Creating a Crust on Bread.

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**Abstract**

The objective of this project is to determine how the various production processes can be used to affect the quality, size, and consistency of the crust on bread. The group studied mathematically the processes occurring in the baking stage of bread in order to model the crust formation. The first step that the group took was a phenomenological approach. Based on the paper by B. Zanoni, C. Peri and S. Pierucci, *A study of the bread-baking process. I: A phenomenological model*, it was possible to establish the main mechanisms that take place in the bread baking process. In order to differentiate crust from the soft center, the so-called crumb, it was decided to model crust as the region of the bread that has lost all its moisture content. Heat conduction and energy conservation are the main physical laws that were considered. This led to a Stephan problem in the dough which uses the heat equation to model the temperature distribution in the crust and the crumb. The moving boundary separating the crust and crumb models the evaporation front. The temperature distribution before the evaporation front appears was found using an asymptotic analysis. This exact solution was used as an initial condition to solve the Stephan problem numerically. This numerical solution showed that the evaporation front evolves almost linearly in time, which agreed with the experimental data found in the paper by Zanoni et al. Finally, the group discussed other mechanisms that could make the model more realistic.
1 Problem Description.

The objective of this project is to determine how the various production processes can be used to affect the quality, size, and consistency of the crust on bread.

The process of conventional bread making is rather simple. The first step is mixing the ingredients, which usually include flour, water, salt, yeast, and/or baking powder. The dough is then kneaded, proved (in which the bread is placed in the refrigerator) and then kneaded again. The dough is then divided and placed in a cooking tin where it stays for a second proving stage. At this point, the dough rises as a result of the yeast or baking powder and fills the tin (see Figure 1). The last step is baking the dough and then letting it cool.

![Figure 1: Left: dough in pan. Right: risen dough.](image-url)

If one wants to produce large-scale amounts of bread this process is slow. In the factory production of bread, the same steps are followed but the process is much quicker. Large, fast machines mix the ingredients at rates of 54000 Kg/hr. These machines perform the first and second kneading stages as well as the first proving stage in about three minutes. The dough is then molded and divided (in standard pieces of 900 grams for sandwich bread) to be placed in tins, the second proving stage takes place when the dough rises to finally be baked and cooled.

In Figure 3, we observe a typical slice of sandwich bread presenting the desired shape of crumb (white part) and crust (boundary of the slice). There are some generally accepted procedures in the process of baking bread. The oven temperatures vary from about 425°F to 450°F. During the first minutes of the baking process the carbon dioxide gas within the dough expands. When the temperature of the loaf reaches 140°F the yeast dies, fermentation ceases and the alcohol produced during fermentation evaporates. Then, the pliability of the dough gradually lessens, the dough becomes set and slowly changes to bread. At this point, some moisture evaporates, the starch becomes gelatinized and the gluten and other proteins become coagulated giving the crumb some structure. After the loaf sets, the intense heat dries out the part exposed to the air and causes a crust to form. The golden brown color of the crust is the result of a chemical change in the starch, sugar and milk known as a
browning reaction (Mailard reaction), which is also known as caramelization. Within the loaf, the crumb near the crust is subjected to a temperature as high as 300°F., whereas the temperature towards the center of the loaf is about 212°F.

In the baking process some problems can arise. Particularly with the crust. Figure 4 shows the most common ones.

The objective is to study mathematically the processes occurring in the baking stage of bread in order to model the crust formation and then use these processes to alter the quality, size, and consistency of the crust. In order to achieve this goal, the group discussed the general mechanisms, read literature, and examined experimental evidence in order to understand the processes that take place. The group then decided the most important mechanisms and set up a mathematical framework in which to describe them. Once this was achieved, the group analyzed and solved the mathematical problem, interpreted the results and finally discussed modifications of the model so it would be more realistic.

2 Analysis.

2.1 Phenomenological Approach.

In order to model mathematically the formation of crust on bread, a phenomenological approach must be taken in order to understand the processes that take place while the bread is being baked. Part of these phenomenological observations are taken from the paper by B. Zanoni, C. Peri and S. Pierucci, *A study of the bread-baking process. I: A phenomenological model*. J. Food Engineering, 19 (1993), pp. 389-398. This paper also provides some experimental data such as the temperature and moisture content in different parts of the dough at different times during the baking process. There are several properties that help us differentiate *crumb* from *crust*. The most relevant ones are the shape of the bubbles, which...
are rounded in the crumb and elongated in the crust (see Figure 3), and the clear change in density, texture and moisture content from one region to the other. Another relevant mechanism that takes place during the baking process is the expansion of the center of the dough as the bread is cooked. Some key temperatures are listed below.

- 21°C: proved dough,
- 60°C: yeast dies,
- 100°C: boiling point of water,
- 150°C: final temperature of crust near pan,
- 220°C: oven temperature (this temperature can be considered uniform along the oven and constant during the baking process).

The most important mechanisms that take place as the bread is baked are the following. Heat diffuses from the oven to the center of the dough. It is worthwhile to mention that at the beginning, the dough conforms to a very uniform bulk. Due to the increment in
Figure 4: (a) Crust with small volume; (b) Crust too thick; (c) Side wall collapse; (d) Coarse crumb texture; (e) Flying on top; (f) Concave bottom; (g) Crust blisters; (h) Streak in crumb.

temperature, air trapped in the dough tends to expand, forming bubbles in the loaf. As regions of the dough reach 100°C water evaporates. Observations confirm that the crust is compressed towards the tin. There must be water movement inside the dough which changes the moisture content. As the raw dough transforms to bread, the heat diffusion rates change in the bulk of the material. Finally, when the temperature reaches 60°C, yeast dies.

One of the main tasks is to determine what physical variable allows us to define crust. Such a variable could be texture, density, moisture content, etc. In the paper by Zanoni et al, crust is defined as the region of bread that has lost all its moisture content, in other words, the region where all the water has evaporated. Based on this definition, we are able to describe the following process. Initially, heat is conducted through the dough. When the outer part of the dough reaches 100°C water evaporates. At this point, all the heat is used to evaporate water (latent heat) creating an evaporation front that moves from the outside of the dough to the center and the region behind this evaporation front is what we define as the crust.

2.2 Problem Formulation.

To model the heat transfer, we assume that heat transports only via conduction. Mathematically, we refer to this as Fourier’s law, which states that the heat flux is proportional to the gradient in temperature of a material,

\[ q = -k \nabla T, \]

where \( q \) is the heat flux, \( T \) is the temperature of the material and \( k \) is the thermal conductivity. This empirical law leads to the heat equation, a partial differential equation describing
how the temperature in a material changes in space and time,

\[
\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},
\]

where \( \kappa = k/(\rho C) \) is the thermal diffusivity of the material (\( \rho \) is the density and \( C \) is the heat capacity). The other principle to be assumed in the model is **energy conservation**, which will be important to model the evaporation front. We can now write mathematically the model for the bread baking process. Before the dough reaches 100°C, the temperature in the dough is modeled using the heat equation,

\[
\frac{\partial T}{\partial t} = \kappa_2 \frac{\partial^2 T}{\partial x^2} \quad 0 \leq x \leq L,
\]

\[-k_2 \frac{\partial T}{\partial x} = h(T_{\text{oven}} - T) \quad \text{at} \ x = 0,
\]

\[T = T_{\text{in}} = 21^\circ C \quad \text{at} \ t = 0,
\]

\[\frac{\partial T}{\partial x} = 0 \quad \text{at} \ x = L.
\]

Here, \( \kappa_2 \) denotes the thermal diffusivity of the dough. For simplicity, the mathematical model is formulated in one dimension taking \( x \) to be the transversal direction of a bread slice (see Figure 5). The boundary with the tin is \( x = 0 \) and the center of the dough is \( x = L \) (approximately equal to 5cm from the tin).

![Figure 5: Coordinate axis for a slice of bread.](image)

The second equation corresponds to the boundary condition of the tin. This is a Robin condition that models heat transfer due to radiation. The heat flux at this boundary is proportional to the difference between the temperature of the dough and the temperature of the oven (denoted as \( T_{\text{oven}} \), approximately 220°C), where \( k_2 \) is the thermal conductivity of the dough. The last equation is a Neumann condition describing the zero net flux of heat at the center of the dough. One could write the model from one end of the dough to the other. In this case, the last equation would be replaced by a condition similar to the second one. This system is equivalent to considering a “half slice” with the Neumann condition at \( x = L \).

As time evolves, the temperature of the dough increases. Once the boundary of the dough has reached 100°C, water starts to evaporate and an evaporation front is formed. The domain
is then divided into two regions, one corresponding to the crust and one corresponding to
the crumb. Mathematically we have

Region 1 (crust) \[
\begin{align*}
\frac{\partial T}{\partial t} &= \kappa_1 \frac{\partial^2 T}{\partial x^2} \quad 0 \leq x \leq s(t), \\
-k_1 \frac{\partial T}{\partial x} &= h(T_{oven} - T) \quad \text{at } x = 0, \\
T &= T_{in2} \quad \text{at } t = t_b, \\
T &= T_b \quad \text{at } x = s(t).
\end{align*}
\]

Region 2 (crumb) \[
\begin{align*}
\frac{\partial T}{\partial t} &= \kappa_2 \frac{\partial^2 T}{\partial x^2} \quad s(t) \leq x \leq L, \\
T &= T_{in2} \quad \text{at } t = t_b, \\
T &= T_b \quad \text{at } x = s(t), \\
\frac{\partial T}{\partial x} &= 0 \quad \text{at } x = L,
\end{align*}
\]

where \( T_b = 100^\circ C, \) \( t_b \) is the time that it takes to reach the boiling temperature and \( T_{in2} \) is the temperature profile at this time. All of the constants with subscript 1 correspond to the crust while those with subscript 2 correspond to the crumb. The position \( s(t) \) represents the boiling front (see Figure 5); its dynamics can be modeled using Fourier’s law and conservation of energy. We have,

\[
\frac{\text{d} \theta}{\text{d} \tilde{t}} = -k_1 \frac{\partial T}{\partial x} \bigg|_{x=s^-} + k_2 \frac{\partial T}{\partial x} \bigg|_{x=s^+},
\]

where \( l \) is the latent heat of water, \( \rho \) is the density of the dough, \( a \) is the fraction of water in the dough and \( k_i, i = 1, 2 \) is the thermal conductivity for the crust and the crumb, respectively. This equation for the front together with the heat equations on either side describes a moving boundary problem, known as a Stephan problem. The particular Stephan problem described here has no known analytical solution, and thus a numerical method is employed to find an approximate solution.

Before formulating a numerical method we nondimensionalize the problem. The following dimensionless variables are introduced,

\[
\tilde{x} = \frac{x}{L}, \quad \tilde{t} = \frac{t}{\tau}, \quad \tilde{\theta} = \frac{T - T_{in}}{T_{oven} - T_{in}}, \quad \tilde{s} = \frac{x}{L},
\]

where \( \tau \) is the baking time, approximately 30 minutes, \( L \) is half the width of a slice of bread. The equations now become (dropping tildes):

\[
\text{Dough} \begin{cases}
\frac{\partial \theta}{\partial \tilde{t}} = B \frac{\partial^2 \theta}{\partial x^2} & 0 \leq x \leq 1, \ 0 \leq t \leq t_b, \\
\frac{\partial \theta}{\partial x} = -Nu(1 - \theta) & \text{at } x = 0, \\
\theta = 0 & \text{at } t = 0, \\
\frac{\partial T}{\partial x} = 0 & \text{at } x = 1.
\end{cases}
\]
This equation corresponds to heat transfer before the outside part of the dough reaches the boiling temperature. Here, \( B = \kappa_2 \tau / L^2 \), \( Nu = hL/k_1 \), and \( t_b \) is the dimensionless time required to reach the boiling temperature. Once the boiling front is created, the equations modeling heat transport become:

**Region 1 (crust)**

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= A \frac{\partial^2 \theta}{\partial x^2} & 0 \leq x \leq s(t), \\
\frac{\partial \theta}{\partial x} &= -Nu(1 - \theta) & \text{at } x = 0, \\
\theta &= \theta_{in2} & \text{at } t = t_b, \\
\theta &= \theta_b & \text{at } x = s(t).
\end{align*}
\]

**Region 2 (crumb)**

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= B \frac{\partial^2 \theta}{\partial x^2} & s(t) \leq x \leq 1, \\
\theta &= \theta_{in2} & \text{at } t = t_b, \\
\theta &= \theta_b & \text{at } x = s(t), \\
\frac{\partial \theta}{\partial x} &= 0 & \text{at } x = 1,
\end{align*}
\]

On the interface

\[
\left\{ \left( \frac{C}{k_1} \right) \frac{ds}{dt} = - \frac{\partial \theta}{\partial x} \bigg|_{x=s^{-}} + \left( \frac{k_2}{k_1} \right) \frac{\partial \theta}{\partial x} \bigg|_{x=s^{+}} \right. 
\]

where \( A = \kappa_1 \tau / L^2 \), \( \theta_{in2} \) is the temperature profile at \( t_b \), \( \theta_b \) is the (dimensionless) boiling temperature, and

\[
C = \frac{lap\rho L^2}{\tau(T_{oven} - T_{in})}
\]

is a dimensionless latent heat.

### 2.3 A Few Analytical Results.

At this point, the group divided into two teams. One team studied various model problems analytically, whereas the other team coded and solved the equations numerically. Some analysis can be done for the problem before the boiling temperature is reached. For

\[
\begin{align*}
v_t &= Kv_{xx} & 0 \leq x \leq 2, \\
v_x - hv &= 0 & x = 0, \\
v_x + hv &= 0 & x = 2, \\
v &= -1 & t = 0,
\end{align*}
\]

the problem has an exact solution given by

\[
v = \sum_{n=1}^{\infty} A_n X_n \exp(-K\alpha^2t),
\]
where
\[ X_n = \cos \alpha_n x + \frac{h}{\alpha_n} \sin \alpha_n x. \]

The discrete eigenvalues are given by
\[ \tanh 2\alpha = \frac{2\alpha h}{\alpha^2 - h^2} \]
and the coefficients, \( A_n \), are given by
\[ -1 = \sum A_n X_n. \]

Even though we have this exact solution, it does not provide much information about the dynamics of the process. A better approach is to use an asymptotic analysis. It is found that the constant \( Nu \) is relatively large compared to the rest of the constants in the non-dimensionalized problem (approximately 4 orders of magnitude). We proceed to rescale the problem as follows,
\[ x = \left( \frac{1}{Nu} \right)^\alpha \tilde{x}, \]
\[ t = \left( \frac{1}{Nu} \right)^\beta \tilde{t}. \]

We introduced these new variables in order to obtain some useful information when considering the limit \( Nu \rightarrow \infty \). Taking \( \alpha = 1 \) and \( \beta = 2 \), we find that
\[
\begin{align*}
\theta_t &= B\theta_{xx} \quad 0 \leq x < \infty, \\
\theta_x &= h(\theta - 1) \quad x = 0, \\
\theta_x &= 0 \quad x = \infty, \\
\theta &= -1 \quad t = 0.
\end{align*}
\]

This problem has an exact solution given by
\[
\theta = 1 - \left( \text{erf} \left( \frac{x}{2\sqrt{Bt}} \right) + \exp (hx + h^2 Bt) \text{erfc} \left( \frac{x}{2\sqrt{Bt}} + h\sqrt{Bt} \right) \right).
\]

At the boiling temperature, \( t_b \) can be calculated from the implicit equation
\[
\theta_b = 1 - \exp (h^2 Bt_b) \text{erfc} \left( h\sqrt{Bt_b} \right).
\]

A profile of the temperature at the boiling time \( t_b \) is shown in Figure 6.

### 2.4 Numerical Solutions.

The previous exact solution provides the initial condition for the numerical solution of the Stefan problem. In order to ensure that the moving boundary is always on a grid point, a change of variables is introduced so that the interval of integration is \([0, 1]\), where \( x = 0.5 \) is
Figure 6: Temperature profile at initial boiling point.

Figure 7: Temperature profile in the two regions, crust (to the left of the evaporation front) and crumb (to the right of the evaporation front).

A representative plot of the temperature profile indicating the boiling front is shown in Figure 7.

A graph of the position of the boiling front as a function of time is shown in Figure 8. We
can observe from this graph that the front evolves nearly linear in time. This result agrees with the experimental data from Zanoni et al.

![Graph showing the evolution of the evaporation front as a function of time.](image)

Figure 8: Evolution of the evaporation front as a function of time.

2.5 Further Analysis.

There are several mechanisms and phenomena that could be taken into account in order to improve the model. In the modeling presented here is was assumed that heat is only transported via heat conduction. A better approach might be to consider other mechanisms such as heat transport due to convection. A different approach can also be taken to model the formation of crust. One could devise a two-front model in which one front models the evaporation of water (as is done here) and a second front models the crust compression that causes the change in density and the formation of elongated bubbles near the crust. The reason for this model is that evaporation of water allows CO$_2$ to escape the bubbles in the dough due to vaporization. This causes the elongation of the bubbles in the crust and allows the expansion of the dough from the center outward.

3 Conclusions.

Much discussion and research was done in order to establish the main mechanisms that have to be considered in order to model crust formation. The paper by Zanoni et al provided useful information not only for the phenomenological approach, but also to corroborate some results. It was decided then, that the region of bread that we know as crust was to be modeled as the region of the dough that has lost all its moisture content. During
baking, heat conduction and energy conservation are the main principles to be considered. Mathematically, this results in the heat equation, which models the temperature distribution in the dough. Once the outer part of the dough has reached 100° C, the domain divides into two parts, crust and crumb. The heat equation still models the temperature distribution in both regions with a moving boundary in between, defining a Stephan problem. Even though an exact solution could be found for the temperature distribution before the evaporation front is created, this solution did not provide much information about the problem. A better approach was to consider an asymptotic limit for the non-dimensionalized equations. In this limit, the group could still find an exact solution which provided an initial condition for the temperature when the front is formed. This profile was used to solve the problem numerically using a finite difference scheme. It could be observed from the numerical solution how the front evolves in time, showing a linear behavior. This result agrees with the experimental data found in the paper by Zanoni et al. Finally, the group discussed what other mechanisms could be incorporated into the model to make it more realistic.