Math 6840
D. Schwendeman

Problem Set 6

Due:
Monday, 11/12/07

1. Consider the linear hyperbolic system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

where A is a constant matrix, and the corresponding upwind method

$$\mathbf{v}_j^{n+1} = \mathbf{v}_j^n - \frac{A_+\Delta t}{\Delta x}\left(\mathbf{v}_j^n - \mathbf{v}_{j-1}^n\right) - \frac{A_-\Delta t}{\Delta x}\left(\mathbf{v}_{j+1}^n - \mathbf{v}_j^n\right)$$

where $A_- + A_+ = A$. For each A given below, find the corresponding matrices A_- and A_+ in the upwind method. (Maple may be helpful for this problem.) If the PDE is defined for $x \ge 0$, how many boundary conditions are needed at $x = 0$ for each case?

(a) $A = \begin{bmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} \frac{1}{4} & -\frac{3\sqrt{2}}{4} & \frac{9}{4} \\ -\frac{3\sqrt{2}}{4} & \frac{5}{2} & -\frac{3\sqrt{2}}{4} \\ \frac{9}{4} & -\frac{3\sqrt{2}}{4} & \frac{1}{4} \end{bmatrix}$

2. Consider the problem

$$u_t + c(x)u_x = -u + a(x), \qquad -2 < x < 2, \quad t > 0$$

with $u(x,0) = f(x)$. Solve the PDE using the two-step Lax-Wendroff scheme discussed in class. Consider the case $c(x) = \operatorname{sech} x$, $a(x) = \sinh x$ and $f(x) = \sin(\pi x)$, and use the boundary conditions $u(-2,t) = u_x(2,t) = 0$. Solve the problem for $0 \le t \le 1$ using grids with $\Delta x = .02$, .01 and .005 and a stable value of Δt for each case. Plot the solution at $t = 1$. Verify the order of accuracy by comparing the numerical solution with the exact solution at $x = 0$ and $t = 1$.

3. Consider the conservation equation

$$\rho_t + f(\rho)_x = 0, \qquad |x| < \infty, \quad t > 0$$

with $\rho(x,0) = \rho_0(x)$. This equation arises as a simple model of traffic flow. In this context $\rho(x,t)$ is the traffic density, i.e. the number of cars per unit distance along a highway, and $f(\rho) = \rho v$ is the flux, where $v(\rho)$ is the velocity of the cars. Various models have been proposed for $v(\rho)$. Typically $v'(\rho) < 0$ which means that the car velocity decreases as the traffic density increases. Let's take the simple choice

$$v(\rho) = v_{\max}(1 - \rho/\rho_{\max})$$

where $v_{\max}$ is the maximum car velocity (the speed limit say) when the traffic is light and $\rho_{\max}$ is the maximum traffic density when the cars are bumper to bumper.

(a) Determine the characteristic form of the equation. What is the characteristic speed?

(b) Determine a formula for the speed U of a jump discontinuity between traffic densities ρ_1 and ρ_2. (Hint: you should find that U is a linear function of ρ_1 and ρ_2.)

(c) Consider the initial conditions

$$\rho_0(x) = \frac{1}{2}(\rho_1 + \rho_2) + \frac{1}{2}(\rho_2 - \rho_1)\tanh x$$

where ρ_1 and ρ_2 are constants. Use the characteristics (just straight lines for this equation) and the fact that $\rho =$ constant along them to sketch solutions qualitatively at various times. Consider the two cases: (i) $0 < \rho_1 < \rho_2 = \rho_{\max}$ and (ii) $0 = \rho_2 < \rho_1 < \rho_{\max}$. Would a shock in the traffic density appear for either case? Interpret the qualitative behavior of the solutions for (i) and (ii) in terms of your experience on the road.