COMPLEX ANALYSIS 6300

Analytic functions: ($z = x + iy$)

A function $\mathbb{C} \to \mathbb{C}$ is analytic (or holomorphic) in $\Omega \subset \mathbb{C}$ (open) if

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} \quad \text{exists}$$

for every $z \in \Omega$.

Let $f(z) \equiv u(x,y) + i v(x,y)$

$\zeta = z + \Delta x$

$$\Rightarrow f'(z) = u_x + i v_x$$

$\zeta = z + i\Delta y$

$$f'(z) = -i u_y + v_y$$

$$\Rightarrow \quad u_x = v_y, \quad u_y = -v_x$$

$\Rightarrow$ Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x$$

$u_x = v_y \qquad v_x = -u_y$

$\Rightarrow u_{xx} = v_{xy} = -u_{yy} \Rightarrow \boxed{u_{xx} + u_{yy} = 0}$

similarly: $v_{xx} + v_{yy} = 0$

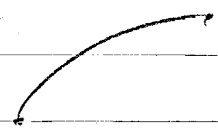
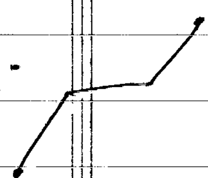
Prop Real and imaginary parts, u & v, of an analytic function f are Harmonic functions, i.e. they satisfy Laplace's equations

$$\Delta u = \Delta v = 0.$$

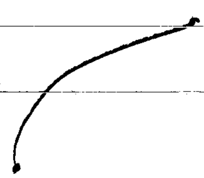
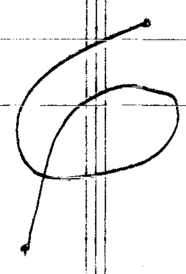
Complex integration:

A curve $\gamma \subset \mathbb{C}$ is

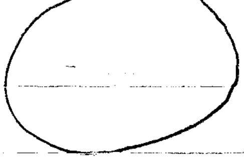
- smooth, if it has a continuously turning tangent

smooth:  non-smooth: 

- simple, if it has no self-intersections

simple:  non-simple: 

(simple curve $\equiv$ Jordan arc)

– closed  (contour)

Parametrization $\gamma = \{z(t),\ a \le t \le b\}$

– smooth: $\dot z(t)$ is continuous

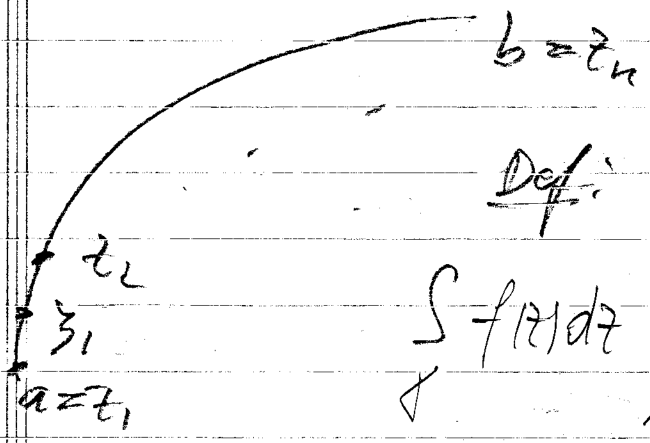
(piecewise smooth: $\dot z$ is piecewise continuous)

– non-simple: $z(t_1) = z(t_2)$ $\quad t_1, t_2 \neq a, b$

– closed: $z(a) = z(b)$

Complex integral:

Let γ be a piecewise smooth curve, let $z_1 = a, z_2, \dots, z_{n-1}, z_n = b$ be n distinct points on γ, ζ_k be any point in $[z_k, z_{k+1}]$.



Def:

$$\int_\gamma f(z)\,dz = \lim_{\max_k |z_{k+1} - z_k| \to 0} \sum_{k=1}^{n-1} f(\zeta_k)(z_{k+1} - z_k)$$

Clear If $\gamma = \{z(t),\ a \le t \le b\}$, then

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\dot z(t)\,dt$$

Usual integration rules apply:
linearity, additivity $\int_{\gamma_1 + \dots + \gamma_n} = \int_{\gamma_1} + \dots + \int_{\gamma_n}$

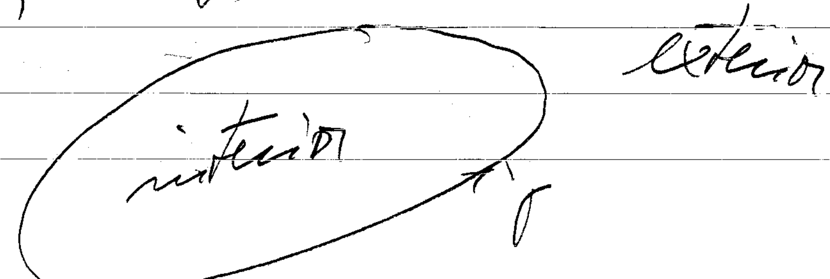
$$\left| \int_\gamma f(z)\,dz \right| \le \int_\gamma |f(z)|\,|dz| \equiv \int_a^b |f(z(t))|\,|\dot z(t)|\,dt$$

$$\left(|dz| = ds = |\dot z(t)|\,dt = \sqrt{(\dot x(t))^2 + (\dot y(t))^2}\,dt \right)$$

Cauchy's theorem

Let γ be a simple closed curve.
It partitions $\mathbb{C}$ into two distinct open sets:

- exterior (contains ∞)
- interior, $\mathrm{Int}(\gamma)$



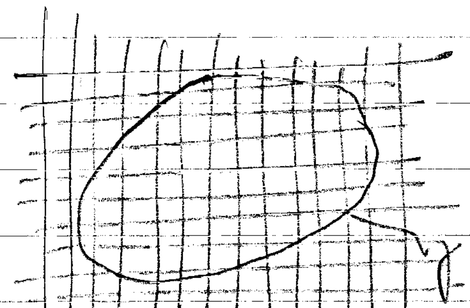
Remarks 1.) Usually γ is parametrized in such a way that as you traverse γ, $\mathrm{Int}(\gamma)$ is on the left

2.) This is a highly non-trivial result, known as Jordan's curve theorem but in most practical situation it is obvious.

Thm Let γ be a closed Jordan curve $f(z)$ analytic on $\Omega \overset{\text{open}}{\subset} \mathbb{C}$ that contains γ and Int γ. Then

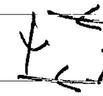
$$\oint_\gamma f(z)\,dz = 0.$$

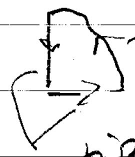
PROOF Draw a mesh of squares with sides parallel to x and y axes and $|\text{diagonal}| < \delta$.



Note

$$\oint_\gamma f(z)\,dz = \sum_{j=1}^{m} \oint_{\gamma_j} f(z)\,dz$$

γ_j — square in $\mathrm{Int}(\gamma)$  γ_j:



— piece of γ

sides of a square

} near γ

Note:  $\int$'als over internal mesh cancel.

Let $z_j \in \mathrm{Int}(\gamma_j)$. Write

$$f(z) = f(z_j) + (z - z_j) f'(z_j) + (z - z_j)\tilde{f}_j(z)$$

w/ $$\tilde{f}_j(z) = \left(\frac{f(z) - f(z_j)}{z - z_j}\right) - f'(z_j)$$

Clear: $\oint_{\gamma_j} dz = 0 \qquad \oint_{\gamma_j} (z - z_j)\, dz = 0$

(use antiderivatives and rules for line integrals)

$$\Rightarrow \left|\oint_\gamma f(z)\,dz\right| \le \sum_{j=1}^{n} \left|\oint_{\gamma_j} f(z)\,dz\right|$$

$$= \sum_{j=1}^{n} \left|\oint_{\gamma_j} (z-z_j)\,\tilde{f}_j(z)\,dz\right|$$

$$\le \sum_{j=1}^{n} \oint_{\gamma_j} |z-z_j|\,|\tilde{f}_j(z)|\,|dz|$$

Now $|z-z_j| < \delta$ for $z \in \gamma_j$

$$\Rightarrow |\tilde{f}_j(z)| = \left|\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right| < \varepsilon_j < \varepsilon$$

$$\left(\varepsilon = \max_{1\le j\le n} \varepsilon_j\right)$$

Now $\delta = \sqrt{2A_j}$, A_j - area of j-th square

$\Rightarrow$ for interior squares

$$\oint_{\gamma_j} |z-z_j|\,|\tilde{f}_j(z)|\,|dz| \le \sqrt{2A_j}\;\varepsilon\;4\sqrt{A_j} =$$

$$= 4\sqrt{2}\,\varepsilon A_j$$

for boundary squares / from the squares

$$\oint_{\gamma_j} |z-z_j'| |\tilde{f}_j(z)| |dz| \le \sqrt{2A_j}\cdot\varepsilon \, (4A_j + L_j)$$

from the piece of γ

$$\Rightarrow \left|\oint_\gamma f(z)\,dz\right| \le \sum_{\substack{j = \text{over} \\ \text{interior squares}}} 4\sqrt{2}\,A_j\,\varepsilon \; +$$

$$+ \sum_{\substack{j = \text{over} \\ \text{boundary} \\ \text{squares}}} \left(4\sqrt{2}\,A_j\,\varepsilon + \sqrt{2}\,\varepsilon\sqrt{A_j}\,L_j\right)$$

$$\le 4\sqrt{2}\,\varepsilon\sum_{j=1}^{n} A_j + \sqrt{2}\,\varepsilon\sqrt{\sum_j A_j}\left(\sum_j L_j\right)$$

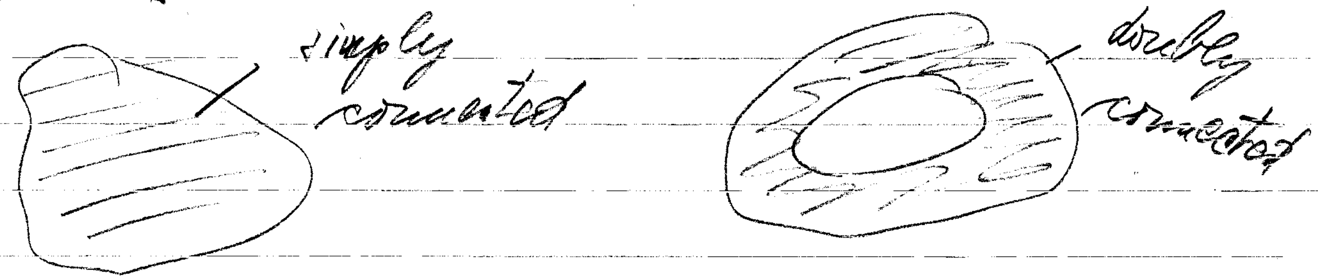
$$= \left(4\sqrt{2}\,A + \sqrt{2}\sqrt{A}\,L\right)\varepsilon$$

A - area enclosed by γ
L - length of γ } finite

$\varepsilon \to 0$, ✓.

Domain connectivity:

$\Omega^{open} \subset \mathbb{C}$ is n-tuply connected if it has $(n-1)$-holes; it is simply connected if it has no holes.



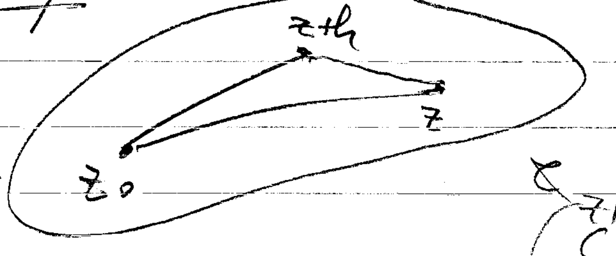
Elementary: $\Omega^{open} \subset \mathbb{C}$ is simply connected if every simple closed curve only contains pts. of Ω in its interior.

Prop A If $f(z)$ is continuous on a simply connected Ω and $\oint f(z)dz = 0$ for all Jordan contours
$\Rightarrow$ then there exists a function $F(z)$, analytic (in Ω) so that $F'(z) = f(z)$

Proof:

Note: $\int_{z_1}^{z_2} f(z)dz$ is path independent.

Let $F(z) = \int_{z_0}^{z} f(z)dz$.



$$\left(\int_{z_0}^{z+h} + \int_{z+h}^{z} + \int_{z}^{z_0}\right) = 0 \Rightarrow$$

$$\int_{z_0}^{z+h} f(z)\,dz - \int_{z_0}^{z} f(z)\,dz = \int_{z}^{z+h} f(z)\,dz$$

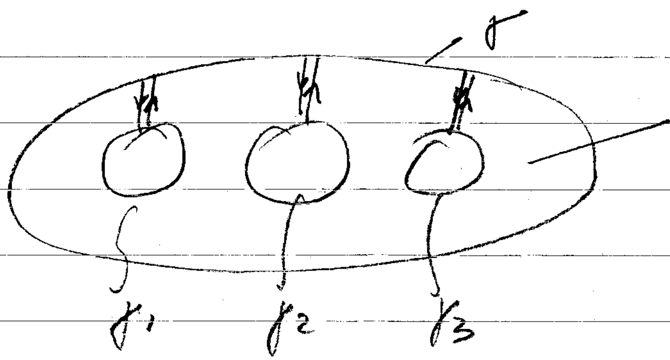
$$\Rightarrow \frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{z}^{z+h} f(z)\,dz$$

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \le \cancel{\frac{1}{|h|}}\,\cancel{|h|} \max_{|\zeta - z| < |h|} |f(\zeta) - f(z)|$$

$$h \to 0 \quad \Rightarrow \quad F'(z) = f(z)$$

Integration on multiply connected domains

$$\oint_{\gamma} f(z)\,dz = \sum_{k=1} \oint_{\gamma_k} f(z)\,dz$$



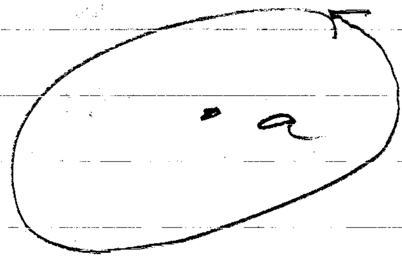
After cuts: simply connected domain

$$\oint f(z)\,dz = 0$$

& $\int$'als over cuts cancel.

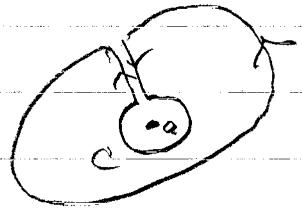
Example:

Let γ enclose a



$$I \equiv \oint_\gamma \frac{dz}{(z-a)^m} = ?$$

Deform γ to a circle:



$$I = \oint_C \frac{dz}{(z-a)^m} \qquad z-a = re^{i\theta}$$

$$dz = re^{i\theta}\, i\, d\theta$$

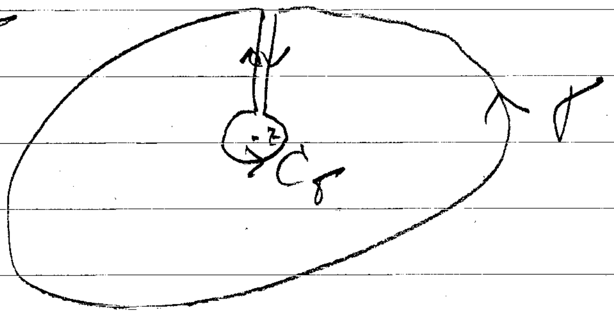
$$I = \int_0^{2\pi} r^{-m+1} e^{i(-m+1)\theta}\, i\, d\theta = \begin{cases} 1, & m=1 \\ 0, & m \neq 1 \end{cases}$$

Cauchy's Formula

Thm If $f(z)$ is analytic on and inside a simple closed contour γ, then for $\forall z \in \mathrm{Int}(\gamma)$

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

PROOF



$$\oint_\gamma = \int_{C_\delta}$$

$$\int_{C_\delta} \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z) \underbrace{\int_{C_\delta} \frac{d\zeta}{\zeta - z}}_{2\pi i} + \int_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta$$

If $|\zeta - z| = \delta$ then $|f(\zeta) - f(z)| < \varepsilon$

b/c $f(z)$ is continuous

$$\left| \oint_{C_\delta} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \right| \leq \oint_{C_\delta} \frac{|f(\zeta) - f(z)|}{|\zeta - z|} |d\zeta|$$

$$\leq \frac{\varepsilon}{\delta} \oint_{C_\delta} |d\zeta| = 2\pi\varepsilon$$

As $\varepsilon \to 0$, $\delta \to 0$.

Thm If $f(z)$ is analytic on a simple closed contour γ and in Int(γ), then all derivatives $f^{(k)}(z)$ exist and

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

Proof $\frac{f(z+h)-f(z)}{h} =$

$$= \frac{1}{2\pi i}\frac{1}{h}\oint_\gamma f(\zeta)\left[\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}\right] d\zeta$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta-(z+h))(\zeta-z)}\, d\zeta =$$

$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\, d\zeta + R$$

$$R = \frac{h}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta-z)^2(\zeta-z-h)}\, d\zeta$$

Let $\min_\gamma |\zeta - z| = 2\delta$. If $|h|<\delta$, for ζ on γ, we have

$$|\zeta-z-h| \geq |\zeta-z| - |h| \geq 2\delta-\delta = \delta$$

Since $|f(\zeta)| < M$ on γ, then

$$|R| \leq \frac{|h|}{2\pi i}\frac{M}{(2\delta)^2\delta} L, \quad L = \text{length of } \gamma$$

and $|R| \to 0$ as $h \to 0$.

$$\Rightarrow f'(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\, d\zeta$$

Almost the same argument proves

$$f''(z) = \frac{2}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta-z)^3}\, d\zeta$$

$\Rightarrow$ $f'(z)$ is analytic

$\Rightarrow$ can use induction to show

$f^{(k)}(z)$ is analytic

$$\Rightarrow f^{(k)}(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f^{(k)}(\zeta)}{\zeta - z}\, d\zeta$$

Integrate by parts k-times

$$f^{(k)}(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta-z)^{k+1}}\, d\zeta$$

Liouville's theorem:

Cauchy's estimates:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

$$C = \{ |\zeta - z| = R \}, \quad |f(z)| < M$$

$$\Rightarrow \left. \begin{aligned} |f^{(n)}(z)| &\le \frac{n!}{2\pi} \oint_C \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \\ &\le \frac{n! M}{2\pi R^{n+1}} \oint_C |d\zeta| = \frac{n! M}{2\pi R^n} \end{aligned} \right\} (*)$$

If $f(z)$ is analytic in $\mathbb{C}$ it is called *entire*.

Liouville's Thm: If $f(z)$ is entire and bounded, including at ∞, it is a constant.

Proof: Use $(*)$ with $n = 1$

$$\Rightarrow \quad |f'(z)| \le \frac{M}{R}$$

Since $f(z)$ is entire, R is arbitrary

$\Rightarrow \quad f'(z) = 0$

$$f(z) = f(0) + \int_0^z f'(z)\,dz = f(0) = \text{const}.$$

Morera's theorem. If $f(z)$ is continuous in Ω open $\subset \mathbb{C}$ and $\oint_\gamma f(\zeta)\,d\zeta = 0$ for every simple contour $\gamma \subset \Omega \Rightarrow f(z)$ is analytic in Ω.

PROOF From prop A $\Rightarrow \exists F(z)$, analytic in Ω s.t. $f(z) = F'(z)$. $\Rightarrow$

$f(z)$ is analytic.

Fundamental theorem of Algebra:

Any polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ has at least one root in $\mathbb{C}$.

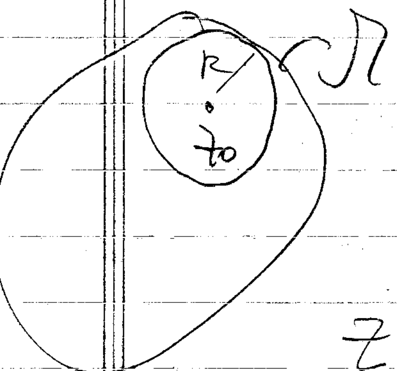
Proof. $Q(z) = 1/P(z)$ would otherwise be analytic and bounded at ∞.

$\Rightarrow$ constant.

Maximum Modulus Principle

If $f(z)$ is analytic on R (open, bounded) and $|f(z)|$ continuous on $\bar{R}$, then $|f(z)|$ attains its maximum on the boundary ∂R.

Proof: Let $|f(z_0)|$ be max, $z_0 \in R$.



$\Rightarrow$ Take a circle with radius R around z_0 that is contained in R.

$z = z_0 + R e^{i\theta}$.

Cauchy's formula: $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R e^{i\theta})\, d\theta$

$\Rightarrow |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + R e^{i\theta})|\, d\theta$

If $|f(z_0)|$ were a maximum

$\Rightarrow |f(z_0)| \ge |f(z_0 + R e^{i\theta})|$

If strict inequality holds at some point, it must hold for an arc

$\Rightarrow \quad f(z_0) > \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$, contradiction

$\Rightarrow$ max $|f(z)|$ must be on ∂R

(Note $|f(z)|$ continuous on compact $\bar{R}$)

Thm Open Mapping If $f(z)$ is analytic, then it maps open neighborhoods into open neighborhoods (Proof later.)

Conformal mapping If $f(z)$ is analytic at z_0, and $z(t)$ a curve through z_0, and $f'(z_0) \neq 0$

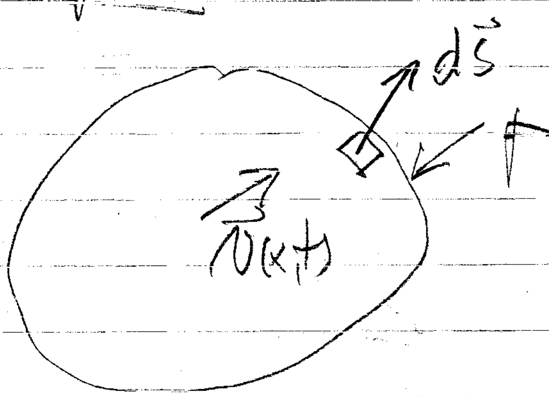
$\Rightarrow \quad w'(t) = f'(z_0)\, z'(t)$

$\Rightarrow \quad \arg w'(t) = \arg f'(z_0) + \arg z'(t)$

$(z = |z| e^{i\arg(z)})$ $\Rightarrow$ all angles increase by the same amount

$\Rightarrow$ angles are preserved $\Rightarrow$ mapping is conformal.

Example Fluid Flow



$\rho =$ const

Newton's law: $\frac{d}{dt}\iiint \rho\vec{v}\,dV = \iint p\,d\vec{S}$

Mass conservation $\frac{d}{dt}\iiint \rho\,dV = \iint \rho\vec{v}\cdot d\vec{S}$

$$\iint p\,d\vec{S} = \vec{i}\left(\iint p\,\vec{n}\cdot\vec{i}\,dS\right) + \vec{j}\iint p\,\vec{n}\cdot\vec{j}\,dS + $$
$$+ \vec{k}\iint p\,\vec{n}\cdot\vec{k}\,dS.$$

Look at $\iint p\,\vec{n}\cdot\vec{i}\,dS = \iint (p\vec{i})\,d\vec{S} =$

$$= \iiint \nabla\cdot(p\vec{i})\,dV = \iiint \frac{\partial p}{\partial x}\,dV$$

$$\Rightarrow \iint p\,d\vec{S} = \iiint \nabla p\,dV$$

Also $\frac{d}{dt}\iiint\limits_{V(t)} f(x,t)\, dV =$

$$= \frac{d}{dt}\iiint\limits_{V_0} f(x(x_0,t),t)\left|\frac{\partial x}{\partial x_0}\right| dV_0$$

$$= \iiint\limits_{V_0}\left\{\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\right)\left|\frac{\partial x}{\partial x_0}\right| + f\frac{\partial}{\partial}\left|\frac{\partial x}{\partial x_0}\right|\right\} dV_0$$

Now: Mass conservation $\Rightarrow$

$$0 = \iiint \operatorname{div}\vec{v}\, dV \Rightarrow \operatorname{div}\vec{v} = 0$$

and $\left|\frac{\partial x}{\partial x_0}\right| = 1 \Rightarrow \frac{d}{dt}\iiint\limits_{V} f\, dV = \iiint\limits_{V}\frac{df}{dt}\, dV$

$$\Rightarrow \frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla p$$

$\boxed{\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p}$ - Euler's equations

$\boxed{\nabla\vec{v} = 0}$ - continuity equation