Math 6840
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Problem Set 7

Due:
Monday, 11/19/07

1. Consider the scalar conservation law $u_t + f(u)_x = 0$ and the MacCormack method

$$v_j^{n+1} = \frac{1}{2}\left(v_j^n + v_j^*\right) - \frac{\Delta t}{2\Delta x}\left(f(v_j^*) - f(v_{j-1}^*)\right)$$

where

$$v_k^* = v_k^n - \frac{\Delta t}{\Delta x}\left(f(v_{k+1}^n) - f(v_k^n)\right), \qquad k = j-1 \text{ or } j$$

(a) Determine the order of accuracy of the method for the case $f(u) = cu$ with $|c|\Delta t/\Delta x \neq 1$.

(b) For a general flux function $f(u)$, show that the method can be written in the conservation form

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\left(G_{\rm mc}(v_j^n, v_{j+1}^n) - G_{\rm mc}(v_{j-1}^n, v_j^n)\right)$$

for a specific numerical flux function $G_{\rm mc}(u_\ell, u_r)$. Give a formula for $G_{\rm mc}$.

(c) Use the conservative MacCormack method with $\Delta x = .01$ to find a numerical solution of the conservation equation $u_t + (u^2/2)_x = 0$ for the initial conditions

$$u(x,0) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Plot your solution at $t = 1$ and $t = 3$. Compare the numerical solution with the exact solution described in class. Is the numerical error dissipative or dispersive?

2. Consider the traffic flow equation

$$\rho_t + f(\rho)_x = 0, \qquad |x| < \infty, \quad t > 0$$

with $\rho(x,0) = \rho_0(x)$ and $f(\rho) = \rho(1-\rho)$. A general conservative discretization is

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(G(w_j^n, w_{j+1}^n) - G(w_{j-1}^n, w_j^n)\right) \tag{1}$$

where w_j^n approximates the cell average of ρ on a grid (x_j, t_n) and G is a numerical flux function.

(a) The Godunov flux is $G_{\rm gd}(w_\ell, w_r) = f(\rho^\star(w_\ell, w_r))$, where ρ^* is found from the exact solution of the Riemann problem for the traffic flow equation. Determine $G_{\rm gd}$ for all possible states (w_ℓ, w_r).

(b) Write a code to solve the traffic flow equation using Godunov's method. Consider a uniform grid with at least 200 grid cells on $[-1, 1]$ and initial conditions of the form

$$\rho_0(x) = \tfrac{1}{2}(\rho_1 + \rho_2) + \tfrac{1}{2}(\rho_2 - \rho_1)\tanh(20x)$$

where ρ_1 and ρ_2 are chosen states on either side of $x = 0$. Run your code for (i) $\rho_1 = .3$, $\rho_2 = 1$ and (ii) $\rho_1 = .8$, $\rho_2 = 0$. Plot the solution at $t = .5$ for each case.