Math 4820  
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Problem Set 7  
Due: 
Monday, 4/12/10

1. Consider the linear BVP

\[ u'' + \frac{1}{t} u' = Q(t), \quad 0 < t \leq 1, \quad u'(0) = 0, \quad u'(1) + \alpha u(1) = \beta \]

(a) Show that \( \lim_{t \to 0} \frac{1}{t} u' = u'' \). (Hint: use L'Hôpital's rule.)

(b) Write a code to solve the BVP numerically using the finite-difference scheme

\[
\frac{1}{h^2} \delta_t^2 v_j + \frac{1}{2h} \delta_{tt} v_j = Q(t_j), \quad \frac{1}{2h} \delta_{tt} v_0 = 0, \quad \frac{1}{2h} \delta_{tt} v_N + \alpha v_N = \beta
\]

with \( t_j = jh, \ h = 1/N \). Note that the finite-difference approximation is singular at \( t_j = 0 \). Use the result in part (a) to redefine the approximation of the ODE when \( t_j = 0 \) (i.e. when \( j = 0 \)). Eliminate the ghost values \( v_{-1} \) and \( v_{N+1} \) from the resulting equations so that the linear system is tridiagonal (similar to what was discussed in class).

(c) Use your code to solve the BVP for the case \( Q(t) = 1 - t^2 \) and \( \alpha = \beta = 1 \). Compare the numerical solution with the exact solution for \( N = 40, 80 \) and 160, and verify the method is second-order accurate. Plot the numerical solution for \( N = 40 \) and the exact solution on the same graph.

2. Consider the nonlinear BVP

\[ u'' - \cosh(\lambda u) = 0, \quad -1 \leq t \leq 2, \quad u(-1) = 1, \quad u(2) = 3 \]

Note that when \( \lambda = 1 \), the BVP agrees with the one considered in Problem Set 6. A finite-difference approximation of the BVP is

\[
\frac{1}{h^2} \delta_t^2 v_j - \cosh(\lambda v_j) = 0, \quad j = 1, 2, \ldots, N-1, \quad v_0 = 1, \quad v_N = 3.
\]

For a path-following procedure, a key step is to determine how \( \mathbf{v} = (v_0, v_1, \ldots, v_N) \) changes with the parameter. Let us consider this step alone (and not the whole path-following procedure). Write a code to determine \( \mathbf{v}_\lambda \), the derivative of \( \mathbf{v} \) with respect to \( \lambda \), when \( \lambda = 1 \). (Hint: this code should be a minor modification of the one used in Problem Set 6.) Use \( N = 60 \) and plot \( \mathbf{v}_\lambda \) versus \( t_j \).

3. Consider the heat equation

\[ u_t = k u_{xx} + Q(x, t), \quad 0 \leq x \leq L, \quad t \geq 0, \]

with initial condition \( u(x, 0) = f(x) \) and Neumann-type boundary conditions \( u_x(0, t) = \alpha(t) \) and \( u_x(L, t) = \beta(t) \). An explicit finite-difference approximation is

\[ v_j^{n+1} = v_j^n + \nu \delta_x^2 v_j^n + \Delta t Q(x_j, t_n), \quad 0 \leq j \leq N, \quad n \geq 0 \]

with \( v_j^0 = f(x_j) \), \( \delta_x v_0^n = 2\Delta x \alpha(t_n) \), and \( \delta_x v_N^n = 2\Delta x \beta(t_n) \). Here \( \nu = k\Delta t/\Delta x^2 \).

(a) Find \( Q(x, t), \ f(x), \ \alpha(t) \) and \( \beta(t) \) so that \( u(x, t) = \cos(2t) \sin(3x) \) is an exact solution of the initial-boundary-value problem (IBVP).

(b) Write a code to solve the IBVP numerically using the finite-difference approximation above. Test your code using \( k = L = 1 \) and the functions found in part (a) so that an exact solution is known. Compute the error in the numerical solution at \( t = .2 \) for \( N = 40, 80 \) and 160 with \( \Delta t \) chosen such that \( \nu \) has the same value (less than 0.5) for each \( N \). Verify the rate of convergence for the approximation.