

MY AIR CONDITIONER? YOU'RE STANDING ON IT!

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Abstract

Installing a ground source heat pump as a means of heating and cooling a house is becoming an increasingly popular alternative to the traditional methods. However, the installation costs being very high, an in-depth study of the configuration of such systems is needed in order to minimize these costs. The relationship between power output and controllable parameters, such as pipe radius and length was investigated to this end.

A derivation of the temperature profile of the soil was done in order to make the most of the temperature difference. The flow of water was modeled first as plug flow, and then as Poiseuille flow, with both models yielding sensibly the same results for the optimal length of the pipe at small radii. The diffusion of heat through the soil was modeled numerically in order to derive a comfortable distance at which pipes can be packed while using the heat capacity of the soil efficiently. A numerical simulation of the whole system was performed and the results for optimal length of the pipe were compared to those obtained from the theoretical model. Further, a value of the heat transfer coefficient of the soil was computed using the numerical simulation.

1 Problem Description

The annual residential energy consumption of the United States reaches up to 9.9 quadrillion BTU, of which 6.3 are used for spacial and water heating and another 0.6 for air-conditioning. The total bill adds up to 160 billion dollars. On average, this means that every household invests \$ 1100 in heating and cooling air and water. These figures reflect the fact that more than 62% of households use natural gas as a source of energy, whereas only 14% use wood, and even fewer use alternate sources.

One very promising such alternative is the exploitation of geothermal energy. Indeed, installing a ground source heat pump as a means of heating and cooling a house costs only \$400 per year, instead of the typical \$1100. However its large installation costs (\$10,000 to \$30,000) are a limiting factor to its widespread use. In this case, it takes about 13 years of operation before the installation of such a system can become profitable for its owner. If the break-even time could be reduced, the use of geothermal pumps could be more appealing to more homeowners, who would start saving money faster.

The geothermal heating system takes advantage of the fact that the temperature of the soil fluctuates slower than that of air, and in fact is almost stable at a certain depth. A series of pipes is buried in the ground following different configurations, and water is circulated through them. Thus the water either heats up or cools down depending on the season. A heat exchanger installed in the house then uses this water to either heat or cool the house and the water re-enters the cycle.

The configuration of the pipes through the ground can be either vertical or horizontal. As shown in figures 1, the pipes can be stretched out or coiled together. A variant of the system is to put run the pipes through a pond of

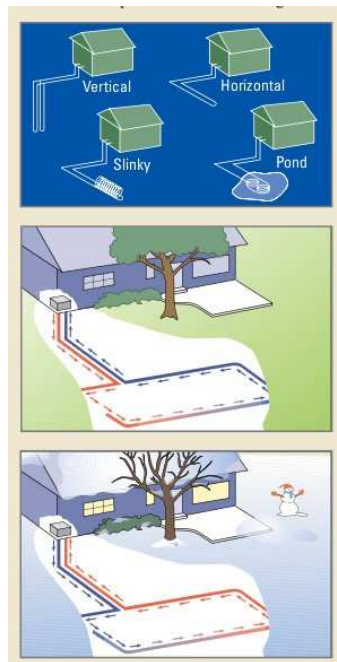


Figure 1: Diagram showing a house regulating its temperature using geothermal energy.

water, for better conductivity. Right now, the choice of the configuration is more or less arbitrary, depending only on such factors as the shape of the yard or the experience of the entrepreneur. A detailed analysis of each of them however could reveal the main differences and thus allow for better choices of the most appropriate configuration.

The efficiency of such a system relies on how much heat can be exchanged with the soil. Generally speaking, the longer the pipe carrying water through the ground, the better the heat exchange. However, other factors, such as flow rate, pipe radius and geometry of the pipe are also to be considered in calculating the heat transfer occurring between water and soil.

Modelling the heat transport in a ground source heat pump will allow for the optimization of its installation costs. As these depend largely on the amount of soil that has to be dug up, the model will be used to minimize the length of the coil and determine the optimal configuration in which it should be laid into the ground for optimal heat transfer to cost ratio.

2 Soil Temperature Profile

The premise of the geothermal heating system is that the soil remains at almost constant temperature at a certain depth. In order to take the best advantage

Table 1: Dimensional Quantities for Soil

Volumetric density	ρ_s	1600 Kg/m ³
Heat transfer coefficient	h	From 150W/Km ² to 1200 W/Km ²
Specific heat	c_s	800 J/KgK
Thermal conductivity	$k_s = \frac{QL}{tA\Delta T}$	0.3 W/Km
Thermal diffusivity	$\alpha_s = \frac{k_w}{\rho c_w}$	$2.34 \cdot 10^{-7} \text{m}^2/\text{s}$

Table 2: Dimensional Quantities for Water

Volumetric density	ρ_w	1,000 Kg/m ³
Dynamic viscosity	μ_w	$8.90 \cdot 10^{-4} \text{Kg/ms}$
Kinematic viscosity	ν_w	$8.90 \cdot 10^{-7} \text{m}^2/\text{s}$
Specific heat	c_w	4,180 J/KgK
Thermal conductivity	$k_w = \frac{QL}{tA\Delta T}$	0.6 W/Km
Thermal diffusivity	$\alpha_w = \frac{k_w}{\rho c_w}$	$1.44 \cdot 10^{-7} \text{m}^2/\text{s}$

Table 3: Dimensionless Quantities

$$P_e = \frac{Q\rho_w c_w}{Rk_w}$$

$$B_i = \frac{hR}{k}$$

of that, one needs to know exactly how the soil responds to the seasonal temperature changes in the air and calculate that depth.

The variation of the temperature in function of the depth in the soil can be set up as a partial differential equation [1]. Indeed, it can be assumed that $\Theta(x, t)$, the temperature in function of the depth and of the time, respects the heat diffusion equation:

$$\Theta_t = k\Theta_{xx}. \quad (1)$$

The seasonal variation of the soil surface temperature yields a periodic boundary condition:

$$\Theta(0, t) = T_A + \Delta T e^{i\sigma t}, \quad (2)$$

where σ^{-1} is proportional to a month. Now, a trial function to transform the PDE in an ODE can be used: $\Theta(x, t) = T_A + A e^{i\sigma t} W(x)$. It then follows that

$$W''(x) = \frac{i\sigma}{k} W(x) \quad (3)$$

with $W(0) = 1$ and $\lim_{x \rightarrow \infty} W(x) = 0$. Trying then $W(x) = e^{mx}$ gives that $m = \pm \sqrt{\frac{\sigma}{2k}}(1 + i)$. Since $W(x)$ decays with x increasing, the negative value of m must be used. Finally, the result lies in the real part of $\Theta(x, t)$:

$$\Re\{\Theta(x, t)\} = T_A + e^{-\sqrt{\frac{\sigma}{2k}}x} \cos\left(-\sqrt{\frac{\sigma}{2k}}x + \sigma t\right). \quad (4)$$

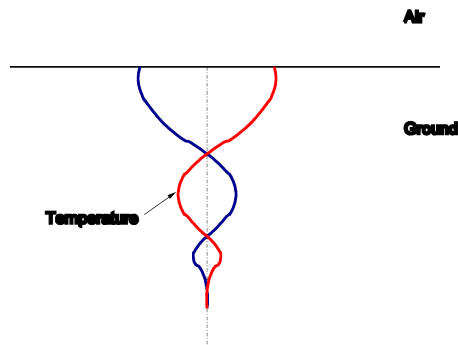


Figure 2: Temperature profile in °C of the ground for both summer and winter as a function of the depth.

Using appropriate values for the constants give that the ground temperature is almost uniformly 13°C below 2.5 m and that there is a temperature inversion at roughly 1.5 m.

3 Theoretical Model of the Geothermal System

In order to find a relationship between the pipe length, pipe radius, flow rate and temperature increase in water, this model assumes that the geothermal system uses an infinitely long pipe, and that the temperature of the soil around the pipe is constant both in time and space. This is not a completely unreasonable assumption, as long as the heat transfer coefficient of the soil is large enough.

Power Consumption

In order to find the power P that a geothermal system can offer, one uses a relation between energy and heat capacity of water:

$$P = \frac{\Delta V}{\Delta t} \rho c_p \Delta T \quad (5)$$

where $\frac{\Delta V}{\Delta t}$ is the volumetric flow rate and ΔT the temperature difference between the water leaving and entering the system. This is based on the assumption that the change in temperature remains constant.

A typical annual household's energy consumption is about 75 MBTU or around 80 MJ. Assuming water enters the system at 3°C and heats up to the temperature of the soil, that is 13°C, a volumetric flow of 30 L/min would be required to power the house. In a typical 1 cm pipes, that means a flow rate of roughly 2 m/s.

Plug Flow

Under the assumption that the soil remains at a constant temperature T_a and that the pipe is straight, it is possible to find an equation for the heat flow Φ into a volume element. First, using the relationship between energy and heat capacity, one has

$$\Phi = \rho_w \Delta V c_{pw} \frac{\Delta T}{\Delta t} \quad (6)$$

for a change of temperature ΔT in a time Δt of a volume element ΔV of water. Now, the heat flux must be related to the heat ϕ_0 transferred from the soil to the pipe and the heat leaving the volume element by convection ϕ_1 :

$$\Phi = \phi_0 - \phi_1 \quad (7)$$

where $\phi_0 = -hS(T - T_a)$ and $\phi_1 = \rho_w Au c_{pw} \Delta T$ with S being the surface area of a volume element, A the cross-sectional area of the pipe and u the velocity of water in the pipe.

Thus the governing equation for the temperature in the volume element is:

$$\rho_w \Delta V c_{pw} \frac{\Delta T}{\Delta t} + \rho_w Au \Delta T c_{pw} = -hS(T - T_a). \quad (8)$$

In order to avoid references to a unit system, the equation should be non-dimensionalized. Note that $\Delta V = A \Delta x = \pi R^2 \Delta x$ and that $S = 2\pi R \Delta x$, with R being the radius of the pipe. Taking the limit in which Δx and Δt go to 0,

$$\rho_w R c_{pw} T_t(x, t) + \rho_w R u T_x(x, t) c_{pw} = -2h(T(x, t) - T_a) \quad (9)$$

with boundary condition being $T(0, t) = T_1$ and the initial condition $T(x, 0) = T_a$, one can define $\tilde{x} = \frac{x}{R}$, $\tilde{T} = \frac{T}{T_a}$, $\tilde{t} = \frac{2h}{\rho_w R c_p} t$ and $\epsilon = \frac{\rho_w c_p u}{2h}$. The equation now becomes

$$\tilde{T}_{\tilde{t}} + \epsilon \tilde{T}_{\tilde{x}} = 1 - \tilde{T} \quad (10)$$

with boundary conditions $\tilde{T}(0, t) = \frac{T_1}{T_a}$ and $\tilde{T}(\tilde{x}, 0) = 1$.

Solving for the steady state of the previous equation,

$$T(x) = (T_1 - T_a) e^{-x/\epsilon} + T_a. \quad (11)$$

Using the previous equation, it is possible to obtain the length of the pipe (as a function of radius, flow rate and initial temperature) needed by the water to reach a given temperature T_2 , by solving for L in $T(L) = \frac{T_2}{T_a}$:

$$L = \frac{Q \rho_w c_{pw}}{2\pi R h} \ln \left(\frac{T_2 - T_a}{T_1 - T_a} \right) \quad (12)$$

where $Q = \pi R^2 u$ is the volumetric flow rate. Note that the length is dependent on h which can vary by up to an order of magnitude, depending on the type of ground.

If $u(t) = 0$ in equation 9, it is possible to solve for $T(x, t)$ since

$$\frac{T_t(x, t)}{T(x, t) - T_a} = \frac{-2h}{Rc_p\rho}. \quad (13)$$

Integrating with respect to t yields

$$T = C(x) \exp\left(\frac{-2h}{Rc_p\rho}t\right) + T_a \quad (14)$$

where $C(x)$ is a function determined by the initial condition.

Poiseuille Flow

A model refinement can be implemented by taking into account non-uniform velocity profile in the pipe. Assuming $u = u(r)$ with Poiseuille flow yields:

$$u = -\frac{\Delta P}{4\mu}(R^2 - r^2). \quad (15)$$

where ΔP is change of pressure (assumed to be constant and negative) of the fluid. Now, the flow rate is

$$Q = 2\pi \int_0^R ur dr \quad (16)$$

and the new energy equation

$$\rho_w c_{pw} \bar{T}_t + \frac{Q\rho_w c_{pw}}{\pi R^2} \bar{T}_x = k_w \bar{T}_{xx} - \frac{2h}{R} (\bar{T} - T_a) \quad (17)$$

where $T(x, t)$ was replaced by its average value over time, $\bar{T}(x, t)$.

Non-dimensionalizing gives

$$\frac{P_e}{B_i} \tilde{T}_t = \frac{1}{B_i} \tilde{T}_{\tilde{x}\tilde{x}} + 1 - \tilde{T} \quad (18)$$

from which the steady-state temperature distribution is

$$T(x) = (T_1 - T_a) e^{\frac{B_i}{P_e}x} + T_a. \quad (19)$$

Thus to get the water to T_2 , the length of the pipe must be

$$L = R \left(P_e - \sqrt{P_e^2 + 2B_i} \right)^{-1} \ln \left(\frac{T_2 - T_a}{T_1 - T_a} \right) \quad (20)$$

4 Numerical Experiments and Results

The previous model assumed that as the water goes through the pipe, the temperature of the soil outside remains constant. This assumption however breaks down if the system runs continuously. Thus another approach is needed to investigate what happens as the soil's temperature changes as it absorbs heat from the pipe.

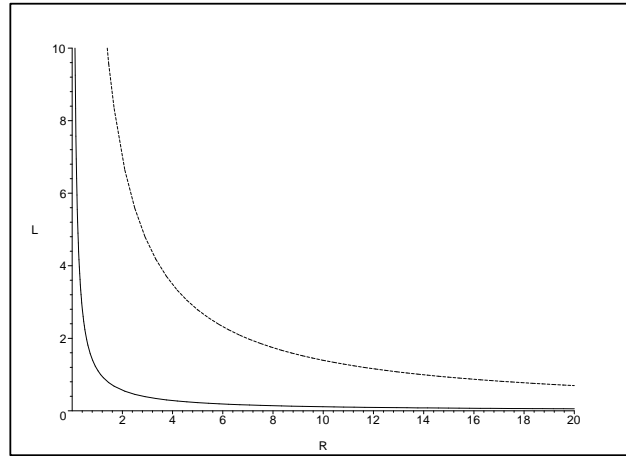


Figure 3: Expected pipe length in meters as a function of the radius of the pipe in meters using plug flow. The lower values seem to be too large to be practical. The top curve uses $h = 55W/Km^2$, numerically obtained, whereas the bottom curve uses $h = 675W/Km^2$.

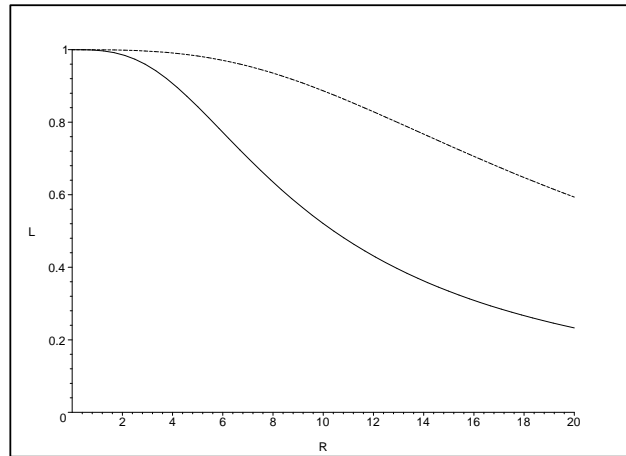


Figure 4: Ratio of the lengths of the pipe without and with Poiseuille flow taken into account as a function of the radius in meters. As long as the pipe is less than roughly 2 m, Poiseuille flow is negligible. The top curve uses $h = 55W/Km^2$, numerically obtained, whereas the bottom curve uses $h = 675W/Km^2$.

Problem Formulation

We can represent the steady state temperature distribution $\theta(x, r)$ around the pipe inside the soil using the axially symmetric polar form of Laplace's equation:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta}{\partial r} \right) = 0 \quad (21)$$

where $0 \leq x \leq L$ and $0 \leq r \leq H$ with the following boundary conditions:

$$\frac{\partial \theta}{\partial x} = 0 \quad \text{for } x = 0 \quad \text{and } x = L \quad (22)$$

$$\frac{\partial \theta}{\partial r} + \frac{\pi}{L} \theta = 0 \quad \text{for } r = H \quad (23)$$

$$\frac{\partial \theta}{\partial x} + C \frac{\partial \theta}{\partial r} = 0 \quad \text{for } r = 0 \quad (24)$$

$$\theta(0, t) = \theta_L \quad (25)$$

By equation (22) we have assumed zero flux through the left and right boundaries. Equation (23) is derived using the approximation:

$$\theta(x, r) \sim \theta(r) \times \cos\left(\frac{\pi x}{L}\right) \quad \text{for } r \gg 1.$$

The constant C that appears in (24) is found to be:

$$C = \frac{k}{2R\rho_w c_w \frac{\Delta V}{\Delta t}}$$

Equation (24) comes from the heat advection through the water inside the pipe via plug flow, together with heat loss to the soil through conduction. We also use prescribed temperature for the water entering the pipe, and we are interested in L , length of the pipe necessary to achieve the desired temperature drop from θ_L .

Finite difference approach

In order to solve the boundary-value problem in (21), we first introduce a uniform grid covering the domain $0 \leq x \leq L$ and $0 \leq r \leq H$. We set

$$x_i = i\Delta x, \quad r_j = j\Delta r, \quad \Delta x = \frac{L}{M}, \quad \Delta r = \frac{H}{N}$$

and let $\theta_{i,j}$ approximate $\theta(x_i, r_j)$ on the grid for $i = 0, 1, \dots, M$ and $r = 0, 1, \dots, N$. The discretized version of (21) becomes

$$\frac{1}{\Delta x^2} (\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}) + \frac{1}{r_j} \times \frac{1}{\Delta r} (\mathcal{F}_{i+1/2,j} - \mathcal{F}_{i-1/2,j}) = 0$$

where $\left\{ \begin{array}{l} \mathcal{F}_{i+1/2,j} = r_{j+1/2} \times \frac{1}{\Delta r} (\theta_{i,j+1} - \theta_{i,j}) \\ \mathcal{F}_{i-1/2,j} = r_{j-1/2} \times \frac{1}{\Delta r} (\theta_{i,j} - \theta_{i,j-1}) \end{array} \right\}$

inside the domain for $i = 1, \dots, M-1$ and $r = 1, \dots, N-1$

The boundary conditions are taken to be

$$\begin{aligned}
 \text{left:} & \quad \frac{1}{\Delta x} (\theta_{1,j} - \theta_{0,j}) = 0 & \text{for } j = 1, \dots, N - 1 \\
 \text{right:} & \quad \frac{1}{\Delta x} (\theta_{N,j} - \theta_{N-1,j}) = 0 & \text{for } j = 1, \dots, N - 1 \\
 \text{top:} & \quad \frac{1}{\Delta r} (\theta_{i,N} - \theta_{i,N-1}) + \frac{\pi}{L} \left(\frac{\theta_{i,N} + \theta_{i,N-1}}{2} \right) = 0 & \text{for } i = 1, \dots, M - 1 \\
 \text{bottom:} & \quad \frac{1}{\Delta x} (\theta_{i,0} - \theta_{i-1,0}) + C \frac{1}{\Delta r} (\theta_{i,1} + \theta_{i,0}) = 0 & \text{for } i = 1, \dots, M - 1
 \end{aligned}$$

Test Setup and Results

As a setup, we have considered a unit test case where the temperature of the water entering the pipe is 1 degree below the ambient temperature which is set to be at 0. Results are shown in figures (4) and (4):

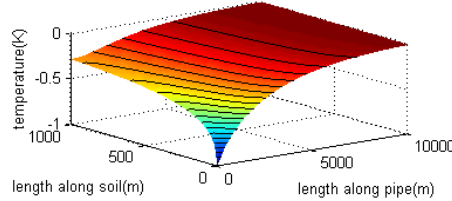


Figure 5: Temperature distribution along the pipe and inside soil.

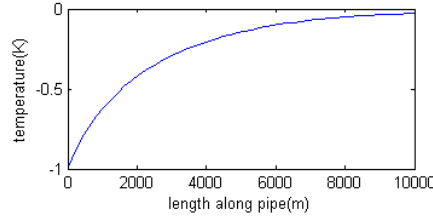


Figure 6: Temperature distribution along the pipe only.

For this run, the radius of the piping used was taken to be 0.1 meters with a volume flow rate of $5 \times 10^{-7} m^3/s$. As it can be seen from this graphs, it takes approximately 10^4 meters of pipe to reduce the temperature of the water entering the pipe to 93% of the temperature difference. From these results, the value of h (heat transfer coefficient) can also be worked out to be approximately $55 W/m^2 K$.

Another experiment was performed to determine the effects of volumetric flow rate on the power output of system. For this, we use the relation:

$$power \propto flowrate \times (\Delta\theta)$$

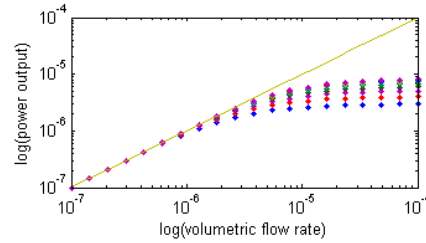


Figure 7: Comparison of power output of varying pipe lengths/flow rates

Maximum (ideal) power output is achieved, when $\Delta\theta = 1$.

In figure 4 the straight line represents the ideal power output, and the dotted lines represent power output for pipes with varying lengths.

As seen from figure 4, as the volumetric flow rate of the water inside the pipe is increased, the power output of the system levels out, and deviates more and more from the ideal. Therefore, there is a limit to the amount of power that can be drawn from the system by increasing the volumetric flow rate.

5 Conclusion

The different models of the geothermal heating system provide a reasonable if not perfect description of it, each one of them focusing on a different aspect.

The two first models aim to calculate the minimal length of pipe needed to heat a typical house. The first one models the heat transfer along the pipe, its main limitation being that it assumes constant temperature outside the pipe. It's predicted pipe length of $113m$ for a $1cm$ radius seems reasonable, although it falls a bit short of the few miles of a similar pipe that are being used in actual heating systems. This is not very surprising considering a straight pipe with continuous flow, which would make the best use of the heat in the ground. This is after all a minimal length, while a real system might well provide more power than what the house actually consumes. An underestimation of this length is however possible, given that the heat transfer coefficients of neither the pipe nor the soil were known, the first one being assumed to be infinite, while the second was taken to be $675W/Km^2$, the average of values cited in literature. An explicit knowledge of these could make the model quite accurate, despite its assumptions.

The second model tries to improve the results of the first one by simulating the heat transfer in the soil around the pipe as well as along the pipe, but only analyzes the steady-state of the system. The length of pipe predicted (of the order of 10^4m long for a $20cm$ thick pipe) however is very far from both the predictions of the first model and of the actual experimental data. In fact, the model predicts that the heat transfer coefficient of the soil is $55W/Km^2$, whereas experiments situate it between 150 and 1200. The discrepancy is probably due to the fact that the model looks at the steady-state of the system, i.e. the state

when the soil's temperature has already increased as much as possible and can therefore absorb less heat.

All models have their particular limitations, but the one common weakness is that they all study the steady state of the system, whereas the behaviour in time seems to be an important factor. For a better understanding of the system, its time dependence should be further investigated.

References

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- [2] US Department of Energy. [<http://www.eia.doe.gov/>], 2007.