1. Use a suitable eigenfunction expansion to determine the temperature $u(r, t)$ satisfying

$$\frac{\partial u}{\partial t} = \frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + Q(r, t), \quad 0 \leq r \leq a, \quad t \geq 0$$

with

$$u(r, 0) = f(r), \quad u(0, t) \text{ bounded}, \quad u(a, t) = 0$$

2. A stretched string with linear restoring force satisfies

$$u_{tt} = c^2 u_{xx} - \alpha u, \quad 0 < x < \pi, \quad t > 0,$$

where $c$ and $\alpha$ are positive constants. The string is at rest initially, i.e. $u(x, 0) = u_t(x, 0) = 0$, and its displacement is fixed on its right end, i.e. $u(\pi, t) = 0$. On the left, the displacement of the string is given by $u(0, t) = A \sin \omega t$, where $\omega$ is the frequency of the periodic displacement. Determine the solution in the form of an eigenfunction expansion. For what values of $\omega$ does resonance occur? (Hint: for this case it is not valid to differentiate the eigenfunction expansion term by term.)

3. Solve

$$u_{xx} + u_{yy} = Q(x, y), \quad 0 \leq x \leq L, \quad 0 \leq y \leq H$$

with

$$u_x(0, y) = u_x(L, y) = 0, \quad u(x, 0) = u(x, H) = 0$$

using (a) an expansion of the form

$$u(x, y) = \sum_n A_n(y) \phi_n(x)$$

where $\phi_n(x)$ are suitable one-dimensional eigenfunctions, and (b) an expansion of the form

$$u(x, y) = \sum_\lambda C_\lambda \Phi_\lambda(x, y)$$

where $\Phi_\lambda(x, y)$ are suitable two-dimensional eigenfunctions of a Helmholtz equation.
4) \[ u_t = \frac{k}{r} (ru_r)_r + \Theta(r, t) \]
\[ u(r, 0) = f(r) \]
\[ u(0, t) \text{ or }, u(a, t) = 0. \]

Let
\[ u = \sum_{n=1}^{\infty} C_n(t) \phi_n(r) \]

where
\[ \phi_n(r) = J_0(\sqrt{\lambda_n} r) \]
\[ \sqrt{\lambda_n} a = 2n, \quad J_0(2n) = 0 \]

Now,
\[ r u_r = \sum_{n=1}^{\infty} C_n r \phi_n' \]
\[ \frac{1}{r} (ru_r)_r = \sum_{n=1}^{\infty} C_n \frac{1}{r} (r \phi_n')' = -\lambda_n \phi_n \]
\[ u_t = \sum_{n=1}^{\infty} C_n' \phi_n \]
So
\[
\sum_{n=1}^{\infty} \left( C_n' + \lambda_n k C_n \right) \phi_n = Q(r, t)
\]

Let
\[
g_n(t) = \frac{1}{N_n} \int_0^a Q(r, t) \phi_n(r) r dr
\]
\[
N_n = \int_0^a \phi_n^2(r) r dr
\]

Gives
\[
C_n' + \lambda_n k C_n = g_n
\]

Solve:
\[
C_n(t) = e^{-\lambda_n k t} \int C_n(0) + \int_0^t e^{\lambda_n k \tau} g_n(\tau) d\tau
\]

where
\[
C_n(0) = \frac{1}{N_n} \int_0^a f(r) \phi_n(r) r dr
\]
5) \[ u_{tt} = c^2 u_{xx} - \alpha u \quad 0 < x < \pi, \quad t > 0 \]
\[ u(x, 0) = u_t(x, 0) = 0 \]
\[ u(0, t) = A \sin \omega t, \quad u(\pi, t) = 0 \]

Let \( u = \sum_{n=1}^{\infty} B_n(t) \sin(nx) \).

\[ B_n = \frac{2}{\pi} \int_{0}^{\pi} u(x, t) \sin(nx) \, dx \]

Multiply PDE by \( \frac{2}{\pi} \sin(nx) \) and \( \int_{0}^{\pi} \, dx \)

\[ \Rightarrow B_n'' = c^2 \frac{2}{\pi} \int_{0}^{\pi} u_{xx} \sin(nx) \, dx - \alpha B_n \]

\[ = c^2 \frac{2}{\pi} \left[ u_x \sin(nx) \right]_0^{\pi} \]

\[ - n \int_{0}^{\pi} u_x \cos(nx) \, dx \bigg| - \alpha B_n \]

\[ = - \frac{nc^2}{\pi} \left[ u \cos(nx) \right]_0^{\pi} \]

\[ + n \int_{0}^{\pi} u \sin(nx) \, dx \bigg| - \alpha B_n \]

\[ = \frac{2c^2}{\pi} A \sin \omega t - nc^2 B_n - \alpha B_n \]
\[ \Rightarrow \beta_n'' + (\alpha + \pi^2 \varepsilon^2) \beta_n = \frac{2\varepsilon^2 n}{\pi^2} A \sin \omega t. \]

\[ \beta_n(0) = \beta_n'(0) = 0 \]

Solu.

\[ \beta_n(t) = K_n \left[ \sin \omega t - \frac{\omega}{\beta_n} \sin \beta_n t \right] \]

where

\[ \beta_n^2 = \alpha + \pi^2 \varepsilon^2. \]

\[ K_n = \frac{2\varepsilon^2 n A}{\pi \left( \beta_n^2 - \omega^2 \right)} \]

Resonance when \( \omega = \beta_n \), \( n = 1, 2, \ldots \)
1) Solve

\[ u_{xx} + u_{yy} = Q(x, y) \quad 0 \leq x \leq L \]
\[ 0 \leq y \leq H \]

with

\[ u(x, 0) = u(x, H) = 0 \]
\[ u(0, y) = u(L, y) = 0 \]

a) Use

\[ u = \sum_{n=0}^{\infty} A_n(y) \phi_n(x) \]

where \( \phi_n = \cos \left( \frac{n\pi x}{L} \right) \), \( n = 0, 1, 2, \ldots \)

Coefficients solve

\[ A_n'' - \left( \frac{n\pi}{L} \right)^2 A_n = g_n(y) \quad A_n(0) = A_n(H) = 0 \]

where

\[ g_n = \frac{1}{N_n} \int_0^L Q(x, y) \phi_n(x) \, dx \]

\[ N_n = \int_0^L \phi_n^2(x) \, dx = \frac{L}{2} \quad \text{if} \quad n = 0 \]
\[ N_n = \frac{L}{4} \quad \text{if} \quad n \neq 0 \]

Solution

\[ A_n(y) = \int_0^H g_n(y, \eta) \delta_n(\eta) \, d\eta \]
\[ g_0 = \begin{cases} -\frac{1}{H} y (H-y) & y > \frac{H}{2} \\ -\frac{1}{H} y (H+y) & y < \frac{H}{2} \end{cases} \]

and for \( n > 0 \),

\[ g_n = \frac{-1}{\pi^2} \frac{\sinh \frac{n \pi y}{L}}{\sinh \frac{n \pi H}{L}} \int \frac{\sinh \frac{n \pi H}{L}}{\sinh \frac{n \pi y}{L}} (H-y) \, dy \]

b) Use

\[ u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \phi_{nm} (x, y) \]

where \( \phi_{nm} = \cos \frac{n \pi x}{H} \sin \frac{m \pi y}{L} \)

\[ \nabla^2 u = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} (-\lambda_{nm} \phi_{nm}) = 0 \]

where \( \lambda_{nm} = \left( \frac{n \pi}{H} \right)^2 + \left( \frac{m \pi}{L} \right)^2 \)

so

\[ C_{nm} = -\frac{1}{\lambda_{nm} N_{nm}} \int_{0}^{H} \int_{0}^{L} Q(x, y) \phi_{nm} (x, y) \, dx \, dy \]

\[ N_{nm} = \int_{0}^{H} \int_{0}^{L} \phi_{nm}^2 \, dx \, dy \]