t) = k_H P(t),
\] (17)
\[
c(r, t) = c_0 \quad r \to \infty.
\] (18)
where \(k_H\) is the Henry's constant for solubility of a gas. The last equation is given by (14)
\[
\frac{R \frac{dP}{dt}}{3} + P \frac{dR}{dt} = \alpha_0 D \frac{\partial c}{\partial r}.
\] (19)

4.2 Non-dimensionalization

Let's introduce dimensionless variables of the form
\[
c = c_0 c', \quad t = t_0 t', \quad r = r_0 r', \quad u = u_0 u', \quad R = r_0 R', \quad P = \frac{c_0}{k_H} P',
\]
where \(u_0\) and \(t_0\) are yet to be determined characteristic velocity and time scales and the primes denote dimensionless quantities. The system of equations describing the model (eq.(15)-(19)) can be rewritten in dimensionless form
\[
\frac{\partial c'}{\partial t'} = \frac{D t_0}{r_0^2 u_0^2} \frac{\partial}{\partial r'} \left( r_0^2 \frac{\partial c'}{\partial r'} \right) - \frac{u_0 t_0}{r_0} u' \frac{\partial c'}{\partial r'},
\] (20)
with the initial condition
\[
c'(r', 0) = 1 \quad r' > R'(t' = 0) = 1,
\] (21)
and boundary conditions
\[
c'(R'(t'), t') = P'(t'),
\] (22)
\[ c'(r', t') = 1 \quad r' \to \infty. \]  

The last equation is given by

\[ \frac{R' dP'}{3 \, dt'} + P' dR' = \alpha_0 D K_H t_0 \frac{\partial c'}{\partial r'}. \]  

(24)

There are 3 dimensionless groups in the above system and we denote them:

\[ \alpha = \frac{D t_0}{r_0}, \quad \beta = \frac{u_0 t_0}{r_0}, \quad \gamma = \frac{\alpha_0 D K_H t_0}{r_0} \]  

(25)

Setting \( \beta = \gamma = 1 \) gives us our characteristic time and velocity

\[ t_0 = \frac{r_0^2}{\alpha_0 D k_H u_0} \quad u_0 = \frac{r_0}{t_0}. \]

then \( \alpha \) takes the form

\[ \alpha = \frac{1}{\alpha_0 k_H}. \]

For simplicity we denote \( \alpha_0 k_H = k'_H \), a dimensionless Henry’s constant, and \( \alpha \) takes the form \( \alpha = \frac{1}{k'_H} \).

Inputting the time and velocity scalings and dropping primes, the dimensionless system of equations (20)-(24) become

\[ \frac{\partial c}{\partial t} = \frac{1}{k_H r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial c}{\partial r} \right) - u \frac{\partial c}{\partial r}, \]  

(26)

with the initial condition

\[ c(r, 0) = 1 \quad r > R(t = 0) = 1, \]  

(27)

and boundary conditions

\[ c(R(t), t) = P(t), \]  

(28)

\[ c(r, t) = 1 \quad r \to \infty. \]  

(29)

The last equation is

\[ \frac{R dP}{3 \, dt} + P \frac{dR}{dt} = \frac{\partial c}{\partial r}. \]  

(30)
The gas under consideration is nitrogen ($N_2$). The dimensionless Henry’s constant for nitrogen dissolved in water at temperature 300 K is $k'_{H} = 1.5 \times 10^{-2}$. Since the quantities $c$, $R$ and $P$ are scaled such that they are $O(1)$, we can simplify the problem assuming $k'_{H} \ll 1$. Taking the limit $k'_{H} \to 0$, equation (26) can be reduced and written in the form

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial c}{\partial r} \right) = 0.$$  \hspace{1cm} (31)

The general solution of the equation (31) is

$$c(r, t) = B(t) - \frac{A(t)}{B(t)},$$  \hspace{1cm} (32)

applying the boundary conditions (28) and (29) gives

$$c(r, t) = 1 - \frac{R(t)(1 - P(t))}{r},$$  \hspace{1cm} (33)

and next we evaluate

$$\frac{\partial c}{\partial r} \bigg|_{r=R(t)} = \frac{(1 - P(t))}{R(t)}. \hspace{1cm} (34)$$

Substituting this last equation into (30) we obtain the following ordinary differential equation for the evolution of the radius of the bubble

$$\frac{R}{3} \frac{dP}{dt} + P \frac{dR}{dt} = \frac{(1 - P(t))}{R(t)},$$ \hspace{1cm} (35)

subject to the initial condition $R(0) = 1$, where $P(t)$ is a given function describing the change in pressure over time. An example with a given $P(t)$ is shown in Fig.2. Here we take $P(t) = 1.1e^{-0.01t}$ so that initially $P > 1$, corresponding to compression, and the radius decreases followed by an increase in the radius as the pressure drops below 1, decompression.

### 5 Scuba Diving Model

Next we considered how gas, specifically, nitrogen ($N_2$) plays a role in scuba diving and decompression sickness.

We wanted to ask, how slowly must a diver return to the surface so that the bubbles inside their tissue do not grow dangerously large.
5.1 Before Diving

Before the diver enters the water, the pressure of the tissue and blood within the diver are the same as the ambient pressure.

5.2 Descending

As the diver descends, the ambient pressure increases causing an increase in the concentration of nitrogen in the blood, due to Henry’s Law. Nitrogen enters the blood through the lungs and experimental data shows that the saturation of the blood with nitrogen occurs very quickly. For the purpose of the model, the blood is assumed to instantly be at saturation for any given pressure.

Then it is necessary to consider the diffusion of nitrogen from the blood into the tissue. This process is analogous to our previous situation of $CO_2$ diffusing out of the soda cup. Eventually the tissues in the body reach saturation of $N_2$.

The blood is assumed to be instantaneously saturated with nitrogen and the concentration of nitrogen is given by Henry’s law

$$c_N(t) = k_H P(t).$$

The nitrogen diffuses from the blood into the tissue and the process is governed by the diffusion equation

$$\frac{\partial c}{\partial t} = D \nabla^2 c,$$  \hspace{1cm} (36)
where $D$ is the diffusion constant. The saturation under the normal atmospheric pressure (1 atm) is $c_0$.

Assumptions:

- the size of the bubble is very small in comparison to the thickness of the tissue,
- the thickness of the tissue is about $3 \times 10^{-3}$m,
- initial size of the bubble is about $10^{-6}$m,
- the bubble experiences an environment of water with concentration $c(x,t)$. The value is obtained by solving the diffusion equation

5.3 Ascending

Now, the diver must ascend to the surface. We assume the existence of very tiny bubbles ($< 10 \cdot 10^{-6} \cdot m$ (size of the average cell)) in the tissue. The 10 micron length was chosen because it is the size of the cell, and should the bubble grow to the size of a cell it could be considered the limit at which damage would occur.

As the diver ascends the pressure decreases, and a situation analogous to the previous model of bubble growth occurs. Moreover, we assume the bubble form a continuum across the tissue which will allow us to find how a bubble would grow at a specific position in tissue.

The initial value problem (35) for growth of bubble is written back in dimensional form to get

$$\frac{R}{3} \frac{dP}{dt} + P \frac{dR}{dt} = \alpha_0 D c_0 - k_H P R$$

where, instead of a constant, we use $c_0 = c_0(z,t)$ which is found by solving (36). This coupling allows us to compute the evolution of the radius of a bubble at any given point inside the tissue.

5.4 Results

Equations (36) and (37) form a coupled system of differential equations, where the coupling is due to the assumption that $c_0 = c_0(z,t)$. Here, the pressure, $P$, is a given function of time which corresponds to the diving profile. The partial differential equation (36) is solved in the same manner as in section 3.2 in the half domain $0 \leq z \leq H/2$, where $H = 3 \times 10^{-3}$ m. At each time step, the current value of $c_0$ is plugged into the ordinary differential equation (37) and is marched forward in time using $ode15s$.

The initial conditions are: $c(z,0) = 0.8k_H P(0) \ g/L$ (since $N_2$ is only about 80% of
the total concentration of gas in blood), where $P(0) = 1$ atm, and $R(z, 0) = 10^{-6}$ m. The boundary conditions for $c$ include the symmetry condition $\frac{\partial c}{\partial z} = 0$ at the centerline of the tissue ($z = 0$) and the Dirichlet condition $c(H/2, t) = c_N$ (the concentration of $N_2$ in the blood). The parameter values used are: $k_H = 1.708 \times 10^{-3}$ kg/(m$^3$ atm); $\alpha_0 = 0.9084$ m$^3$ atm/kg; $D = 0.353 \times 10^{-9}$ m$^2$/s.

We consider two pressure profiles; one during which bubbles grow too large and another during which the bubbles do not break our 10 micron threshold, after which we believe cell damage occurs. The profiles considered are shown in Fig.3 - the computations are stopped after the diver has been at the surface for two hours. The profiles are broken up into five parts, each of which is linear. First is the descent to a given depth/pressure followed by a pause at this depth. After the pause the diver ascends to the surface in two parts, each part with a different slope. Finally, the diver reaches the surface and remains there for two hours. Note that pressure profiles are identical except for the slope of the final ascent thus the profiles for $c$ and $R$ will be identical until the second ascent.

![Figure 3: Diver profiles considered for numerical computations. Left: ‘bad’ profile; Right: ‘good’ profile](image)

5.4.1 Descent

During the descent portion of the dive profile we see in Fig.4 growth in the concentration of $N_2$ in the tissue everywhere, predominantly at the wall. Note that during the descent our model does not predict any change in the radius of the bubbles.

5.4.2 Pause

During the pause segment of the dive profile we see in Fig.5 even more growth in the concentration of $N_2$ in the tissue everywhere, except this time the concentration reaches
a maximum at the wall and continually grows in the center. Note that the model again does not predict any change in the radius of the bubbles.

5.4.3 First Ascent

During the first ascent, where the pressure is now decreasing, we see in Fig.6 a decrease in the concentration of $N_2$, as expected, in the tissue everywhere. Still the model does not predict any change in the radius of the bubbles.
Figure 6: Change in concentration of $N_2$ in tissue during first ascent.

5.4.4 Second Ascent

During the second ascent of the dive profile, where the two profiles now differ, we see in Fig. 7 a continual decrease in the concentration of $N_2$, however, $c$ decreases faster in the ‘bad’ profile since the pressure change is more dramatic. This fast change in pressure in the ‘bad’ profile causes the radius of the bubbles towards the center to rapidly increase whereas $R$ does not change much for the ‘good’ profile, see Fig. 8.

Figure 7: **Left**: Change in concentration of $N_2$ during second ascent of ‘bad’ profile. **Right**: Change in $c$ of ‘good’ profile.
Finally, we watch as both $c$ and $R$ change over two hours after reaching the surface. We see in Fig.9 a continual decrease in the concentration of $N_2$. It is during this final stage we are able to see why we call one profile ‘bad’ and the other ‘good’. In Fig.10 the figure on the left shows that the radius of the bubbles near the center of the tissue grow to larger than 10 $\mu$m which is larger than the average cell size and we would consider bubbles of this size to be dangerous, hence ‘bad’ profile. In Fig.10 on the right we see that for the ‘good’ profile bubbles grow to at most half the size of a cell and thus not hazardous.
6 Summary

We have investigated the change in size of bubbles due solely to pressure change and also the motion of gas due to diffusion alone. We concluded that, while pressure and diffusion are major contributors to the growth of bubbles and the change in concentration of gas, there must be other factors which contribute to decompression sickness. Diffusion is then combined with the fact that gas is carried with the fluid to describe the flux of the concentration of gas at any point in the fluid which is then used to conserve mass. The ideal gas law is then coupled with the aforementioned conservation equation through use of the change in mass of a bubble. This coupling leads to a model which describes the growth of an isolated gas bubble in a fluid, however, the system has a moving boundary condition which is difficult to solve. Dimensional analysis is employed to simplify the problem and thus give an idea as to what is happening to leading order. However, this simplified problem does not take into account spatial effects. Thus, finally, we consider a simple model which accounts for spatial effects in the change in the growth of a bubble. The model is subjected to several pressure profiles and the results agree with what should be expected.

Future work would include the solving of the full moving boundary problem. This problem can be solved by allowing the boundary to move or, since it is only one-dimensional, the spatial terms can be easily be scaled by the moving boundary thus leading to a fixed-boundary problem, which is much easier to solve. The one downside to the models considered is that there is no equation for pressure; the pressure profile must be prescribed. This limitation does not allow one to find the optimum dive profile, however, the upside is that our model is simple and if given some data we could easily determine a safe profile which would not lead to significant growth of bubbles and thus no damage inflicted and decompression sickness avoided.
References
