1. Solve $\nabla^2 u = 0$ in the annular cylinder $1 \leq r \leq 2, -\pi \leq \theta \leq \pi, 0 \leq z \leq 1$ subject to the boundary conditions

$$u(1, \theta, z) = 0, \quad u(2, \theta, z) = \gamma(z) \cos \theta, \quad u_z(r, \theta, 0) = u_z(r, \theta, 1) = 0.$$  

(Hint: consider $u(r, \theta, z) = v(r, z) \cos \theta$ and note that $r$ is bounded away from zero for this problem.)

2. Determine expressions for the eigenvalues and eigenfunctions for the following eigenvalue problems.

   (a) $\rho^2 R'' + 2 \rho R' + (\lambda \rho^2 - 2) R = 0, \quad 0 < \rho \leq 3, \quad R(0) \text{ bounded and } R(3) = 0$

   (b) $(1 - x^2) g'' - 2x g' + \lambda g = 0, \quad 0 \leq x < 1, \quad g(0) = 0 \text{ and } g(1) \text{ bounded}$

3. The steady state temperature $u(\rho, \phi)$ in a spherical shell (assuming axial symmetry) satisfies

$$\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) = 0, \quad 1 \leq \rho \leq a, \quad 0 \leq \phi \leq \pi$$

with $u(1, \phi) = 0$ and $u(a, \phi) = f(\cos \phi)$. Let $x = \cos \phi$ and $u = R(\rho) g(x)$, and find the separated equations for $R(\rho)$ and $g(x)$. Solve the eigenvalue problem for $g(x)$ and the Cauchy-Euler equation for $R$. Determine the solution for the steady state temperature $u(\rho, \phi)$.

4. Radially symmetric oscillations in a spherical cavity satisfy

$$u_{tt} = \frac{c^2}{\rho^2} \left( \rho^2 u_{\rho} \right)_\rho, \quad 0 < \rho \leq a, \quad t \geq 0$$

with

$$u(\rho, 0) = \alpha(\rho), \quad u_t(\rho, 0) = 0, \quad u(a, t) = 0$$

Determine $u(\rho, t)$ and an expression for the frequencies of vibration.