1. Consider the linear system $A v = d$ generated by the finite difference equations

$$\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} - p(t_j)\frac{v_{j+1} - v_{j-1}}{2h} - q(t_j)v_j = r(t_j), \quad j = 1, \ldots, N - 1$$

where $v_0 = \alpha$, $v_N = \beta$, and $p(t)$, $q(t)$, and $r(t)$ are smooth functions. Show that $A$ is nonsingular if $q(t) > 0$ and $h \leq 2/\max|p(t)|$.

2. Consider the linear boundary-value problem

$$\frac{t^2}{3} u'' - \frac{t}{2} u' - \frac{1}{2} u = \frac{1}{t^2}, \quad 1 \leq t \leq 2, \quad u(1) = \frac{7}{15}, \quad u(2) = \frac{19}{30}.$$ 

Find a numerical solution of the BVP using the finite-difference approximation discussed in class. Solve the difference equations using $N = 40, 80$ and $160$ grid cells. Find the global error $E = \max|v_j - u(t_j)|$ for each case and use this error to verify the order of accuracy of the approximation. Plot the numerical solution and the exact solution on the same graph for $N = 40$.

3. Consider the boundary-value problem

$$u'' + f(t, u) = 0 \quad a \leq t \leq b$$

with boundary conditions $u(a) = \alpha$ and $u(b) = \beta$, and where $f$ is a smooth function.

(a) Consider the finite-difference approximation

$$\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + f(t_j, v_j) = 0, \quad j = 1, \ldots, N - 1$$

with $v_0 = \alpha$ and $v_N = \beta$. Find the first two nonzero terms in the expansion of the truncation error $\tau_j$.

(b) An alternate finite-difference approximation to the one in (a) is

$$\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + \frac{f(t_{j+1}, v_{j+1}) + 10f(t_j, v_j) + f(t_{j-1}, v_{j-1})}{12} = 0, \quad j = 1, \ldots, N - 1$$

with $v_0 = \alpha$ and $v_N = \beta$. Determine the order of accuracy of this approximation.

4. Solve the nonlinear BVP

$$u'' - \cosh(u) = 0, \quad -1 \leq t \leq 2, \quad u(-1) = 1, \quad u(2) = 3$$

using the second-order finite-difference approximation in problem 3, part (a) with $h = 3/N$ and $N = 60$. Use Newton’s method to solve the nonlinear algebraic equations as discussed in class. Note that there is always an issue of the starting guess for Newton’s method. For this case, it is sufficient to assume that $v_j = 1 + 2(t_j + 1)/3$ as a starting guess. Verify that your implementation of Newton’s method converges quadratically and plot solution.