1. Consider the constant coefficient system $y' = Ay$ for the two cases
   
   (i) $A = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$,  
   (ii) $A = \begin{bmatrix} -1 & 5 \\ -2 & -3 \end{bmatrix}$

   (a) For each system, find the largest value of the step size $h$ such that the Euler’s method is absolutely stable.

   (b) Repeat part (a) for the explicit midpoint method.

2. The trapezoidal method is an implicit 2-stage RK method which can be written in the form
   
   $$w_n = w_{n-1} + \frac{h}{2} [f(t_{n-1}, w_{n-1}) + f(t_n, w_n)]$$

   (a) Apply the trapezoidal method to the test equation $y' = \lambda y$ to show that
   
   $$w_n = \left(1 + \frac{h\lambda}{2}\right) w_{n-1}$$

   (b) Show that the trapezoidal method is A-stable.

3. Consider the initial-value problem
   
   $$\frac{d}{dt} s(t) = -s + (s + 1) c, \quad s(0) = 1$$
   $$\epsilon \frac{d}{dt} c(t) = s - (s + 2) c, \quad c(0) = 0$$

   where $\epsilon$ is a positive constant. This IVP arises as a model of a biochemical system. Let us take $\epsilon = .02$, and consider the interval $0 \leq t \leq 1$. Note that for small values of $\epsilon$ this is a stiff problem. There is an initial layer on the time scale $t \sim O(\epsilon)$ whose solution satisfies
   
   $$\epsilon \frac{d}{dt} c(t) = s - (s + 2) c, \quad c(0) = 0, \quad \text{with } s = 1$$  \hspace{1cm} (1)

   to a first approximation, and an “outer” solution on an $O(1)$ time scale determined by the equations
   
   $$\frac{d}{dt} s(t) = -s + (s + 1) c, \quad s(0) = 1, \quad \text{with } s - (s + 2) c = 0. \hspace{1cm} (2)$$

   (a) Solve equations (1) and (2) to find $c(t)$ and $s(t)$ in the initial layer and outer region, respectively.

   (b) The backward Euler method for this IVP is
   
   $$s_n = s_{n-1} - h \left[ s_n - (s_n + 1) c_n \right], \quad s_0 = 1$$
   $$c_n = c_{n-1} + \frac{h}{\epsilon} \left[ s_n - (s_n + 2) c_n \right], \quad c_0 = 0$$

   Compute a solution for $0 \leq t \leq 1$ using $h = 1/100$. Plot your numerical solution for $s(t)$ and $c(t)$ versus $t$ and compare it with the solutions found in part (a). Describe your results. Are the solutions accurate and/or stable? (Hint: even though the backward Euler method is implicit, you can solve for $s_n$ and $c_n$ explicitly for this problem. To do this, first eliminate $c_n$ from the two equations. The resulting equation is quadratic in $s_n$ and can be solved using the quadratic formula.)