Normal Families

Def: Functions $f$ in a family $F$ are equicontinuous on $E$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, for all $f \in F$ (i.e., each $f$ is uniformly continuous).

Def: $F$ is normal in $E$ if every sequence of functions $f_n \in F$ contains a subsequence converging uniformly on every compact subset of $E$.

Arzela-Ascoli Theorem: A family $F$ of continuous functions $C \to C$ is normal in the region $E$ if and only if

(i) $F$ is uniformly continuous on every compact subset of $E$,

(ii) for any $x \in E$, the values $f(x)$, $f \in F$, lie in a compact subset of $C$. (i.e., are bounded)
If $F$ is not equivalent on $E$, there is an $e > 0$ such that for every sequence $z_n, z'_n \in E$, and for $e \in F$
and $\lim_{n \to \infty} |z_n - z'_n| = 0$, and

$$|f_n(z_n) - f_n(z'_n)| \geq e \quad \text{for all } n.
$$

By $E$ is compact, we can choose subsequences of $z_n$ and $z'_n$, converging to the same $z \in E$
and by $F$ is normal, there is a subsequence of $f_n$ converging uniformly on $E$. We can choose all these subsequences to have the same subscripts, $n_k$. $f = \lim_{k \to \infty} f_{n_k}$ is uniformly continuous on $E$.

Hence, we can find $a, b > 0$ such that

$$|f_{n_k}(z_{n_k}) - f(z)|, \quad |f(z_{n_k}) - f(z'_n)|, \quad |f(z_{n_k}) - f(z'_n)|$$

$$< \frac{e}{3}$$
Thus, \(|f_{m_k}(x_{n_k}) - f_{m_k}(x_{n_k}')| < \varepsilon\)

contradiction

Necessity of (iii): Let \(\{x_{n_k}\}\) be a subsequence in the closure of the set \(\{f(x) | f \in F\}\). For each \(m_k\) take \(f_{m_k} \to f\) so that \(|f_{m_k}(x) - m_k| < \frac{\varepsilon}{2}\).

By normality, there is a convergent subsequence \(\{f_{m_{k_l}}(x)\}\) and \(\{x_{n_{k_l}}\}\) converge to the same value. Therefore, the above closure is compact.

Sufficiency of (i) and (iii): Let \(\{x_{n_k}\}\) be a dense sequence (such as all rational points). Let \(\{f_{m_k}\} \subset F\).

(iii) shows \(f_{m_k}(x) \to f(x)\) has a
convergent subsequence, then \( f_n \to f \).

If \( f_n \) has a subsequence, then
\( \exists f_{n_k} \) such that \( f_{n_k} \to f_{n_k} \) converges.

Continue this to find sequences
\( \exists f_{n_2} \) such that \( f_{n_2} \to f_{n_2} \)
and \( f_{n_2} \to f_{n_2} \) converges.

Of course, also \( \exists f_{n_3} \to f_{n_3} \) converges.

Let \( E \subset \mathbb{R} \) be compact and \( f \) equicontinuous on \( E \). We will show: \( f_n \) converges uniformly on \( E \). Let \( f_n \to f \) (not the original \( f \)). Given \( \epsilon > 0 \), choose \( \delta > 0 \) such that for all \( \varepsilon, \varepsilon' \in E \),
\( |f(\varepsilon) - f(\varepsilon')| < \frac{\epsilon}{3} \).
Since \( E \) is compact, you can cover it by a finite number of \( \frac{1}{2^n} \) neighborhoods. Choose a \( \frac{1}{2^n} \) in each of them. Since \( \frac{1}{2^n} \) tends to zero as \( n \to \infty \), it implies 
\[
|f(m(\frac{1}{2^n})) - f(m(\frac{1}{2^n}))| < \frac{\epsilon}{2}
\]
for all small \( \frac{1}{2^n} \). For every \( x \in E \),
\[
|f(x) - f(m(\frac{1}{2^n}))| < \frac{\epsilon}{2}
\]
\[
\Rightarrow |f(x) - f(m(\frac{1}{2^n}))| < \frac{\epsilon}{2}
\]
so \( f(x) \) converges uniformly on \( E \).

**Remark**: For \( \hat{f} \) compact with respect to the metric
\[
\hat{f}(h) = \sum_{k=1}^{\infty} \frac{1}{Z_k} \sum_{\mathbf{z} \in E_k} \frac{1 + \hat{f}(\mathbf{z}) - \hat{f}(x)}{\hat{f}(\mathbf{z}) - \hat{f}(x)}
\]
\[
\hat{E}_k = \mathbb{S}^{2} \setminus \{ x \in \mathbb{S}^{2} : d(\mathbf{z}, \mathbf{x}) \geq \frac{1}{k} \}
\]
\( \hat{E}_k \) is compact. \( J = \bigcup \hat{E}_k \).
This a family $F$ of analytic functions is normal iff all these functions are uniformly bounded on every compact set.

Let $F$ be equicontinuous on compact sets. Given $f \in F$, there exists $2\alpha \leq \alpha < \frac{1}{2}$ and $\bar{f}$ is equicontinuous on $D$. Choose $\varepsilon$ and thus $\bar{f}(c \in F)$, $|f(\alpha) - \bar{f}(c)| \leq \varepsilon$ for all $f \in F$ (by boundedness of $\alpha$). Thus $|f(\alpha)| \leq M + \varepsilon$ for $|\alpha - \beta| < \frac{1}{2}$. Oser this compact set by a finite number of neighborhoods each. This property and yet the functions of $F$ are uniformly bounded on every compact set (the bound depends on the set).

This uniform boundedness implies equicontinuity on compact sets.
Let $C$ be a circle of radius $r$ in $\mathbb{R}$.
If $z, \bar{z}$ are inside $C$, by Cauchy
\[ f(z) - f(\bar{z}) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z - z'} dz' \]
\[ = \frac{1 - z}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z} d\theta \]
If $|f| \leq M$ on $C$, replace $z, \bar{z}$ to a concentric disk with radius $\frac{r}{2}$

\[ |f(z) - f(\bar{z})| \leq \frac{4M \pi \frac{r}{2}}{r} \]

$\Rightarrow f$ - equicontinuous on the smaller disk.

If $E \subset \mathbb{R}$ is compact and joint $f$ on the center of a disk with radius $r(\delta)$ for some $\delta \delta$ as above.
Given an open ball $B(x, r)$ for any open cover of $E$, to a finite subcover exists; each with center $x_k$, radius $r_k$ and $|f(x) - f(x_k)| \leq M_k$ for $|x - x_k| \leq r_k$. Let $r = \min \{r_k\}$, $M = \max \{M_k\}$. Given $\varepsilon > 0$, let $\delta = \min \left\{\frac{\varepsilon}{4r}, \frac{\varepsilon}{4M} \right\}$. Then $|f(x) - f(x_k)| \leq \frac{M_k}{4} < \frac{1}{4} \varepsilon$ for $|x - x_k| < \varepsilon + \frac{r_k}{4} = r_k/2$. For (\ref{eq:property}), holds and $|f(x) - f(y)| \leq \frac{M}{r} \delta < \varepsilon$.

Property (\ref{eq:property}) is called local boundedness.
For a locally bounded family of analytic functions has locally bounded derivatives.

If \( f \) is \( C \subset \mathbb{R}^2 \) in a circle of radius \( r \)

\[ f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z')}{(z - z')^2} \, dz' \]

So \( \frac{1}{2\pi i} \int_{C_r} \frac{f(z')}{(z - z')^2} \, dz' \) in a concentric circle with radius \( r/2 \) (with \( |f| \leq M \))

\[ \Rightarrow f \text{ - locally bounded} \]

Classical definition:

\[ \text{def } F \text{ of analytic f: } \mathbb{R} \rightarrow \mathbb{C} \]

normal if every sequence either contains a subsequence converging uniformly on every compact \( E \subset \mathbb{R} \)

or a subsequence that tends
uniformly to \( \infty \) on compact sets.

This is the same as uniform convergence on Riemann sphere.

**Lemma**: If a sequence of meromorphic functions converges (in the sense of spherical distance) uniformly on compact sets, the limit function is meromorphic or identically \( 0 \).

**f**: Suppose \( f(z) \to f(\infty) \) in the sense of Lemma. Then \( f(z) \) is continuous in spherical metric.

If \( f(z) \neq \infty \) then \( f(z) \) is bounded near \( 0 \) and \( f(n) \to \) there for large \( n \).
By Weierstrass, $f(z)$ is analytic near $z_0$. If $f(z_0) = 0$, look at
\[
\frac{f(z)}{z - z_0} = \lim_{m \to 0} \frac{f(z)}{z - z_0}.
\]
Then $f(z)$ is analytic near $z_0$ to $f(z_0)$ and meromorphic. If $f_n$ are analytic and \( \lim_{m \to 0} f_n(z_0) \) is not, then
\[
f = 0 \quad \text{by Hartogs and } f = \infty.
\]

Counterexample. Derivatives of a normal family need not be a normal family:
\[
f_n(z) = n(z^2 - n)
\]
\[
f_n \to \infty \quad \text{uniformly on } \mathbb{C}
\]
But $f_n(z) = 2n + z \to \infty$ $\to \infty$
\[
f \to \text{ not normal.}
\]
They are analytic or meromorphic functions if in the classical sense if
\[
C(f) = \frac{2|f(z)|}{1 + |f(z)|^2}
\]
are locally bounded.

Comment: distance on the Riemann sphere
where
\[
d(f(z_1), f(z_2)) = \frac{2|f(z_1) - f(z_2)|}{\sqrt{(1 + |f(z_1)|^2)(1 + |f(z_2)|^2)}}
\]

f + stereographic projection map are f on an image of length
\[
\int |C(f(z))| dz
\]
\[
f \quad (z_1, z_2)
\quad (z_1 \leq z \leq z_2)
\]
\[
\Rightarrow d(f(z_1), f(z_2)) \leq |z_1 - z_2|
\]
\[
\Rightarrow \text{equicontinuity}
\]
(now necessary: \( c(f) = c(f') \) (simple))

Let \( F \) be normal, but \( c(f) \) unbounded on compact \( F \). Consider

\( \{f_n\} \subset F \) such that \( \max\ c(f) \to \infty \)

Let \( f = \lim f_n \) (which must exist) around every \( z \in E \), find a closed disk \( c \subset \mathbb{R} \) on which \( f_n \to f \) is analytic. If \( f \) is analytic, it is bounded by the spherical convergence \( \{f_n\} \) has no poles in disk for large \( n \). By

Weierstrass, \( c(f_n) \to c(f) \)

on some smaller disk, uniformly.

Since \( c(f) \) is continuous,

\( c(f_n) \to c(f) \) bounded there.
If \( f \) is analytic, same proof applies to \( C(\frac{1}{f_n}) = C(\frac{1}{f_n}) \).

Since \( E \) is compact, it is covered by finitely many smaller disks, so \( C(f_n) \) is bounded on \( E \), contradiction.
Picard's Theorem

Given any simply connected region \( \mathbb{D} \) in the plane \( \mathbb{C} \) and \( \mathbb{D} \neq \mathbb{C} \), there exists a unique analytic function \( f \) on \( \mathbb{D} \) such that \( f(z_0) = 0 \), \( f'(z_0) > 0 \), and \( f \) maps \( \mathbb{D} \) onto \( \{ w : |w| < 1 \} \) in a 1-1 way.

If Existence Consider the family of functions satisfying

1. \( f \) is analytic and 1-1 on \( \mathbb{D} \).
2. \( |f(z)| \leq 1 \) in \( \mathbb{D} \).
3. \( f(z_0) = 0 \) and \( f'(z_0) > 0 \).

The function we seek is \( f \) with maximal \( f'(z_0) \).
1. \( F \neq \emptyset \): There is an \( a + \infty \) at \( \infty \).

Since \( \mathbb{R} \) is simply connected, \( \sqrt{z-a} \) has a single-valued branch in \( \mathbb{R} \). We never have

\[ h(z_1) = h(z_2), \quad \text{nor} \quad h(z_1) = -h(z_2) \text{ in } \mathbb{R}. \]

\( h(\mathbb{R}) \) covers a disk \( |w - h(\infty)| < \rho \)

so doesn't meet \( |w + h(\infty)| < \rho \).

\[ |h(z_1) + h(\infty)| = \rho, \quad z \in \mathbb{R}, \]

\[ \Rightarrow z \geq |h(\infty)| \geq \rho \]
Then

$$
\varphi(t) = \frac{c}{4} \left( \frac{l_1(t_0) - l_1(t)}{l_1(t_0)^2} - \frac{l_1(t) - l_1(t_0)}{l_1(t)^2} \right)
$$

belongs to $F$:

$$
\varphi(t) = \text{Linear transformation of } F
$$

+ $1$ function $\varphi$

$\Rightarrow \varphi(t) \text{ is } 1-1.$

$$
\varphi(t_0) = 0 \text{ and } \varphi'(t_0) = \frac{c}{4} \frac{l_1(t_0)}{l_1(t_0)^2} > 0
$$

$\therefore$

$$
\begin{align*}
&\frac{l_1(t_0) - l_1(t_0)}{l_1(t_0) + l_1(t_0)} = l_1(t_0) \left( \frac{1}{l_1(t_0)} - \frac{2}{l_1(t_0) + l_1(t_0)} \right) \\
&\leq 4 \frac{l_1(t_0)}{c}
\end{align*}
$$

So

$$
|\varphi(t)| \leq 1 \text{ in } \mathbb{R}.
$$
For \( p + F \) have a sup \( B \),
which could be \( B = +\infty \).

There exists a sequence \( g_n \in F \)
such that \( g_n \to B \).

For compact \( B \) of \( g_n \) uniformly bounded \( \Rightarrow \) there is a subsequence \( g_{n_k} \), \( g_{n_k} \to f \), analytic, uniformly on compact sets.

If \( |f(z)| \leq 1 \) and
\( b \leq f' \) are \( f(0) = 0 \) and
\( f'(0) = B \) \( \Rightarrow B < +\infty \).

We must still show \( f \) is 1-1
\( \Rightarrow f \in F \) with real derivative.
\[ f = \text{const b/c } f'(b) = B > 0. \]

Choose \( \varepsilon > 0 \) such that \( g(x) = g(b) - \varepsilon \) for all \( x \in F. \)

They are \( \neq 0 \) in \( x = B. \) By

Hermite's theorem, \( f(B) - f(x) = \lim_{x \to B} g(x) = 0 \) for \( x = 0 \)

and clearly not \( = 0 \) so \( \neq 0 \)

\[ \Rightarrow f(B) \neq f(\xi) \text{ for } \xi \neq B, \]

\[ \Rightarrow f \text{ is 1-1.} \]

Show that \( f(x) = |x| < 1 \),

let \( f(\xi) = 0 \) for some \( 0 < \xi < 1 \)

since \( \xi \) \( \Rightarrow \) simply connected.
we can define a triple-valued branch $\mathcal{F}$

\[
\mathcal{F}(\mathcal{L}) = \sqrt{\mathcal{F}(\mathcal{L}) - \mathcal{M}_0 \mathcal{F}(\mathcal{L})} \frac{1}{1 - \mathcal{M}_0 \mathcal{F}(\mathcal{L})}
\]

\[\mathcal{F}(\mathcal{L}) = 1 - \mathcal{I}\] if a linear transformation

and $|\mathcal{F}| \leq 1$ (can show)

that $\mathcal{M}_0 \mathcal{F}(\mathcal{L}) = \delta |\mathcal{F}| < 1$ into step (i)

Normalize $\mathcal{F}$ by forming

\[
G(\mathcal{L}) = \frac{|\mathcal{F}^*(\mathcal{L})| \mathcal{F}(\mathcal{L}) - \mathcal{F}^*(\mathcal{L})}{\mathcal{F}^*(\mathcal{L}) \mathcal{F}(\mathcal{L}) - 1 - \mathcal{F}^*(\mathcal{L}) \mathcal{F}(\mathcal{L})} \]

\[G(\mathcal{L}) = 0\]
$$G(z_0) = \frac{1 + |w_0|}{1 - |F(z_0)|^2} = \frac{1 + |w_0|}{2 \sqrt{|w_0|}} \quad |s > B$$

$\Rightarrow$ contraction

$\Rightarrow$ f assumes all values in $|w| < 1$, $|w| < 1$.

Remark (1) & (2) let us express $f(z)$ as a single-valued analytic function of $W = g(z)$ which maps $|W| < 1$ into itself. If $|F(z_0)| < 1$ good follows by $f$. This follows from Schwartz's lemma.

Uniqueness: if $f_1, f_2$ are two such functions, $f_1, f_2(W)$ in a 1-1 conformal map of $|W| < 1$ into itself, $f_1[f_2^{-1}(0)] = 0$.
So, let \( h = f_2^{-1} \). Then

\[
\frac{h(z)}{z} \text{ is analytic in } |z| < 1, \text{ including at } z = 0. \text{ Since } |h(z)| \leq 1 \text{ and } h(0) = 0, \text{ we follow the lemma to show } \left| \frac{h(z)}{z} \right| \leq 1.
\]

Like above, \( h = f_2 \circ f_1^{-1} = h \). Then

\[
\left| \frac{h(z)}{z} \right| \leq 1 \Rightarrow \left| h(z) \right| \leq 1 \Rightarrow h(z) = h(z)
\]

\[
\Rightarrow \frac{h(z)}{z} = \text{ constant } = e^{\alpha z}
\]

\[
\Rightarrow h(z) = z e^{\alpha z}
\]

\[
h(0) > 0 \Rightarrow e^{\alpha 0} = 1
\]

\[
\Rightarrow h(z) = z \Rightarrow f_2 = f_1
\]