1. (a) Find the Green’s function $G(x, \hat{x})$ solving

$$G_{xx} - k^2 G = \delta(x - \hat{x}), \quad G(0, \hat{x}) = G_x(L, \hat{x}) = 0$$

where $k$ is a real positive constant

(b) Find the Green’s function $G(\rho, \hat{\rho})$ solving

$$(\rho^2 G_\rho)_\rho - n(n + 1)G = \delta(\rho - \hat{\rho}), \quad |G(0, \hat{\rho})| < \infty, \quad G(1, \hat{\rho}) = 0$$

where $n$ is a non-negative integer.

2. The steady state temperature $u(\rho, \phi)$ in a sphere (assuming axial symmetry) satisfies

$$\frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) = Q(\rho, \cos \phi), \quad 0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi$$

where $Q$ is a given heat source and $u(1, \phi) = 0$. Using $x = \cos \phi$ gives

$$\frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{\partial}{\partial x} \left( (1 - x^2) \frac{\partial u}{\partial x} \right) = Q(\rho, x), \quad 0 \leq \rho \leq 1, \quad -1 \leq x \leq 1$$

with $u(1, x) = 0$. Find a solution of the form

$$u(\rho, x) = \sum_{n=0}^{\infty} R_n(\rho) P_n(x)$$

where $P_n(x)$ is the $n$th degree Legendre polynomial. (Hint: you can use the Green’s function in problem 1(b) to find the solution of the differential equation for $R_n(\rho)$.)

3. Consider

$$u_{xx} + u_{yy} = f(x, y) \quad x \geq 0, \quad y \geq 0$$

with

$$u_x(0, y) = 0 \quad \text{and} \quad u(x, 0) = g(x)$$

(a) Obtain the corresponding Green’s function using the method of images.

(b) Solve for $u(x, y)$ using Green’s formula. (You may assume that $u$ and its derivatives are well-behaved at infinity.)