PLANAR SYSTEMS

$$\dot{x} = P(x,y) \qquad \dot{y} = Q(x,y)$$

Trajectories can be obtained from

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

If you can solve this equation globally, you get a conserved quantity.

EXAMPLE: The Lotka - Volterra model

$$\dot{x} = kx - axy \quad \text{- prey} \qquad k, a, l, b > 0$$
$$\dot{y} = -ly + bxy \quad \text{- predator} \qquad x, y > 0$$

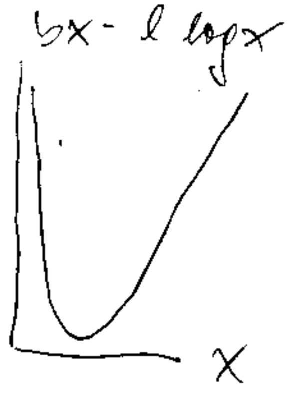
(a.k.a. predator - prey model)

$$\frac{dy}{dx} = \frac{y(bx-l)}{x(k-ay)}$$

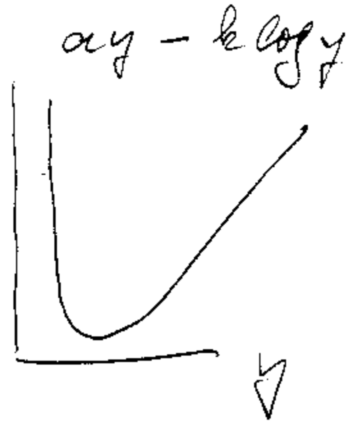
$$\left(\frac{k}{y} - a\right) dy = \left(b - \frac{l}{x}\right) dx$$

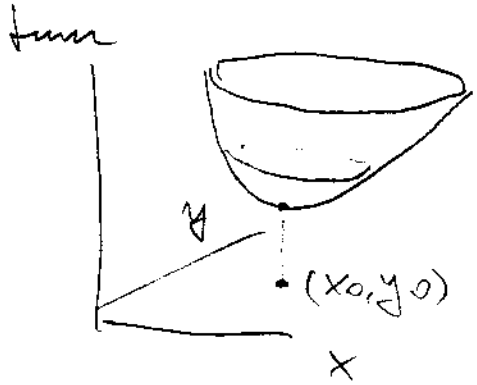
$$k \log y - ay = bx - l \log x + \text{CONST}$$

$$ay - k \log y + bx - l \log x = C$$

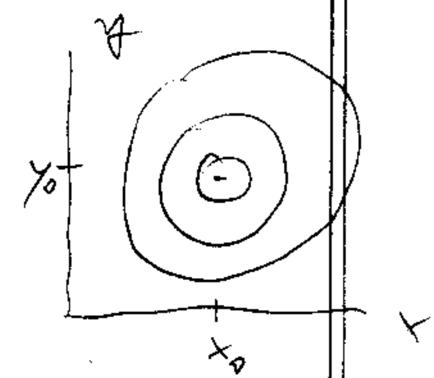


both are convex for $x,y>0$





$\Rightarrow$



Trajectories

Let $\vec{r} = (x,y)$, $\vec{R}(\vec{r}) = (P(x,y), Q(x,y))$

Consider $\dot{\vec{r}} = \vec{R}(\vec{r})$

Let $\vec{r} = \vec{r}_0$ be an equilibrium: $\vec{R}(\vec{r}_0) = 0$,

and let $A = \frac{\partial \vec{R}}{\partial \vec{r}}(\vec{r}_0)$.

<u>THM</u> If $\det A \neq 0$ then for any smooth $\vec{\rho}(\vec{r})$ and $\forall \varepsilon$ small enough, $\exists$ an equilibrium $\vec{r}(\varepsilon)$ of

$$\dot{\vec{r}} = \vec{R}(\vec{r}) + \varepsilon \vec{\rho}(\vec{r}).$$

<u>PF.</u> Implicit function theorem.

Holds in $\mathbb{R}^n$, too.

Write $\dot{\vec{r}} = \vec{R}(\vec{r})$ as

$$\dot{\vec{r}} = A\vec{r} + f(\vec{r}), \qquad (1)$$

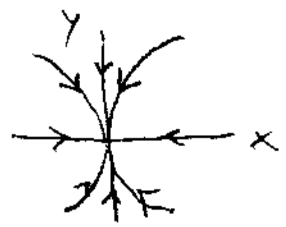
where $f(\vec{r}) = O(|\vec{r}|^2)$

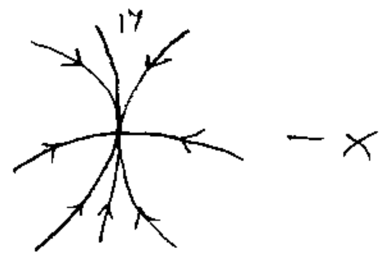
(here we assume $\vec{r}_0 = 0$)

We say that $(0,0)$ is a <u>hyperbolic equilibrium</u> of (1) if neither eigenvalue of A lies on the imaginary axis.

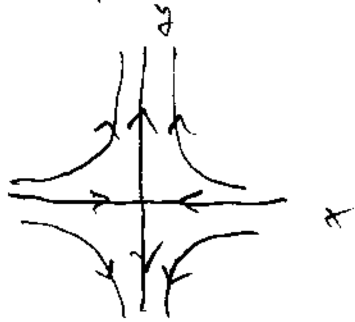
<u>CLAIM</u>: Let $(0,0)$ be a hyperbolic equilibrium of (1) and let $\lambda_1 \neq \lambda_2$ (λ_1, λ_2 - eigenvalues of A). Then the phase portrait of $\dot{\vec{r}} = A\vec{r}$ is only slightly deformed for $\dot{\vec{r}} = A\vec{r} + f(\vec{r})$, and the size of the deformation is $O(|\vec{r}|^2)$

E.g.: $\dot{\vec{r}} = A\vec{r}$ $\qquad\qquad$ $\dot{\vec{r}} = A\vec{r} + f(\vec{r})$

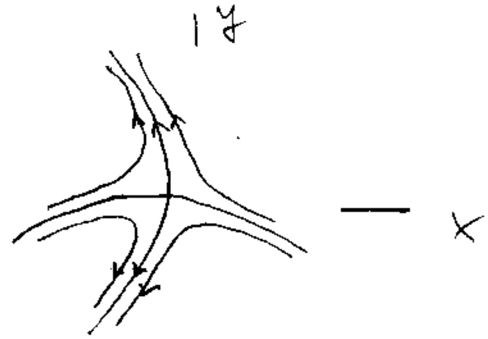




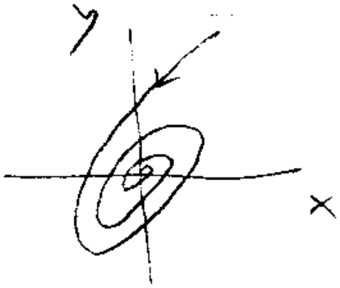
$\dot{\vec{r}} = A\vec{r}$



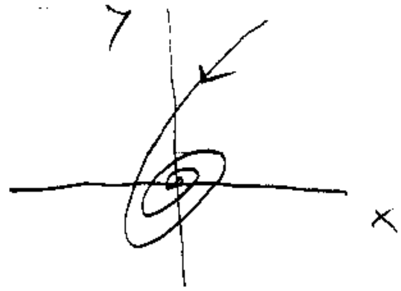
$\dot{\vec{r}} = A\vec{r} + f(\vec{r})$



$\dot{\vec{r}} = A\vec{r}$

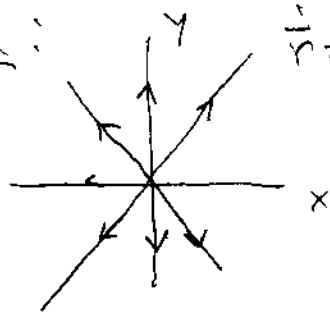


$\dot{\vec{r}} = A\vec{r} + f(\vec{r})$

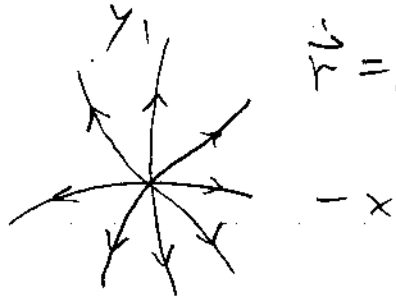


Even:

$\lambda_1 = \lambda_2$



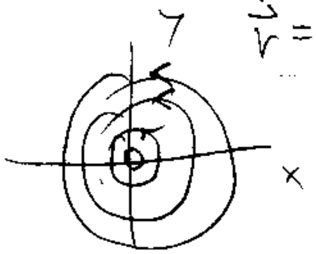
$\dot{\vec{r}} = A\vec{r}$



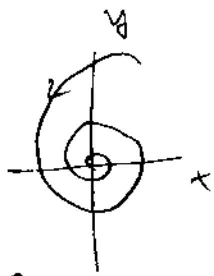
$\dot{\vec{r}} = A\vec{r} + f(\vec{r})$

But: centers may turn into spiral sources or sinks or something worse

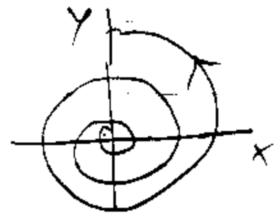
$\dot{\vec{r}} = A\vec{r}$



$\dot{\vec{r}} = A\vec{r} + f(\vec{r})$



or



or worse

sometimes

EXAMPLES:

1.) Competition between two species

rabbits versus sheep:

Species compete for the same limited amount of food.

Assumptions
1.) Logistic growth for one species in the absence of the other
2.) Interaction is proportional to the size of each population, and reduces the growth

$x(t)$ - rabbits $y(t)$ - sheep   , $x, y > 0$

$$\dot{x} = x(3 - x - 2y) \qquad \dot{y} = y(2 - x - y)$$

rabbits multiply faster

rabbits get pushed aside by the sheep and are to more negatively affected by competition

sheep multiply more slowly

sheep are less negatively affected by the competition.

EQUILIBRIUM POINTS:

$x(3-x-2y)=0 \qquad y(2-x-y)=0$

1.) $x=y=0 \Rightarrow (0,0)$

2.) $x=0,\ y=2 \Rightarrow (0,2)$

3.) $y=0,\ x=3 \Rightarrow (3,0)$

4.) $\left.\begin{array}{l} x+2y=3 \\ x+y=2 \end{array}\right\} \Rightarrow y=1,\ x=1 \Rightarrow (1,1)$

Stability: Compute the Jacobian matrix (linearization matrix)

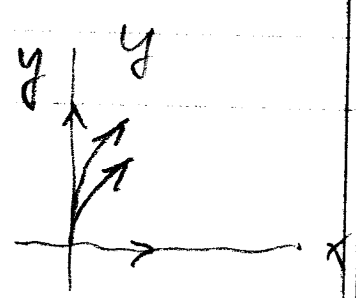
$$A(x,y)=\begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}=\begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

$(0,0)$: $A=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

eigenvalues: $\lambda_1=3,\ \lambda_2=2$

eigenvectors: $e_1=\begin{pmatrix}1\\0\end{pmatrix},\ e_2=\begin{pmatrix}0\\1\end{pmatrix}$

unstable node (source)

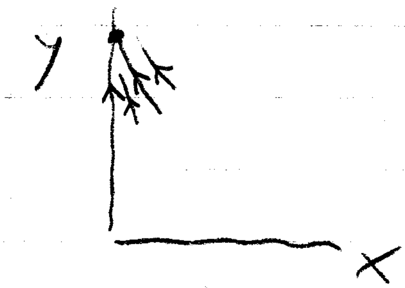


(0,2): $A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$

eigenvalues: $\lambda_1 = -1$ $\lambda_2 = -2$

eigenvectors: $e_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

stable node
(sink)

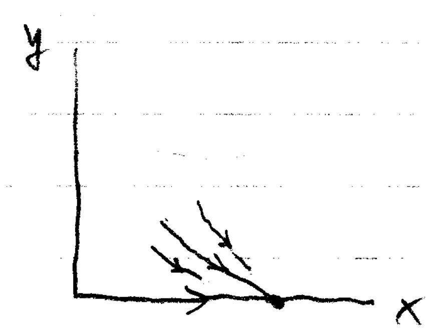


(3,0): $A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$

eigenvalues $\lambda_1 = -3$ $\lambda_2 = -1$

eigenvectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

stable node
(sink)



(1,1):

$A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

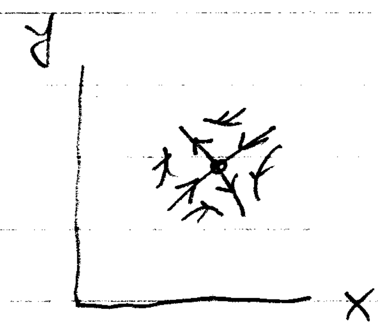
$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & -2 \\ -1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 2 = 0$$

$$(1+\lambda)^2 = 2 \qquad 1+\lambda = \pm\sqrt{2} \qquad \lambda_{1,2} = -1 \pm \sqrt{2}$$

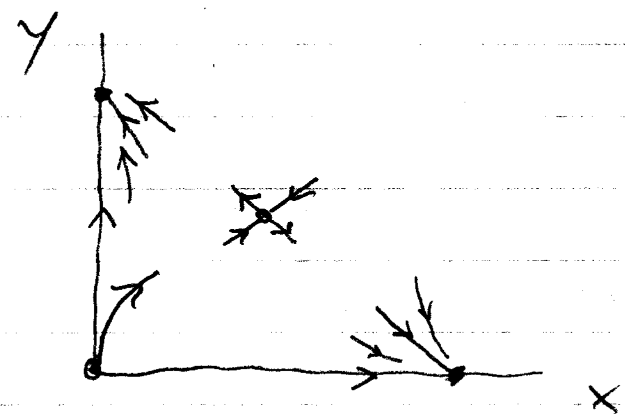
eigenvalues: $\lambda_1 = -1+\sqrt{2}$ $\quad \lambda_2 = -1-\sqrt{2}$

eigenvectors: $e_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} \sqrt{2} \\ \end{pmatrix}$

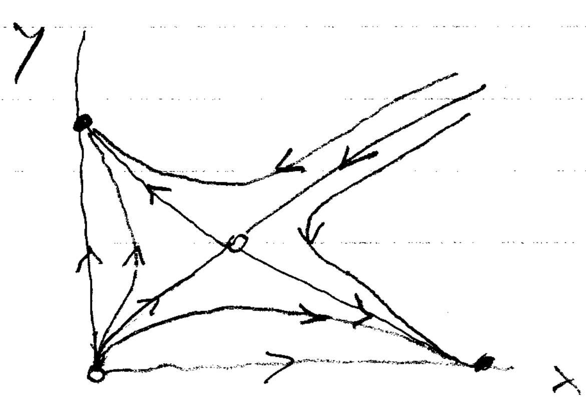
saddle



$\Rightarrow$



USE common sense to connect these points



No coexistence: either rabbits or sheep become extinct.

DAMPED PENDULUM: $\dot{x} = y \qquad \dot{y} = -\sin x - \beta y$

$\beta \geq 0$

Equilibria; $x = 0, \pi, \quad y = 0$

Jacobian: $A(x,y) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos x & -\beta \end{bmatrix}$

$(0,0)$: $A = \begin{bmatrix} 0 & 1 \\ -1 & -\beta \end{bmatrix}$

stability: $\lambda(\lambda + \beta) + 1 = 0$

$\lambda^2 + \beta\lambda + 1 = 0$

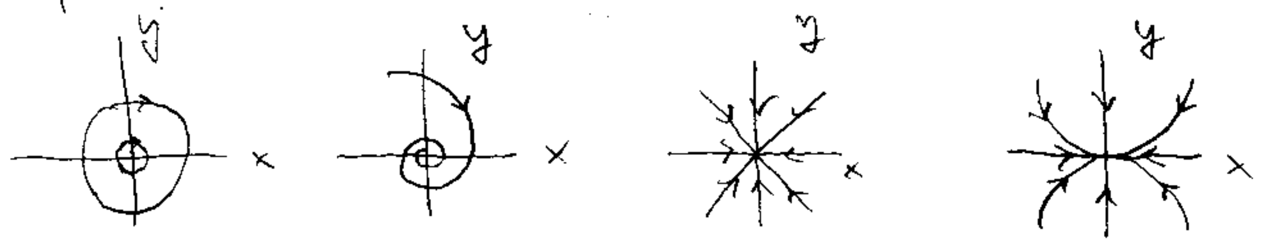
$\lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 - 1}$

$\beta = 0$: $\lambda_{1,2} = \pm i$ center

$0 < \beta < 1$: $\lambda_{1,2} = -\frac{\beta}{2} \pm i\sqrt{1 - \left(\frac{\beta}{2}\right)^2}$ spiral sink

$\beta > 1$: $\lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 - 1}$ stable node (sink)

$\beta = 1$: $\lambda_{1,2} = -\frac{\beta}{2}$ stable star



$\beta = 0 \qquad 0 < \beta < 1 \qquad \beta = 1 \qquad \beta > 1$

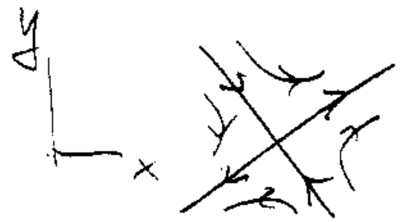
$\underline{(0,\pi)}$:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -\beta \end{pmatrix}$$

stability: $\lambda(\lambda+\beta) - 1 = 0$

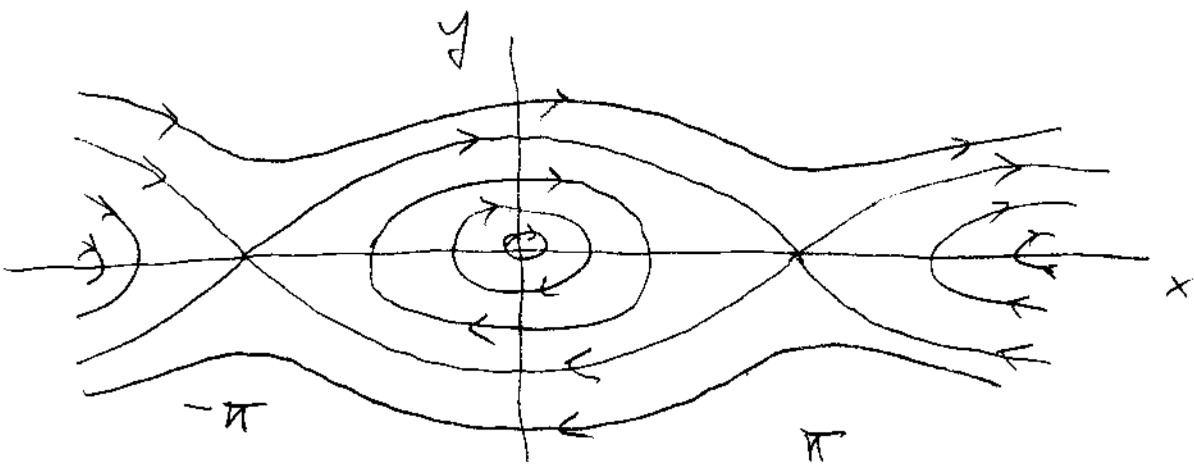
$$\lambda^2 + \beta\lambda + 1 = 0$$

$$\lambda_{1,2} = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + 1}$$ saddle



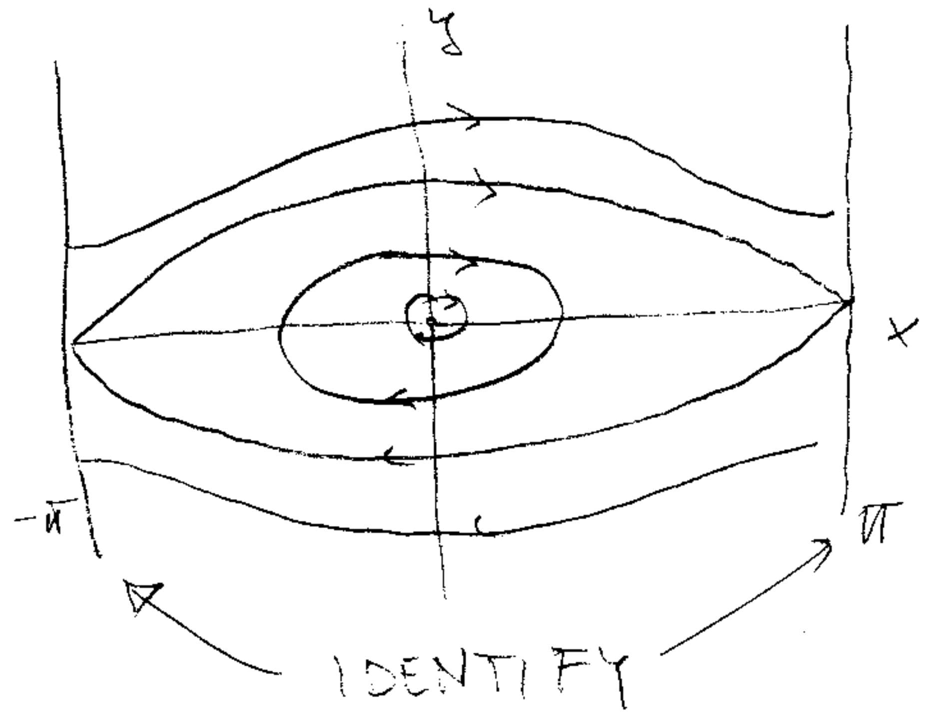
$\underline{\beta = 0}$: This is a Newtonian system:

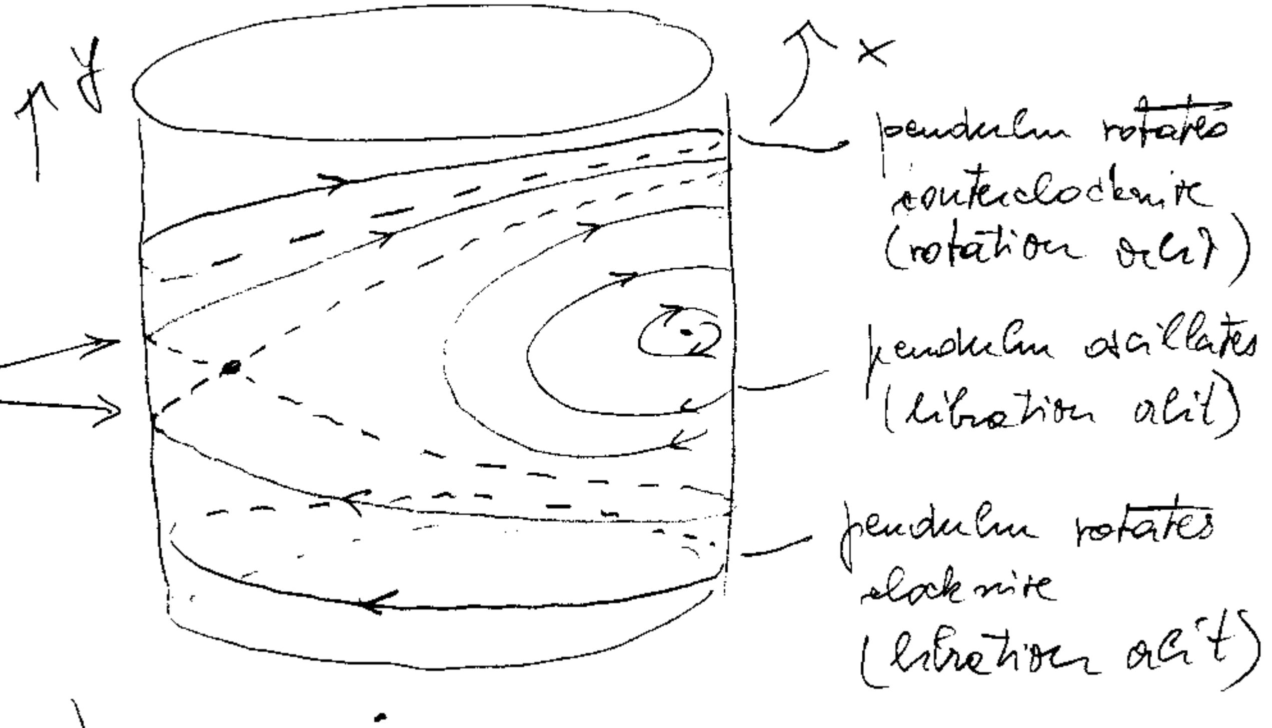
energy $E = \frac{1}{2} y^2 - \cos x = \text{const}$



The phase portrait is periodic in x with period 2π, because the equations are

$\Rightarrow$ roll it into a cylinder



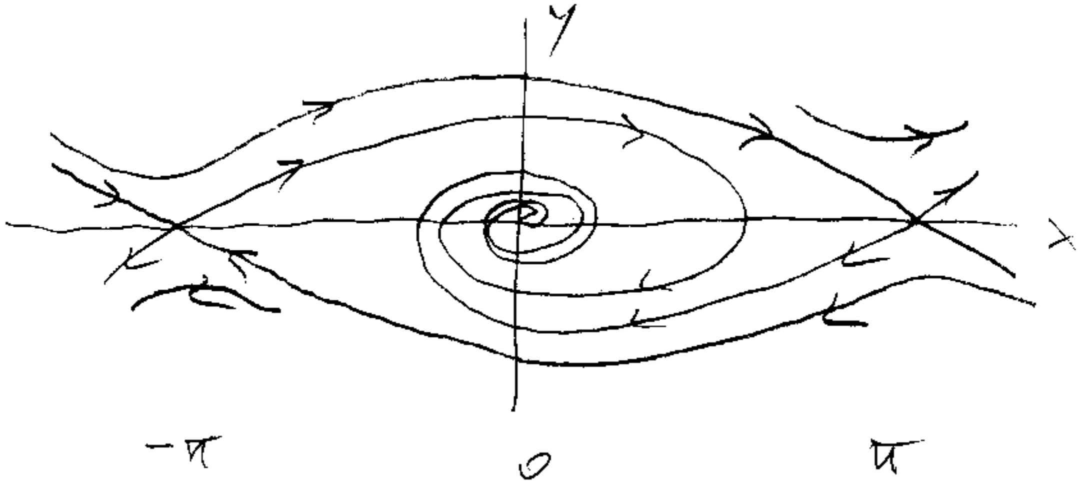


$0 < \beta < 1$

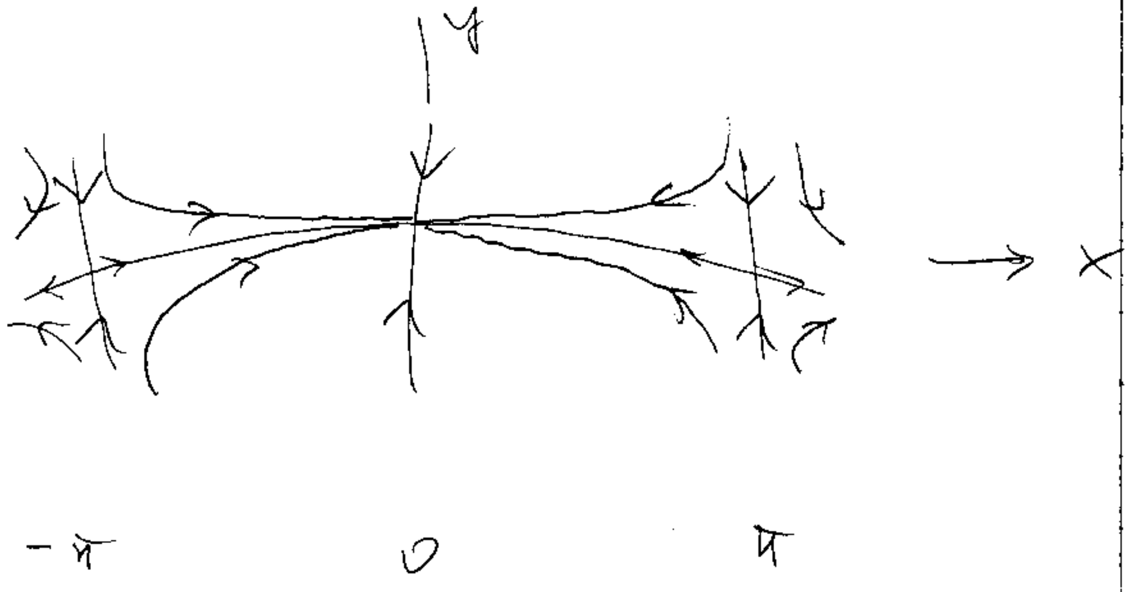
$$\dot{E} = y\dot{y} + \sin x\,\dot{x} =$$

$$= y(-\sin x - \beta y) + \sin x\, y$$

$$= -\beta y^2 < 0$$



$\underline{\beta > 1}$



GLOBAL THEORY OF PLANAR SYSTEMS

Index theory

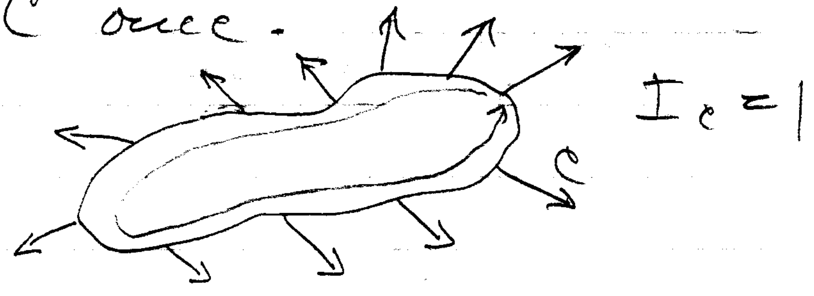
The purpose of this section is to prove the theorem: Let $\dot{x} = f(x)$. Then

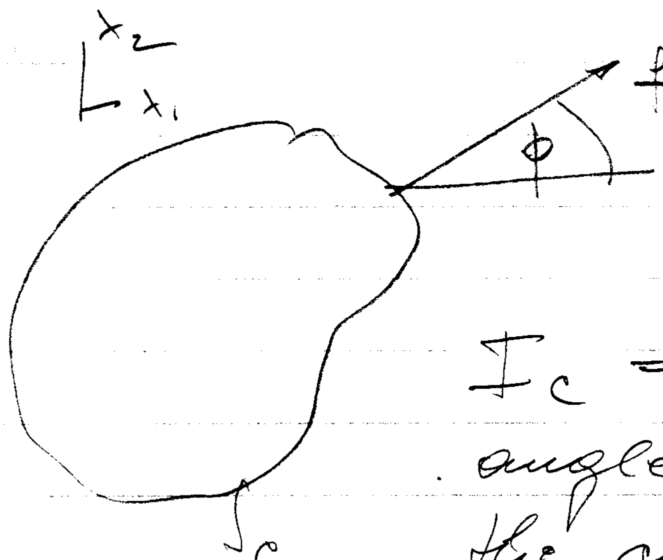
THM Inside every periodic orbit in $\mathbb{R}^2$ there must be at least one equilibrium point.

Let $f(x)$ be a vector field on $\mathbb{R}^2$ (i.e. a function $f: \mathbb{R}^2 \to \mathbb{R}^2$): $\bar{x}$ is a critical point of f if $f(\bar{x}) = 0$, i.e. if it is an equilibrium point of $\dot{x} = f(x)$.

Index of a closed curve

The index of a closed curve C is the number of revolutions made by the vector field $f(x)$ in traversing the curve C once.





In other words,

I_C = change of the angle ϕ after traversing the curve C once

$$\phi = \arctan \frac{\dot{x}_2}{\dot{x}_1}$$

Since ϕ changes continuously the change in ϕ is an integer multiple of 2π

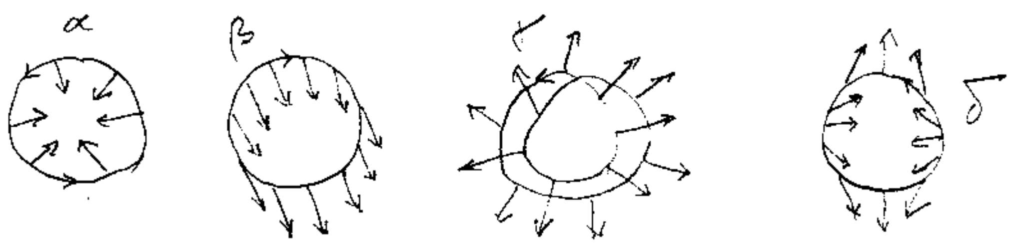
$$I_C = \frac{1}{2\pi} \Delta\phi \Big|_C$$ — change of ϕ along C.

$$I_C = \frac{1}{2\pi} \oint_C d\phi =$$

$$= \frac{1}{2\pi} \oint_C d\left(\arctan \frac{f_2(x)}{f_1(x)}\right) =$$

$$= \frac{1}{2\pi} \oint_C \frac{f_2\, dx_1 - f_1\, dx_2}{f_1^2 + f_2^2}$$

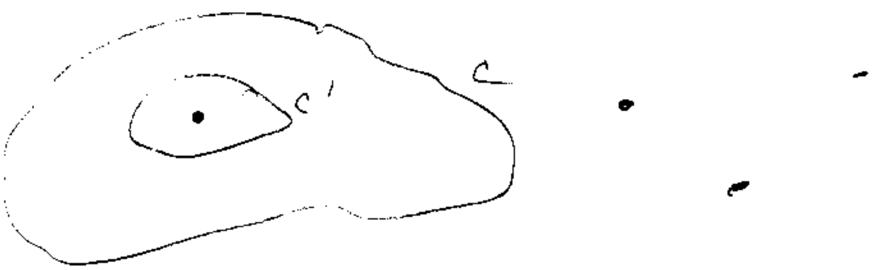
Curves with various indices:



$I_\alpha = 1$ $I_\beta = 0$ $I_\gamma = 2$ $I_\delta = -1$

PROPERTIES OF THE INDEX

1.) If C can be continuously deformed into C' without passing through any critical pts. of $f(x)$, $I_C = I_{C'}$.



PROOF: I_C is an integer on the one hand, and varies continuously on the other $\Rightarrow$ I_C = CONST.

2.) If C doesn't contain any critical points of $f(x)$, then $I_C = 0$

PROOF: You can shrink C to a point at which $f(x)$ is just one vector, which does not rotate at all.

3.) For $-f(x)$ and any closed curve C, I_c is the same as for $f(x)$

Proof: All angles change from ϕ to $\phi+\pi$. Hence $\Delta\phi|_c$ stays the same.

4.) If C is a closed orbit then $I_c = 1$
$\Rightarrow$ C must enclose at least one critical pt.

Proof: See the picture.
We omit the formal proof.

Index of an equilibrium point

Let $\bar{x}$ be an equilibrium point of $\dot{x} = f(x)$, i.e., a critical pt. of $f(x)$.

The index of $\bar{x}$, $I(\bar{x})$, is the index I_c of any closed curve c ~~that~~ encloses $\bar{x}$ and no other equilibrium point.

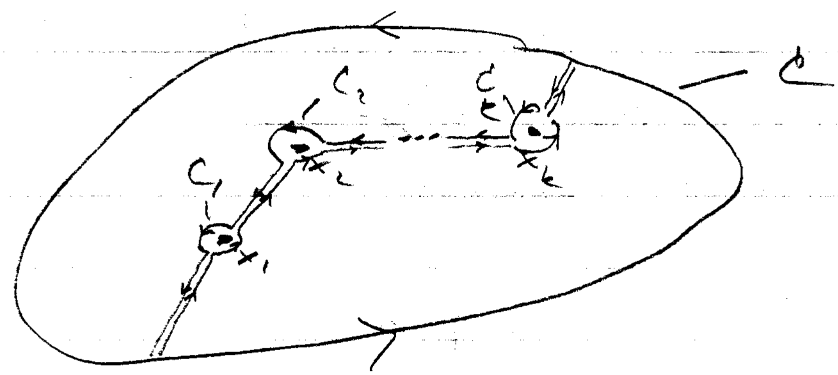
An equilibrium point $\bar{x}$ of $\dot{x}=f(x)$ is simple if $\frac{\partial f}{\partial x}(\bar{x})$ has both eigenvalues nonzero.

THM: The index of any simple equilibrium point except for the saddle is $=+1$. The index of a saddle is $=-1$.

(Nodes, spiral points, centers: $I=+1$)

THM: Let C be a closed curve. Its index I_C is the sum of the indices of the equilibria enclosed by C

PROOF



$$I_C = \sum I_{C_k}$$

THM Any closed orbit in $\mathbb{R}^2$ must enclose equilibria whose indices sum to $+1$.

⇒ THM Any closed orbit C of $\dot{x} = f(x)$ must contain at least one equilibrium point $\bar{x}$ of $\dot{x} = f(x)$.

PROOF: Above properties 3.) and 4.)

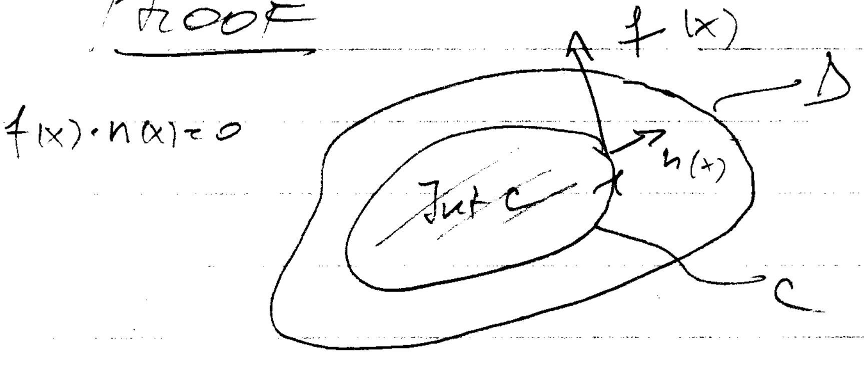
BENDIXSON CRITERION

When can we rule out the existence of limit cycles of $\dot{x} = f(x)$?

Let $\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$

THM If, in a simply connected (no holes) region $D \subset \mathbb{R}^2$, $\nabla \cdot f(x)$ is of one sign, then $\dot{x} = f(x)$ has no closed orbits in D.

PROOF



Let C be a limit cycle of $\dot{x} = f(x)$

Then

$$0 = \oint_C f(x) \cdot n(x)\, dx =$$

$$= \iint_{\mathrm{Int}\, C} \nabla \cdot f(x)\, dx_1 dx_2 \gtrless 0 \text{ — contradiction}$$

<u>EXAMPLES</u> 1) $\dot{x} = Ax$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\nabla \cdot Ax = a + d$$

This system has no closed orbits if $a+d \neq 0$. Now, the characteristic polynomial of A is

$$\lambda^2 - (a+d)\lambda + ad - bc,$$

so $a+d = \lambda_1 + \lambda_2$.

If $a+d=0$, $\lambda_1 = -\lambda_2$, so either $\lambda_1 = -\lambda_2 = \alpha \in \mathbb{R}$, and $x=0$ is a saddle ($\Rightarrow$ no closed orbits) or $\lambda_1 = -\lambda_2 = i\beta$, $\beta \in \mathbb{R}$, and $x=0$ is a center ($\Rightarrow$ only closed orbits)

2) $\dot{p} = -\frac{\partial H}{\partial q} - \alpha p$, $\dot{q} = \frac{\partial H}{\partial p} - \alpha q$

$$\nabla \cdot f = -\frac{\partial^2 H}{\partial p \partial q} - \alpha + \frac{\partial^2 H}{\partial p \partial q} - \alpha = -2\alpha$$

$\Rightarrow$ Damped planar Hamiltonian systems can have no closed orbits.

POINCARÉ - BENDIXSON THEOREM

How can we show that a periodic orbit exists for $\dot{x} = f(x)$?

THM Suppose that

(1) R is a closed, bounded subset of $\mathbb{R}^2$

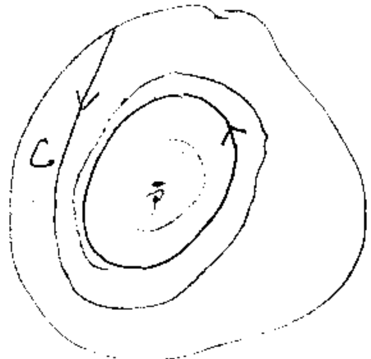
(2) $\dot{x} = f(x)$ is a C^1 vector field on an open set containing R

(3) R contains no equilibrium points

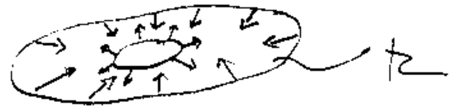
(4) There exists a trajectory C that is "confined" in R in the sense that it starts in R and stays in R for all $t \geq 0$.

Then, R contains a closed orbit.

In fact either C is a closed orbit or it spirals towards one.



How do we ensure that C exists? Construct a trapping region R, i.e. a closed connected set s.t. $f(x)$ points inwards everywhere on the boundary of R. Then all trajectory in R are confined. If there are no equilibria in R, Poincare - Bendixson Thm. applies.



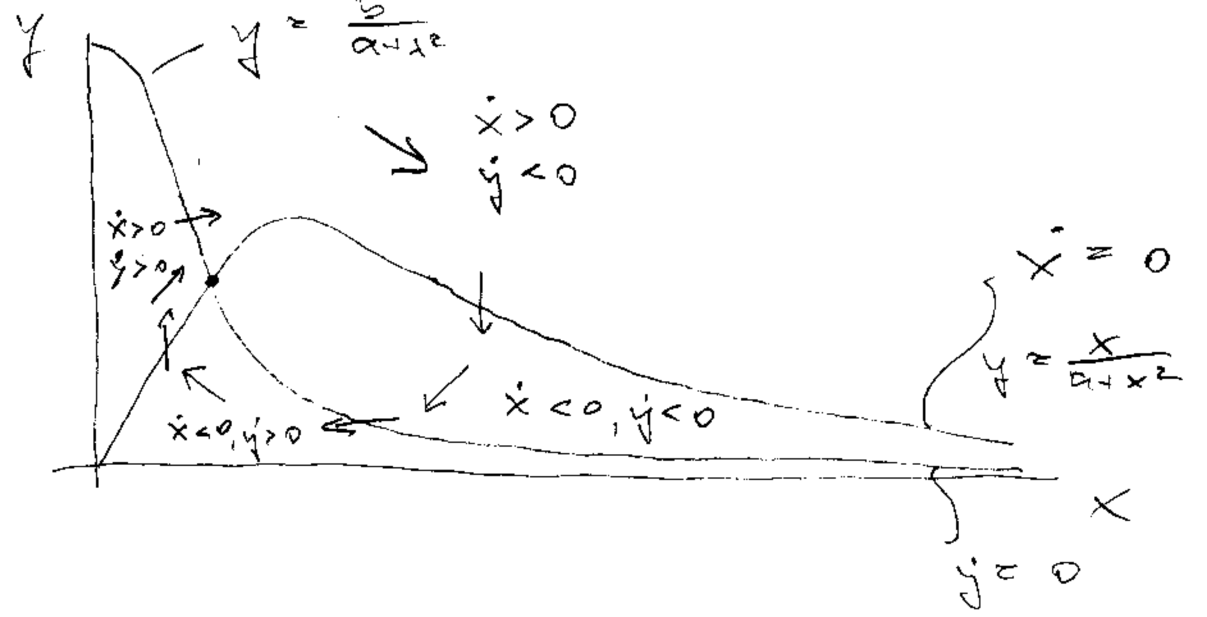
EXAMPLE : Used to describe glycolysis, the process of breaking down sugar to get energy within a cell

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$

Check for where $\dot{x} = 0$ or $\dot{y} = 0$

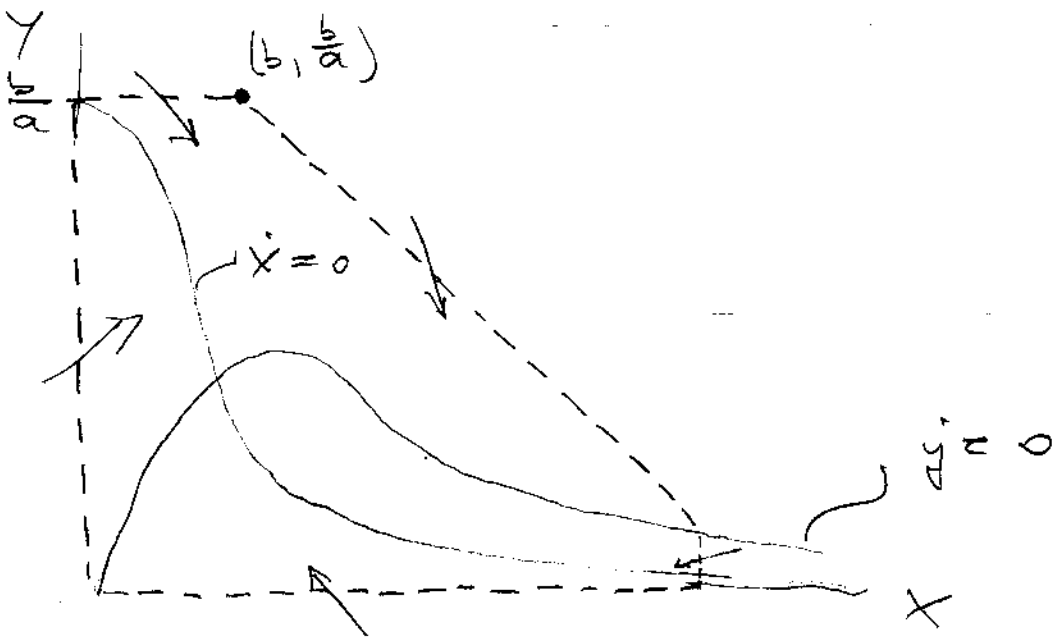
$$\dot{x} = 0 : y = \frac{x}{a+x^2}$$

$$\dot{y} = 0 : y = \frac{b}{a+x^2}$$



Appears like there may be a limit cycle.

CONSIDER THE REGION:



The tricky part was to get the line with slope -1 extending from $(b, \frac{b}{a})$ to the curve with $\dot{y} = 0$.

GET THE INTUITION: Large x: $\dot{x} = x^2 y$, $\dot{y} = -x^2 y$

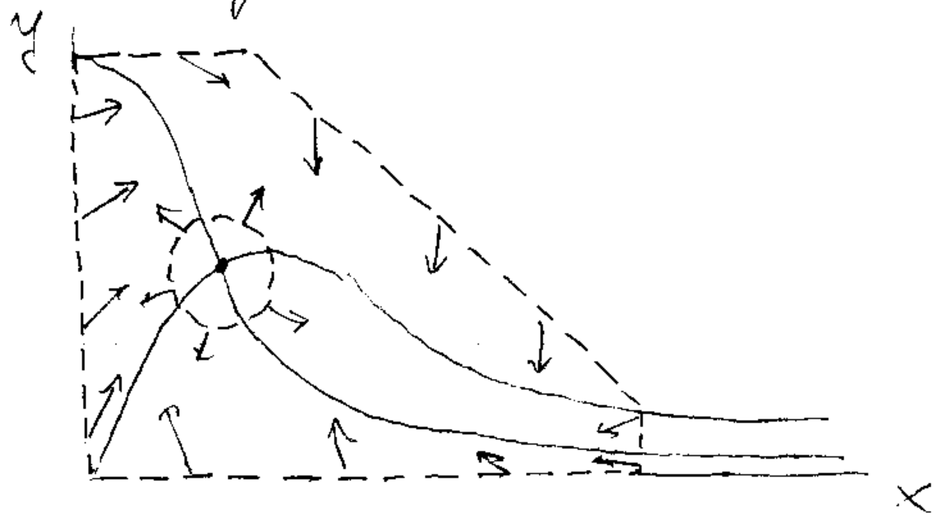
$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -1$$

$$\Rightarrow \text{LOOK AT} \quad \dot{x} - (-\dot{y}) = -x + ay + x^2 y + (b - ay - x^2 y)$$
$$= b - x$$

$\Rightarrow -\dot{y} > \dot{x}$ if $x > b \Rightarrow \frac{dy}{dx} < -1$ on the line with slope -1.

$\Rightarrow$  is a trapping region

$\Rightarrow$ there is at least one closed orbit in the trapping region if the only equilibrium in this region is a source



The equilibrium: $x = b$, $y = \frac{b}{a+b^2}$

Stability matrix: $A = \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -(a+x^2) \end{bmatrix}$

$$A\left(b, \frac{b}{a+b^2}\right) = \begin{bmatrix} -1+\frac{2b^2}{a+b^2}, & a+b^2 \\ -\frac{2b^2}{a+b^2} & -(a+b^2) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{b^2-a}{a+b^2} & a+b^2 \\ -\frac{2b^2}{a+b^2} & -(a+b^2) \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{bmatrix}$$

CHARACTERISTIC POLYNOMIAL

$$P(\lambda) = \lambda^2 - (A_{11} + A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21}$$

$$= \lambda^2 - \mathrm{Tr}\,A\,\lambda + \mathrm{Det}\,A =$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$\lambda_1\lambda_2 = \mathrm{Det}\,A = -\frac{b^2-a}{a+b^2}(a+b^2) + \frac{2b^2}{a+b^2}(a+b^2) =$$

$$= a + b^2 > 0$$

$\Rightarrow$ λ_1 & λ_2 have the same sign

$$\lambda_1 + \lambda_2 \lessgtr 0 \quad \text{iff} \quad \mathrm{Tr}\,A \lessgtr 0$$

$$\mathrm{Tr}\,A = \frac{b^2-a}{a+b^2} - a - b^2 = \frac{b^2 - a - (a+b^2)^2}{a+b^2}$$

$$\mathrm{Tr}\,A > 0 \quad \text{iff} \quad b^2 - a > (a+b^2)^2$$

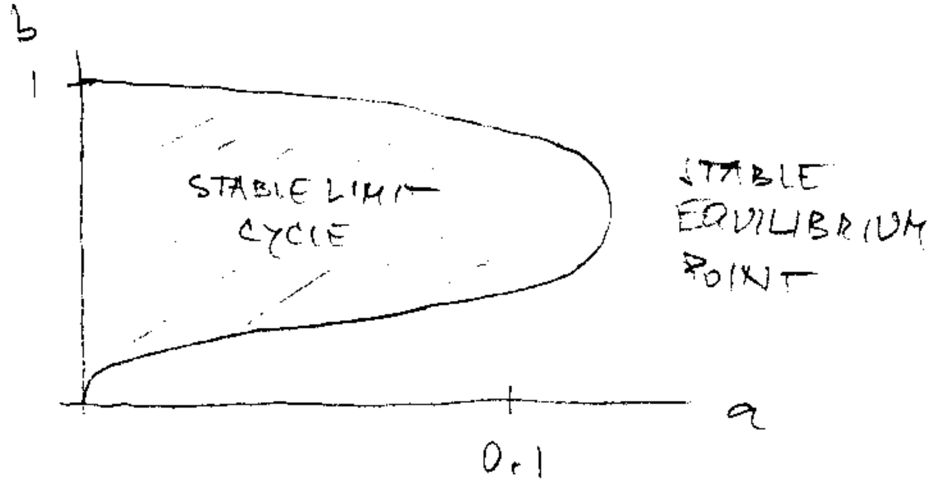
$$b^2 - a - a^2 - 2ab^2 - b^4 > 0$$

$$b^4 + (2a-1)b^2 + a^2 + a < 0$$

$\mathrm{Tr}\,A < 0$ iff

$$b^2 = \frac{1}{2}\left(1-2a \pm \sqrt{(2a-1)^2 - 4a^2 - 4a}\right)$$

$$= \frac{1}{2}\left(1-2a \pm \sqrt{1-8a}\right)$$



NOTE: There could be more than one stable limit cycle: there could be two, with an unstable one in the middle. We can only exclude this possibility by numerical computation.

Small perturbations of nonconservative systems

EXAMPLE: $\dot{x}_1 = x_2 + \varepsilon f_1(x_1, x_2)$ $\varepsilon \ll 1$

$\dot{x}_2 = -x_1 + \varepsilon f_2(x_1, x_2)$ $x_1^2 + x_2^2 \le R^2$

$\underline{\varepsilon = 0}$ $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$

$x_1 = A\cos(t - t_0)$, $x_2 = -A\sin(t - t_0)$

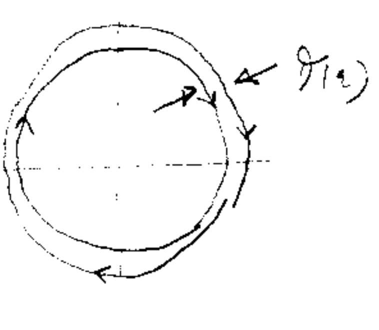
$\underline{\varepsilon \neq 0}$: By differentiability, for small $\varepsilon = \varepsilon(T)$, trajectories stay close to the circle of radius A in the interval $0 \le t \le T$.

For $\varepsilon \neq 0$, trajectories may be spirals.

Consider how the energy $E = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ changes after one circuit around the origin:

$$\dot{E}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = \varepsilon(x_1 f_1 + x_2 f_2)$$

$$\Delta E = \int_0^{2\pi} \dot{E}(A\cos t, -A\sin t)\,dt + O(\varepsilon^2)$$



$$\Delta E = \varepsilon \oint_{x_1^2 + x_2^2 = A^2} f_2\,\overset{x_2\,dt}{\overset{\shortparallel}{dx_1}} - f_1\,\overset{-x_1\,dt}{\overset{\shortparallel}{dx_2}} + O(\varepsilon^2)$$

$$= \varepsilon F(A) + O(\varepsilon^2)$$

$F(A) > 0 \Rightarrow$ OUTWARD SPIRAL

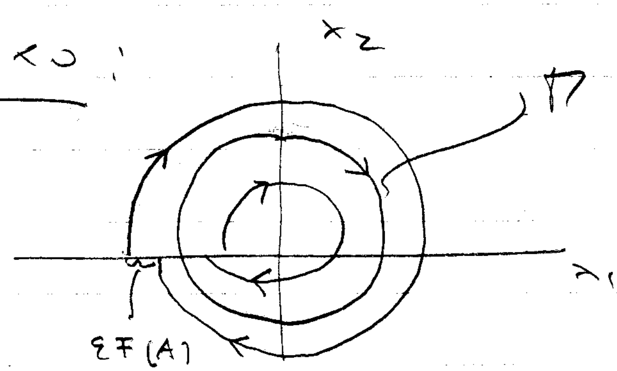
$F(A) < 0 \Rightarrow$ INWARD SPIRAL

If $F(A_0) = 0$, $F'(A_0) \neq 0$ (F changes sign), the implicit function theorem shows that there is a closed orbit Γ near the circle $x_1^2 + x_2^2 = A_0^2$, and it is a limit cycle.

Stable, if $F'(A_0) < 0$

Unstable, if $F'(A_0) > 0$

$\underline{F'(A_0) < 0}$:



$\Delta E = \varepsilon F(A) < 0$ if $A > A_0$

$\Delta E = \varepsilon F(A) > 0$ if $A < A_0$

$\Rightarrow \Gamma$ is stable

EXAMPLE Van Der Pol Oscillator

$$\ddot{x} + \varepsilon(x^2-1)\dot{x} + x = 0$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \varepsilon x_2(1-x_1^2)$$

$$f_1 = 0, \quad f_2 = x_2(1-x_1^2)$$

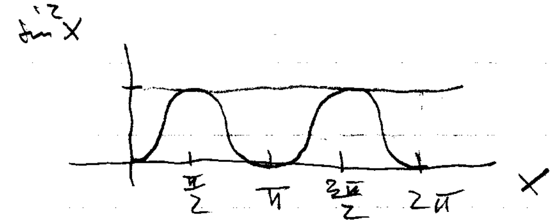
$$F(A) = \oint_{x_1^2+x_2^2=A^2} x_2(1-x_1^2)\,dx_1$$

Let $x_1 = A\cos t$, $x_2 = -A\sin t$

$$F(A) = \int_0^{2\pi} (-A\sin t)(1-A^2\cos^2 t)(-A\sin t)\,dt$$

$$= A^2\int_0^{2\pi}(\sin^2 t - A^2\sin^2 t\cos^2 t)\,dt$$

$$= A^2\int_0^{2\pi}\sin^2 t\,dt - \frac{A^4}{8}\int_0^{4\pi}\sin^2(2t)\,d(2t)$$



$$\int_{n\pi/2}^{(n+1)\pi/2} \sin^2 x\,dx = \frac{\pi}{4}$$

$$F(A) = A^2\pi - \frac{A^4}{8}\cdot 2\pi = \pi A^2\left(1-\frac{A^2}{4}\right)$$

$F(2) = 0$, $\quad F'(2) = \pi(2A - A^3) = -4\pi < 0$

$\Rightarrow$ STABLE LIMIT CYCLE NEAR $x_1^2+x_2^2 = 4$