Math 4820  
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Problem Set 10  
Due:  
Monday, 5/10/10

1. The first-order wave equation is  
\[ Lu = u_t + cu_x = 0. \]

Find the leading terms in the truncation error for the following approximations:

(a)  
\[ L_h v^n_j = \frac{1}{\Delta t} \left( v^{n+1}_j - \frac{v^n_{j+1} + v^n_{j-1}}{2} \right) + \frac{c}{2\Delta x} (v^n_{j+1} - v^n_{j-1}) = 0 \]

(b)  
\[ L_h v^n_j = \frac{1}{\Delta t} \left( v^{n+1}_j - v^n_j \right) + \frac{c}{2\Delta x} (v^n_{j+1} - v^n_{j-1}) - \frac{c^2\Delta t}{2\Delta x^2} (v^n_{j+1} - 2v^n_j + v^n_{j-1}) = 0 \]

What is the order of accuracy of each approximation assuming \( \Delta t \) is proportional to \( \Delta x \).

2. The Lax-Wendroff method for the first-order wave equation \( u_t + cu_x = 0 \) has the form

\[ v^{n+1}_j = v^n_j - \sigma \frac{\Delta t}{\Delta x} v^n_j + \frac{\sigma^2}{2} \frac{\Delta t}{\Delta x} v^n_{j+1} + \frac{\sigma^2}{2} \frac{\Delta t}{\Delta x} v^n_{j-1}, \quad \sigma = \frac{c\Delta t}{\Delta x}. \]

Consider Fourier modes of the form \( v^n_j = a^n e^{i\alpha x_j}, \) where \( a \) is an amplitude factor, \( \alpha \) is a wave number, and \( x_j = j\Delta x \), to determine a constraint on the parameter \( \sigma \) such that the approximation is stable. Hint: consider \( |a|^2 = (\text{Re}(a))^2 + (\text{Im}(a))^2 \) and find \( \sigma \) such that \( |a|^2 \leq 1 \).

3. Consider the initial-boundary-value problem

\[ u_t + cu_x = 0, \quad 0 < x < L, \quad t > 0 \]

with \( u(x, 0) = x \sin(x) \) and \( u(0, t) = 0 \). Write a code to solve the IBVP using the Lax-Wendroff method given in problem 2. Use

\[ v^n_0 = x_j \sin(x_j), \quad v^n_0 = 0, \quad v^n_N = 2v^n_{N-1} - v^n_{N-2}, \]

where \( x_j = j\Delta x, \Delta x = L/N, j = 0, 1, \ldots, N \). Run your code for the case \( c = 0.95 \) and \( L = 10 \). Compare the numerical solution with the exact solution at \( t = 1 \) for \( N = 500, 1000 \) and 2000. Use \( \Delta t = \Delta x \) for each case and and verify that the method is second-order accurate. Plot the numerical solution for \( N = 500 \) and the exact solution on the same graph.