Math 6840 **Due:**
D. Schwendeman **Problem Set 1** Monday, 9/10/07

1. For each of the four PDEs below, determine whether the equation has real characteristics or not. If it has characteristics, specify the region of the (x, y) plane where the characteristics exist, and find equations for these curves.

(a) $$yu_x + (6x + y)u_y + e^{-y}u = \sin x$$

(b) $$\begin{cases} u_x + u_y - 3v_y = 0 \\ v_x + 2u_y + v_y = 0 \end{cases}$$

(c) $$yu_{xx} + (x + y)u_{xy} + xu_{yy} = \sin(xy)u + \cos(x + y)$$

(d) $$u_{xx} + xu_{yy} = 0$$

2. Consider the boundary-value problem

$$u_x + \left(\frac{x+2}{y+1}\right) u_y + u = 0, \qquad x \geq 0, \quad y \geq 0$$

with $u(x, 0) = \cos x$ and $u(0, y) = g(y)$, where $g(y)$ is a given function.

(a) Find the characteristic curves for the first-order PDE in the form $y = y(x)$.

(b) Determine the exact solution of the boundary-value problem. Under what conditions on the function $g(y)$ is the solution continuous?

3. Consider the problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0$$

with boundary conditions $u(0, t) = 0$ and $u(1, t) = 1$, and with initial conditions

$$u(x, 0) = x + \sin(\pi x) + \sin(3\pi x), \qquad 0 \leq x \leq 1$$

(a) Find the exact solution.

(b) Consider the finite difference scheme

$$v_j^{n+1} = v_j^n + r\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right), \qquad j = 1, 2, \ldots, N-1, \quad n = 0, 1, \ldots$$

where $v_j^n \approx u(x_j, t_n)$ on the mesh $x_j = j\Delta x$, $t_n = n\Delta t$, $\Delta x = 1/N$, and where $r = \Delta t/(\Delta x)^2$. The boundary conditions for the difference scheme are $v_0^n = 0$ and $v_N^n = 1$, and the initial conditions are

$$v_j^0 = x_j + \sin(\pi x_j) + \sin(3\pi x_j), \qquad j = 0, 1, \ldots, N$$

Write a code to solve the difference equations for given values for N and Δt. Run your code for cases $(N, \Delta t) = (20, 1/1000)$, $(40, 1/4000)$ and $(80, 1/16000)$. Plot the solution v_j^n of the finite difference equations at $t = 0.02$ for each case, and also plot the error $v_j^n - u(x_j, t_n)$. Comment on the results. (Note that the value for r is the same for each case.)

(c) Run your code in part (b) for $N = 40$ and for several values of Δt smaller and bigger than $1/4000$, the value used in part (b). Comment on the results. In particular, comment on the behavior for $\Delta t > 3.125 \times 10^{-4}$.