1. Solve the initial-value problem

\[ x' = \begin{bmatrix} 7 & 9 \\ -6 & -8 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \]  

(1)

Describe the behavior of the solution as \( t \to \infty \).

We start by assuming solutions of the form \( ze^{rt} \) and solve the system

\[
\begin{bmatrix} 7 - r & 9 \\ -6 & -8 - r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(2)

for eigenvalues \( r \) and eigenvectors \( z \). For (2) to hold, the matrix must be singular (zero determinant).

\[
\begin{vmatrix} 7 - r & 9 \\ -6 & -8 - r \end{vmatrix} = (7 - r)(-8 - r) - (-6)(9) = r^2 + r - 2
\]

(3)

\[
= (r + 2)(r - 1) \quad \Rightarrow \quad r = 1, -2.
\]

(4)

(5)

\( r_1 = 1: \)

\[
\begin{bmatrix} 6 & 9 \\ -6 & -9 \end{bmatrix} \Rightarrow 6z_1 + 9z_2 = 0 \Rightarrow z_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.
\]

(6)

\( r_2 = -2: \)

\[
\begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix} \Rightarrow 9z_1 + 9z_2 = 0 \Rightarrow z_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

(7)

Thus, the general solution is

\[ x = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}. \]

(8)

Lastly we apply the initial condition.

\[
\begin{bmatrix} 4 \\ -3 \end{bmatrix} = x(0) = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

(9)

\[
\begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} \Rightarrow c_1 = c_2 = 1.
\]

(10)

So the solution for the initial value problem is

\[ x = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}. \]

(11)
\[
\lim_{t \to \infty} x = \lim_{t \to \infty} \left( \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \right) = \begin{bmatrix} \infty \\ -\infty \end{bmatrix}
\] (12)

2. Consider the system
\[
x' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} x.
\] (13)

Find the general solution of the system and express this solution in terms of real-valued functions.

We start by assuming solutions of the form \( ze^{rt} \) and solve the system
\[
\begin{bmatrix} 1 - r & -1 \\ 5 & -3 - r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (14)

for eigenvalues \( r \) and eigenvectors \( z \). For (14) to hold, the matrix must be singular (zero determinant).
\[
\begin{vmatrix} 1 - r & -1 \\ 5 & -3 - r \end{vmatrix} = (1 - r)(-3 - r) - (-1)(5) = r^2 + 2r + 2 = 0 \quad \Rightarrow \quad r = -1 \pm i
\] (15)

\( r_1 = -1 + i: \)
\[
\begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 5 & -2 - i \\ 5 & -2 - i \end{bmatrix} \Rightarrow 5z_1 - (2 + i)z_2 = 0
\] (17)

\[
z_1 = \begin{bmatrix} 1 \\ 2 - i \end{bmatrix}
\] (18)

Note that we only need one eigenvalue/eigenvector pair, but if we did need \( z_2 \), it would be \( z_2 = \overline{z_1} \), since \( r_2 = \overline{r_1} \). Since we have \( r_1 \) and \( z_1 \), then one solution is
\[
x_1 = \begin{bmatrix} 1 \\ 2 - i \end{bmatrix} e^{(-1+i)t}
\] (19)

\[
= \begin{bmatrix} 1 \\ 2 - i \end{bmatrix} e^{-t} (\cos t + i \sin t)
\] (20)

\[
= e^{-t} \begin{bmatrix} \cos t \\ 2 \cos t + \sin t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin t \\ 2 \sin t - \cos t \end{bmatrix}
\] (21)

Thus our general solution is
\[
x = c_1 e^{-t} \begin{bmatrix} \cos t \\ 2 \cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ 2 \sin t - \cos t \end{bmatrix}
\] (22)

3. Consider the mass-spring-damper equation
\[
mu'' + \gamma u' + ku = 0
\] (23)
(a) Let $x_1 = u$ and $x_2 = u'$ and write the second-order equation for $u(t)$ as a first-order system for $x(t)$.

First we may note that $x_1' = x_2$. Also, using these substitutions, (23) becomes

\[ mx_2' + \gamma x_2 + kx_1 = 0 \]  
\[ x_2' = -\frac{\gamma}{m} x_2 - \frac{k}{m} x_1. \]  

Using these two equations, we can form a system for $x = [x_1, x_2]^T$.

\[ x' = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix} x. \]  

(b) Let $m = 1$ and $\gamma = 4$ and find the eigenvalues of the coefficient matrix in part (a). How do the eigenvalues depend on the spring constant $k > 0$? Find a critical value of $k$ where the eigenvalues change type.

If $m = 1$ and $\gamma = 4$ then the coefficient matrix becomes

\[ \begin{bmatrix} 0 & 1 \\ -k & -4 \end{bmatrix}. \]  

This matrix has the following eigenvalues:

\[ \begin{vmatrix} -\lambda & 1 \\ -k & -4 - \lambda \end{vmatrix} = -\lambda (-4 - \lambda) + k = \lambda^2 + 4\lambda + k = 0 \]  
\[ \lambda = -2 \pm \sqrt{4 - k}. \]  

If $k < 4$, the eigenvalues are real. If $k > 4$, the eigenvalues are imaginary.

(c) Sketch the behavior of solutions in the phase plane for the cases

(See Table 9.1.1, pg. 494 in the textbook).

(i) $m = 1$, $\gamma = 4$, $k = 3$

Eigenvalues are $-1$ and $-3$, both negative. The sketch should show an asymptotically stable node at $(0, 0)$ (solution curves into the origin). Below is the solution with $x(0) = (0, 2)$. 
(ii) \( m = 1, \gamma = 4, k = 5 \)

Eigenvalues are \(-2 \pm i\). These are complex with a negative real part, so there is an asymptotically stable spiral point at \((0, 0)\) (solution spirals into the origin). Below is the solution with \( x(0) = (0, 2) \).