1.1) Let

$$E_1 = \begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1/2 & \\ & & & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) result of described op.s $= E_5 E_3 E_2 B E_1 E_4 E_6 E_7$

b) result $= ABC$

where $A = E_5 E_3 E_2 = \begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$$C = E_1 E_4 E_6 E_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1.2

$$f_1 = k_{12}(x_2 - x_1 - l_{12})$$
$$f_2 = -k_{12}(x_2 - x_1 - l_{12}) + k_{23}(x_3 - x_2 - l_{23})$$
$$f_3 = -k_{23}(x_3 - x_2 - l_{23}) + k_{34}(x_4 - x_3 - l_{34})$$
$$f_4 = -k_{34}(x_4 - x_3 - l_{34})$$

a) $$\underline{F} = \underline{\underline{K}}\,\underline{x} + \underline{z}$$

$$\underline{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}, \quad \underline{\underline{K}} = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ k_{12} & -k_{12} - k_{23} & k_{23} & 0 \\ 0 & k_{23} & -k_{23} - k_{34} & k_{34} \\ 0 & 0 & k_{34} & -k_{34} \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} -k_{12} l_{12} \\ k_{12} l_{12} - k_{23} l_{23} \\ k_{23} l_{23} - k_{34} l_{34} \\ k_{34} l_{34} \end{bmatrix}$$

b) $\dim[\underline{\underline{K}}] =$ force / length.

c) $\dim[\det \underline{\underline{K}}] = (\dim[\underline{\underline{K}}])^4$

d) $\underline{\underline{K}}' = 1000\,\underline{\underline{K}}$, $\det \underline{\underline{K}}' = 10^{12} \det \underline{\underline{K}}$

1.3) Suppose $\underline{\underline{R}}$ = $n \times n$ upper triangular, nonsingular matrix

Thus $\underline{\underline{M}}$ exists such that

$$\underline{\underline{R}}\,\underline{\underline{M}} = \underline{\underline{I}} \quad \text{, i.e., } \underline{\underline{M}} = \underline{\underline{R}}^{-1}$$

Consider this matrix eqn column-wise

$$\underline{\underline{R}}\,\underline{m}_j = \underline{e}_j \quad , \; j = 1, \ldots, n$$

or

$$\underbrace{\left[\begin{array}{c|c} \underline{\underline{R}}_1 & \underline{\underline{B}} \\ \hline 0 & \underline{\underline{R}}_2 \end{array}\right]}_{\underline{\underline{R}}} \underbrace{\left[\begin{array}{c} \underline{x} \\ \hline \underline{y} \end{array}\right]}_{\underline{m}_j} = \underbrace{\left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right]}_{\underline{e}_j} \longleftarrow j^{th} \text{ row}$$

So,

$$\underline{\underline{R}}_2\,\underline{y} = 0 \quad \Rightarrow \underline{y} = 0$$

$$\underline{\underline{R}}_1\,\underline{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \text{determines } \underline{x} \neq 0$$

Thus

$$\underline{m}_j = \left[\begin{array}{c} \underline{x} \\ \hline 0 \end{array}\right] \quad \Rightarrow \underline{\underline{M}} = \underline{\underline{R}}^{-1} \text{ is upper triangular}$$

1.4) Have

$$\sum_{j=1}^{8} c_j f_j(i) = d_i \quad , \quad i = 1, \ldots, 8$$

and for any set $\{d_i\}$ there exists a set $\{c_j\}$ s.t. the equality holds.

a) Linear eqns read

$$\underline{\underline{F}}\, \underline{c} = \underline{d}$$

Since there is a soln. for any $\underline{d}$ $\Rightarrow$ 8 columns of $\underline{\underline{F}}$ span $\mathbb{R}^8$ $\Rightarrow$ columns are linearly ind $\Rightarrow$ $\underline{\underline{F}}$ is nonsingular $\Rightarrow$ $\underline{d}$ determines $\underline{c}$ uniquely

b) Given that $\underline{\underline{A}}$ maps $\underline{d}$ to $\underline{c}$ thus $\underline{\underline{A}}^{-1} = \underline{\underline{F}}$ maps $\underline{c}$ to $\underline{d}$

$$\Rightarrow \quad [\underline{\underline{A}}^{-1}]_{i,j} = f_j(i)$$

2.7) Let

$$\underline{Q} = \begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{bmatrix} \times & \times & \times & \\ & \times & \times & \\ & & \times & \\ & & & \ddots \end{bmatrix} \end{array}$$

upper triangular say.

If $\underline{Q}$ is also unitary, then all columns are orthogonal.

e.g.

$$q_1^* q_2 = 0 \Rightarrow \underline{q_2} = \begin{bmatrix} 0 \\ \times \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$q_1^* q_3 = q_2^* q_3 = 0 \Rightarrow \underline{q_3} = \begin{bmatrix} 0 \\ 0 \\ \times \\ 0 \\ \vdots \end{bmatrix}$$

and so on.

$\Rightarrow \underline{Q}$ is diagonal.

2.2)

a) $$\|x_1 + x_2\|^2 = (x_1 + x_2)^*(x_1 + x_2)$$

$$= x_1^* x_1 + \underbrace{x_1^* x_2 + x_2^* x_1}_{0} + x_2^* x_2$$

$$= \|x_1\|^2 + \|x_2\|^2 \quad ✓$$

b) $$\|x_1 + \cdots + x_n\|^2 = (x_1 + \cdots + x_n)^*(x_1 + \cdots + x_n)$$

$$= (x_1 + \cdots + x_{n-1})^*(x_1 + \cdots + x_{n-1})$$

$$+ \underbrace{(x_1 + \cdots + x_{n-1})^* x_n}_{\to 0}$$

$$+ \underbrace{x_n^*(x_1 + \cdots + x_{n-1})}_{\to 0} + x_n^* x_n$$

$$= \underbrace{\|x_1 + \cdots + x_{n-1}\|^2}_{} + \|x_n\|^2$$

$$\hookrightarrow = \|x_1\|^2 + \cdots + \|x_{n-1}\|^2$$

by induction.

2.3) $Ax = \lambda x$, $x \neq 0$, $A^* = A$

a)

$$x^* A x = \lambda \|x\|^2$$

take $*$ $\Rightarrow$ $x^* \underset{= A}{A^*} x = \bar{\lambda} \|x\|^2$

$$\Rightarrow \|x\|^2 (\lambda - \bar{\lambda}) = 0 \Rightarrow \lambda = \bar{\lambda} \checkmark$$

b) $Ay = \mu y$, $y \neq 0$ say.

$$y^* A x = \lambda y^* x$$

$$x^* A y = \mu x^* y \rightarrow y^* A x = \mu y^* x$$

$$\Rightarrow 0 = \underbrace{(\lambda - \mu)}_{\neq 0} y^* x \Rightarrow y^* x = 0 \checkmark$$

2.6)

$$A = I + uv^*$$

$$A^{-1} = I + \alpha uv^*$$

If A is nonsingular, then

$$I = AA^{-1} = (I + uv^*)(I + \alpha uv^*)$$

$$= I + (1+\alpha)uv^* + \alpha u \underbrace{v^* u}_{\text{scaler}} v^*$$

$$= I + \underbrace{(1 + \alpha + \alpha v^* u)}_{0} uv^*$$

$$\Rightarrow \alpha = \frac{-1}{1 + v^* u}$$

A is singular if $v^* u = -1$ (i.e. $\alpha \to \infty$)

For this case,

$$Ax = x + uv^* x = 0 \quad \text{if} \quad x \propto u$$

so, $\text{null}(A) = x = \beta u \;, \beta \neq 0.$