Math 4820
D. Schwendeman

Problem Set 8
Due: Monday, 4/19/10

1. Consider the heat equation

\[ Lu = u_t - ku_{xx} = 0, \]

where \( k > 0 \) is a constant thermal diffusivity.

(a) Consider the finite-difference approximation

\[ L_h v^n_j = \frac{1}{\Delta t} \delta_{t} v^n_j - \frac{k}{\Delta x^2} \left[ \theta \delta_{x}^2 v^{n+1}_j + (1 - \theta) \delta_{x}^2 v^n_j \right] = 0, \]

where \( \theta \in [0, 1] \) is a parameter. Show that \( L_h v^n_j = 0 \) is a consistent approximation of the heat equation \( Lu = 0 \) for any choice of \( \theta \). Find a value of \( \theta \) such that the approximation is second-order accurate in both space and time.

(b) Consider the finite-difference approximation

\[ L_h v^n_j = \frac{1}{2\Delta t} \delta_{0t} v^n_j - \frac{k}{\Delta x^2} \left[ v^n_{j-1} - (v^n_{j-1} + v^n_{j+1}) + v^n_{j+1} \right] = 0. \]

Find the order of accuracy of the approximation assuming that \( k\Delta t/\Delta x^2 = \text{constant} \) (independent of \( \Delta t \) and \( \Delta x \)).

2. A consistent approximation of the heat equation \( u_t = ku_{xx} \) has the form

\[ v^{n+1}_j = v^n_j + \frac{\nu}{3} \left[ \delta_r^2 v^{n+1}_j + 2\delta_r^2 v^n_j \right], \quad \nu = \frac{k\Delta t}{\Delta x^2} \]

Consider Fourier modes of the form \( v^n_j = a^n e^{i\alpha x_j} \), where \( a \) is an amplitude factor, \( \alpha \) is a wave number, and \( x_j = j\Delta x \), to determine a constraint on the parameter \( \nu \) such that the approximation is stable, i.e. \( |a| \leq 1 \).

3. Heat flow in a unit disk with circular symmetry solves the PDE

\[ u_t = k \left( u_{rr} + \frac{1}{r} u_r \right) + Q(r), \quad 0 < r \leq 1, \quad t > 0 \]

with initial and boundary conditions taken to be

\[ u(r, 0) = f(r), \quad u_r(0, t) = 0, \quad u_r(1, t) + \alpha u(1, t) = \beta \]

Here, \( r \) is radial distance from the center of the disk and \( t \) is time as before.

(a) The steady state solution is the long-time solution of the differential equation with \( u_t \) set to zero and satisfying the boundary conditions. Find the steady state solution for the case \( k = \alpha = \beta = 1 \) and \( Q(r) = r^2 - 1 \). (Hint: recall one of the problems in Problem Set 7.)

(b) Write a code to solve the initial-boundary-value problem using the finite-difference approximation

\[ v^{n+1}_j = v^n_j + k\Delta t \left( \frac{1}{\Delta r^2} \delta_r^2 v^n_j + \frac{1}{2r_j \Delta r} \delta_{0r} v^n_j \right) + \Delta t Q(r_j), \]

where \( r_j = j\Delta r, \Delta r = 1/N, \) and \( v^0_j = f(r_j) \), and with suitable approximations at the boundaries \( r = 0 \) and \( r = 1 \). Compute the solution for the case \( k = \alpha = \beta = 1, Q(r) = r^2 - 1, \) and \( f(r) = r^2/3 \). Use \( N = 50 \) and verify that the numerical solution approaches the correct steady state solution as \( t \) increases. Plot \( u(0, t) \) versus \( t \) from your numerical solution.