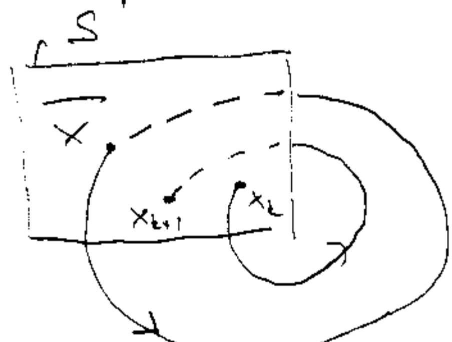
POINCARÉ MAPS

Useful for investigating existence and stability of periodic solutions of $\dot{x} = f(x)$

Surface of section: S, $(n-1)$-dimensional surface, transverse to the flow of $\dot{x} = f(x)$, i.e., all trajectories starting on S flow through it, not parallel to it



(Usually only defined locally.)

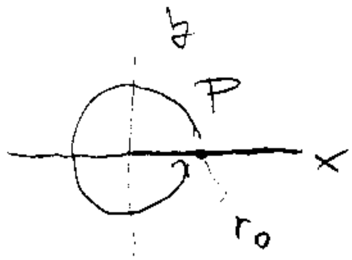
Poincaré map: $P : S \Rightarrow S$, obtained by following trajectories from one intersection with S to the next.

$x_k \in S$, k-th intersection $\Rightarrow x_{k+1} = P(x_k)$

Fixed point: $\bar{x}$, s.t. $P(\bar{x}) = \bar{x}$

$\bar{x}$ corresponds to a closed orbit (periodic solution) of $\dot{x} = f(x)$, see picture.

EXAMPLES: 1) $\dot{r} = r(1-r^2) \qquad \dot{\theta} = 1 \qquad S = \{(x,0) \mid x>0\}$



r_0 – I.C. on S

$\dot{\theta} = 1 \Rightarrow$ first return to S after time of flight $t = 2\pi$

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_0^{2\pi} dt = 2\pi$$

$$\int \frac{dr}{r(1-r^2)} = \int \left(\frac{A}{r} + \frac{B}{1-r} + \frac{C}{1+r} \right) dr$$

$$= \int \frac{A(1-r^2) + B\,r(1+r) + C\,r(1-r)}{r(1-r^2)}\, dr$$

$A = 1 \qquad B + C = 0 \Rightarrow C = -B$

$-A + B - C = 0$

$2B = A \Rightarrow B = \frac{1}{2}, \quad C = -\frac{1}{2}$

$$\int \frac{dr}{r(1-r^2)} = \frac{1}{2} \ln \frac{r^2}{|1-r^2|} = \frac{1}{2} \ln \frac{1}{\left|\frac{1}{r^2} - 1\right|}$$

$$\Rightarrow \ln \left| \frac{1}{\frac{1}{r_1^2} - 1} \right| - \ln \left| \frac{1}{\frac{1}{r_0^2} - 1} \right| = 4\pi$$

$$\left|\frac{1}{r_1^2}-1\right| = \left|\frac{1}{r_0^2}-1\right| e^{-4\pi}$$

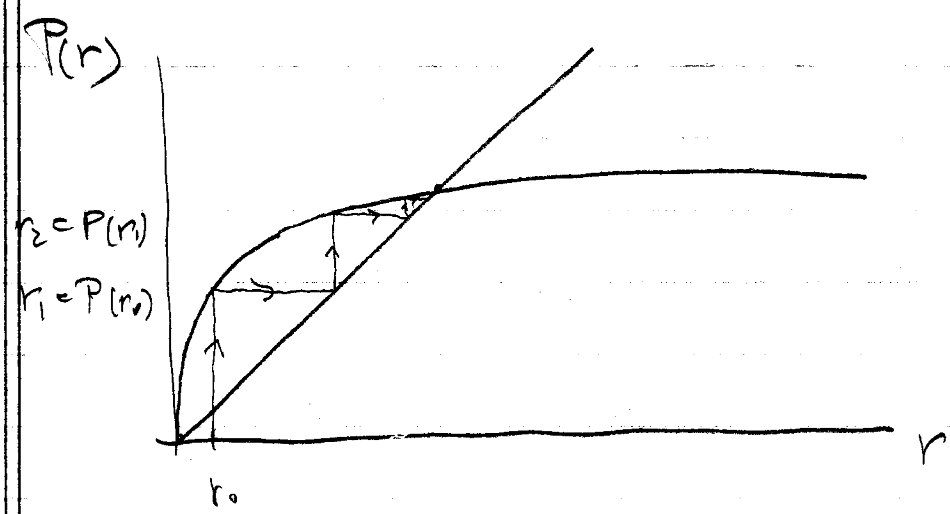
Since there is an equilibrium of $\dot r = r(1-r^2)$ at $r=1$, both $\left|\frac{1}{r_i^2}-1\right|$, $i=1,2$ must be of the same sign

$$\frac{1}{r_1^2}-1 = \left(\frac{1}{r_0^2}-1\right)e^{-4\pi}$$

$$r_1^2 = \frac{1}{1+\left(\frac{1}{r_0^2}-1\right)e^{-4\pi}}$$

$$r_1 = \frac{1}{\sqrt{1+\left(\frac{1}{r_0^2}-1\right)e^{-4\pi}}}$$

$$\Rightarrow P(r) = \frac{1}{\sqrt{1+\left(\frac{1}{r^2}-1\right)e^{-4\pi}}}$$



$r=1$ is a stable limit cycle

2.) Sinusoidally forced RC circuit

$$\dot{x}+x=\sin\omega t\,, \quad \omega>0$$

$$x(t)=c_1 e^{-t}+c_2\sin\omega t+c_3\cos\omega t$$

c_2 and c_2 can be found from the equation:

$$\omega c_2\cos\omega t-\omega c_3\sin\omega t+$$
$$+c_2\sin\omega t+c_3\cos\omega t=\sin\omega t$$

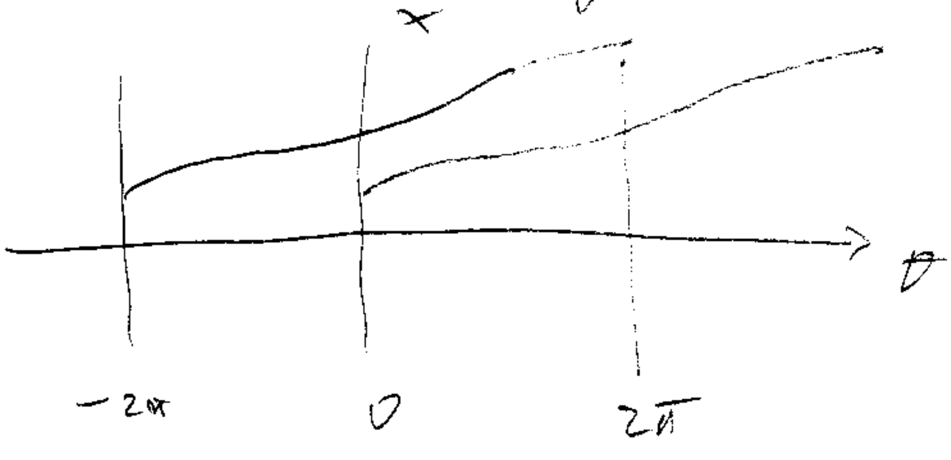
$$\omega c_2+c_3=0 \quad\Rightarrow\quad c_3=-\omega c_2$$
$$-\omega c_3+c_2=1$$
$$c_2=\frac{1}{1+\omega^2}\,,\quad c_3=-\frac{\omega}{1+\omega^2}$$

$$x(t)=c_1 e^{-t}+\frac{1}{1+\omega^2}\left[\sin\omega t-\omega\cos\omega t\right]$$

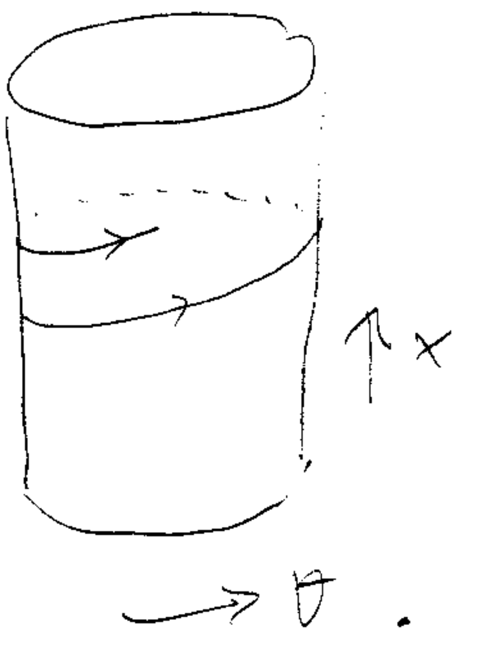
Let $\theta=\omega t \Rightarrow$

$$\dot{\theta}=\omega\,,\quad \dot{x}+x=\sin\theta$$

$\theta - x$ PHASE PLANE: Translations in θ by 2π map integral curves into integral curves



$\Rightarrow$ We can take $\theta \bmod 2\pi$ =
= roll the phase plane onto a cylinder



POINCARÉ SECTION:

$S = \{(\theta, x) \mid \theta = 0 \bmod 2\pi\}$

Time of flight
$t = \frac{2\pi}{\omega}$

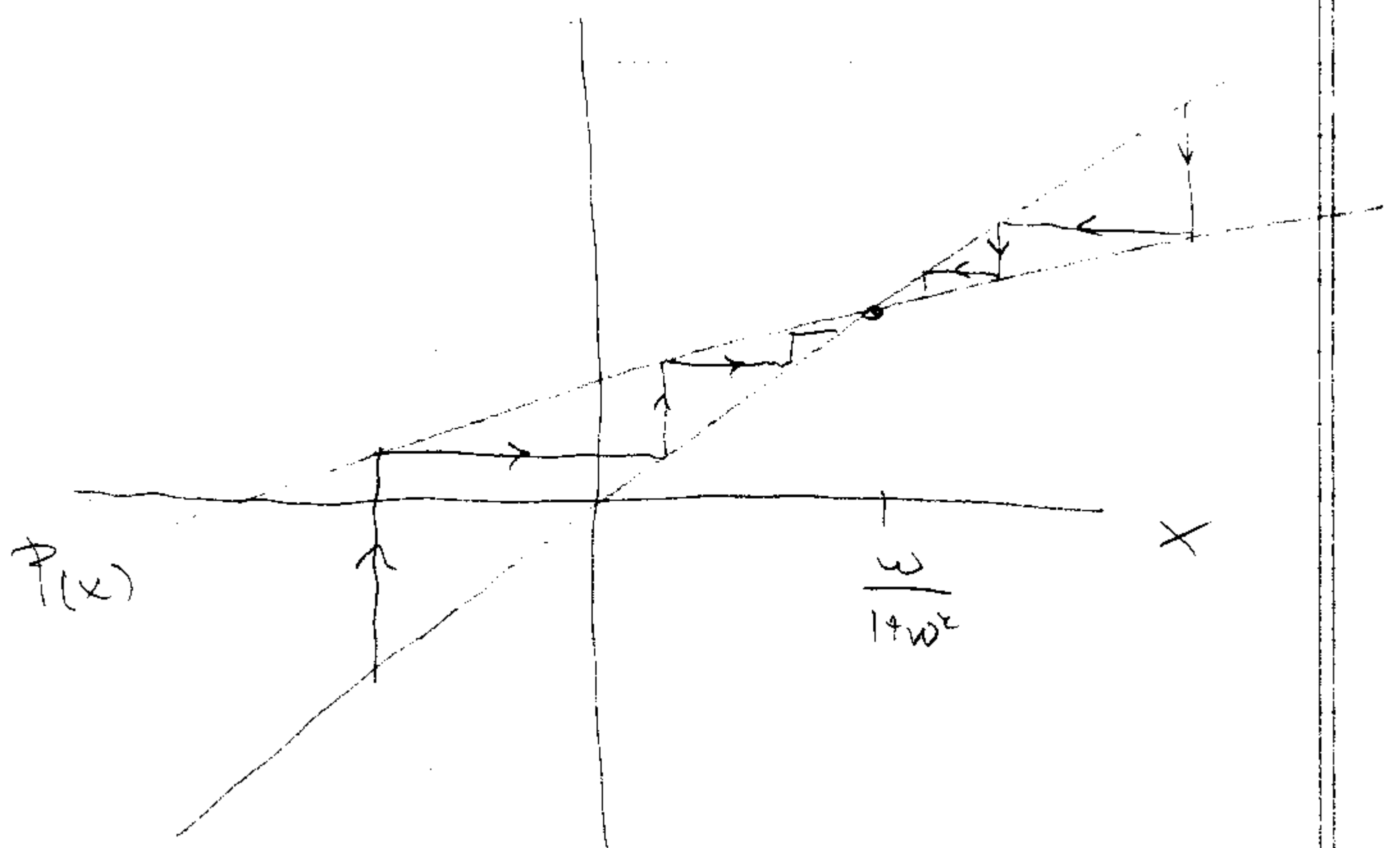
Let $x(0) = x_0 \Rightarrow c_1 = x_0 + \frac{\omega}{1+\omega^2}$

$$P(x_0) = x\left(\frac{2\pi}{\omega}\right) = \left(x_0 + \frac{\omega}{1+\omega^2}\right) e^{-\frac{2\pi}{\omega}} - \frac{\omega}{1+\omega^2}$$

$$= x_0 e^{-\frac{2\pi}{\omega}} - \frac{\omega}{1+\omega^2}\left(1 - e^{-\frac{2\pi}{\omega}}\right)$$

FIXED POINT $P(\bar{x}_0) = \bar{x}_0$

$$\bar{x}_0 = \frac{\omega}{1+\omega^2}$$

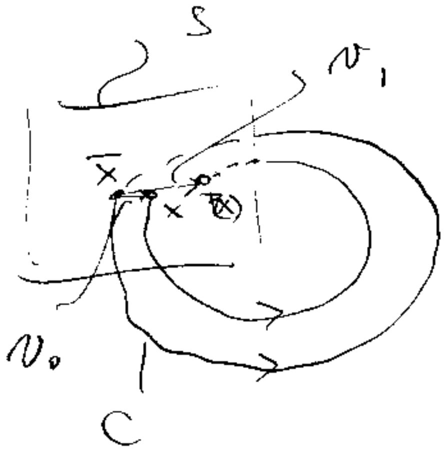


$\Rightarrow$ STABLE LIMIT Cycle

LINEAR STABILITY OF PERIODIC ORBITS

$\dot{x} = f(x)$ with a closed orbit C

C is stable $\Longleftrightarrow$ $\bar{x}$ of the Poincaré map P near C is stable



Let $x = \bar{x} + v_0$

$P(x) = \bar{x} + v_1 =$

$= P(\bar{x}+v_0) = P(\bar{x}) +$

$+ \underbrace{\frac{\partial P}{\partial x}(\bar{x})}_{(n-1)\times(n-1)\text{ matrix}} v_0 + \mathcal{O}(\|v_0\|^2)$

$\Rightarrow v_1 = \frac{\partial P}{\partial x}(\bar{x})\, v_0$

provided we can neglect $\mathcal{O}(\|v_0\|^2)$

PROP The closed orbit C is linearly stable iff $|\lambda_j| \leq 1$ for all eigenvalues λ_j of $\frac{\partial P}{\partial x}(\bar{x})$, $j = 1, \dots, n-1$.

PROOF If there are no repeated eigenvalues of $\frac{\partial P}{\partial x}(\bar{x})$

$\Rightarrow \exists$ eigenvector basis $\{e_j\}$

$$v_0 = \sum_{j=1}^{n-1} \nu_j e_j$$

Floquet multipliers

$$v_1 = \frac{\partial P}{\partial x}(\bar{x}) \sum_{j=1}^{n-1} \nu_j e_j = \sum_{j=1}^{n-1} \nu_j \lambda_j e_j$$

$\Rightarrow$ Iterating the map k times gives

$$v_k = \sum_{j=1}^{n-1} \nu_j (\lambda_j)^k e_j$$

If $|\lambda_j| < 1$ for $\forall j \Rightarrow \|v_k\| \to 0$

If some $|\lambda_j| > 1 \Rightarrow$ some $\|v_k\| \to \infty$

In both cases we say that C is <u>hyperbolic</u>.

<u>CLAIM</u>: If C is hyperbolic linear stability or instability imply the same for the nonlinear system.

EXAMPLES 1) $\dot r = r(1-r^2)$ $\dot\theta = 1$
has a stable limit
cycle at $\bar r = 1$.

Floquet multiplier: linearize about $\bar r = 1$

$r = 1+\eta \Rightarrow \dot r = \dot\eta = (1+\eta)(1-(1+\eta^2)) =$

$= -2\eta + \mathcal{O}(\eta^2)$

$\dot\eta = -2\eta$ – linearization

$\eta(t) = \eta_0 e^{-2t}$

Time of flight $t = 2\pi$: $\eta_1 = \eta_0 e^{-4\pi}$

$e^{-4\pi}$ – Floquet multiplier

$|e^{-4\pi}| < 1 \rightarrow$ cycle is linearly stable.

We can check that $\left.\frac{\partial P(r)}{\partial r}\right|_{r=1} = e^{-4\pi}$

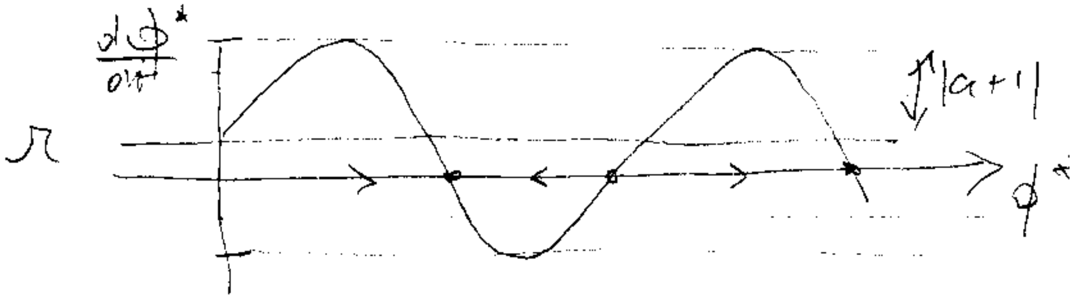
$$\left.\frac{\partial P(r)}{\partial r}\right|_{r=1} = \left.\frac{1}{\left(1+\left(\frac{1}{r^2}-1\right)e^{-4\pi}\right)} \frac{1}{r^3}\right|_{r=1} e^{-4\pi} = e^{-4\pi}$$ ✓

2.) $\dot{\phi}_i = \Omega + a \sin\phi_i + \frac{1}{N}\sum_{j=1}^{N} \sin\phi_j \,, \quad i=1,\dots,N$

(series array of overdamped Josephson junctions in parallel with a resistive load)

Technologically important: In-phase solution $\phi_j(t) = \phi^*(t) \quad j=1,\dots,N$

$$\Rightarrow \quad \frac{d\phi^*}{dt} = \Omega + (a+1)\sin\phi^*$$



$|\Omega| < |a+1|$

Equilibrium points exist, no solution $\phi^*(t)$ s.t. $\phi^*(t+T) = \phi^*(t) + 2\pi$ for some T. (This is a periodic solution on the circle)

If $|\Omega| > |a+1|$, such a solution exists

$\phi_i(t) = \phi^*(t) + \eta_i(t)$

$\Rightarrow$

$$\dot{\eta}_i = [a \cos\phi^*(t)]\eta_i + [\cos\phi^*(t)] \cdot \frac{1}{N}\sum_{j=1}^{N}\eta_j$$

Let $\mu = \frac{1}{N}\sum_{j=1}^{N}\eta_j$, $\quad \xi_i = \eta_{i+1} - \eta_i$, $i = 1, \ldots, N-1$

$\Rightarrow \quad \dot{\xi}_i = [a \cos\phi^*(t)]\xi_i$

$\Rightarrow \quad \frac{d\xi_i}{\xi_i} = a \cos\phi^*(t)\, dt = \frac{[a \cos\phi^*]\, d\phi^*}{\Omega + (a+1)\sin\phi^*}$

After one circuit around $\phi^*(t)$

$$\oint \frac{d\xi_i}{\xi_i} = \int_0^{2\pi} \frac{a \cos\phi^*\, d\phi^*}{\Omega + (a+1)\sin\phi^*}$$

$$\ln\frac{\xi_i(T)}{\xi_i(0)} = \frac{a}{a+1}\ln\left[\Omega + (a+1)\sin\phi^*\right]_0^{2\pi} = 0$$

$\Rightarrow \quad \xi_i(T) = \xi_i(0)$

Also $\dot{\mu} = (a+1)\cos\phi^*(t)\mu \Rightarrow \mu(T) = \mu(0)$

$\Rightarrow \lambda_j = 1 \quad$ for $\forall j = 1, \dots, n$

Linear stability test is inconclusive

Let $\varphi_j = \psi_j + \frac{\pi}{2}$, $\sin\varphi_j = \sin(\psi_j + \frac{\pi}{2}) =$

$= \cos\psi_j$

$$\Rightarrow \dot{\psi}_j = \Omega + a\cos\psi_j + \frac{1}{N}\sum_{j=1}^{N}\cos\psi_j$$

If we let $t \to -t$, $\psi_j \to -\psi_j$, the equation stays the same $\Rightarrow$

$\{\psi_j(t)\}$ - solution $\Rightarrow$ so is $\{-\psi_j(-t)\}$

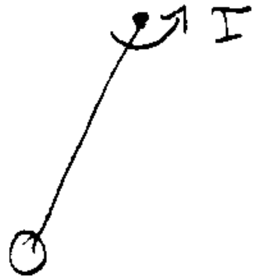
Let $\psi^*(t) = \varphi^*(t) - \frac{\pi}{2}$.

$$\Rightarrow \frac{d\psi^*}{dt} = \Omega + (a+1)\cos\psi^*$$

$\psi^*(t)$ and $-\psi^*(-t)$ satisfy the same 1-d equation; they are translates: $-\psi^*(-t) = \psi^*(t+c)$

But if $\psi^*(t)$ is stable, $-\psi^*(-t)$ is unstable $\Rightarrow$ $\psi^*(t)$ is neither attracting nor repelling but neutrally stable

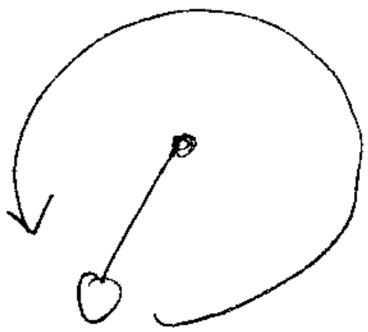
EXAMPLE PENDULUM WITH CONSTANT TORQUE



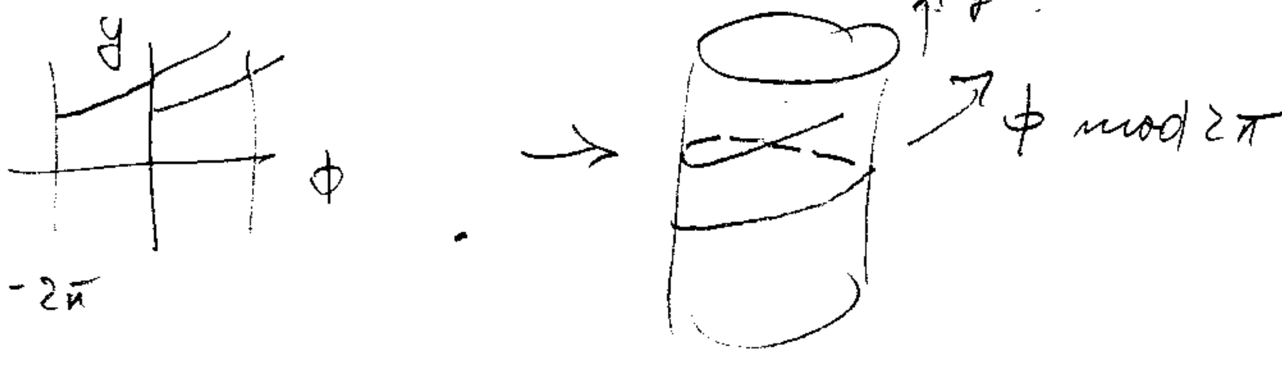
$$\left.\begin{aligned} \dot{\phi} &= y \\ \dot{y} &= I - \sin\phi - \alpha y \end{aligned}\right\} (1)$$

No equilibria for $I > 1$.

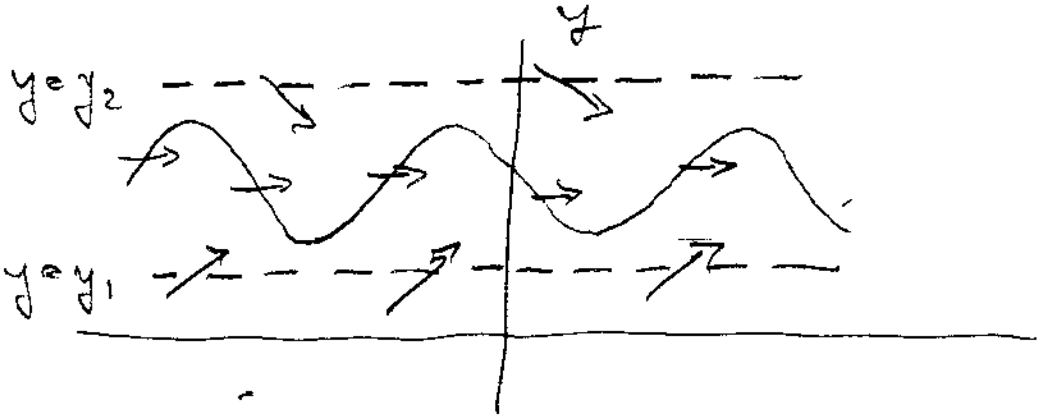
We want to show that there is a unique attracting solution, such that the pendulum rotates.



PHASE PLANE for (1) can be rolled into a PHASE CYLINDER b/c R.H.S. IS PERIODIC with 2π in ϕ



NOTE $\dot{y} = 0$ for $y = \frac{1}{a}(I - \sin\phi)$



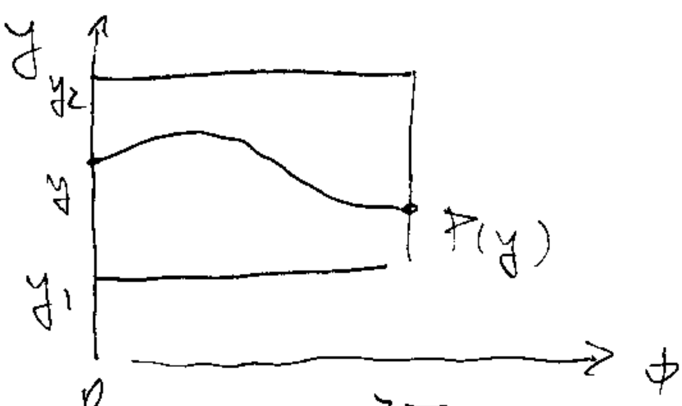
$y_1, y_2 > 0$

$\dot{y} < 0$; $y > \frac{1}{a}(I - \sin\phi)$

$\dot{y} > 0$; $y < \frac{1}{a}(I - \sin\phi)$

$\dot{\phi} > 0$ if $y > 0$

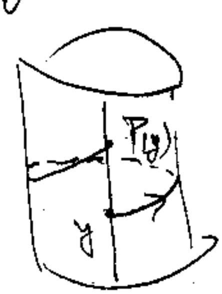
$y_1 \le y \le y_2$ is a trapping region



POINCARE MAP

$S: \ \phi = 0 \mod 2\pi$

$P(y|\phi=0)) = y(\phi = 2\pi)$

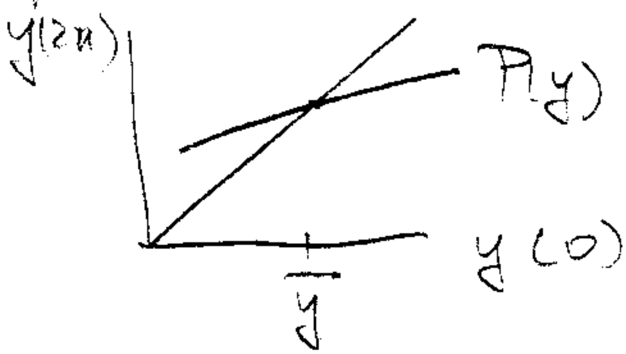


Now $\quad P(y_1) > y_1$

$\quad P(y_2) < y_2$

(because the vector field points strictly inwards)

$P(y)$ is monotonic, otherwise two trajectories would have to cross.

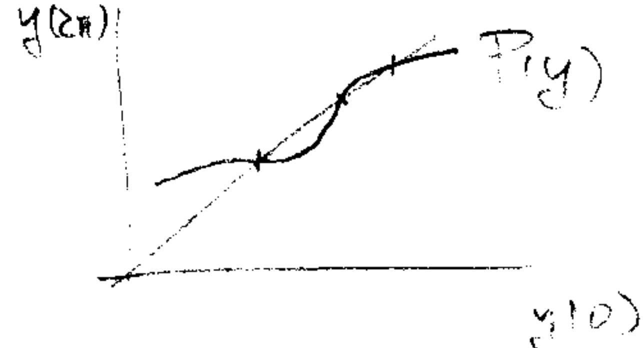


P - continuous

For some $y = \bar{y}$, $P(\bar{y}) = \bar{y} \Rightarrow$

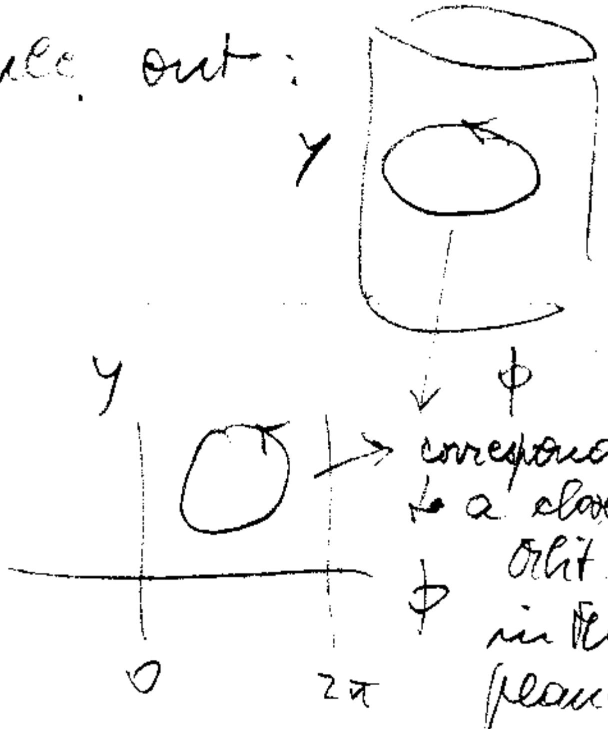
limit cycle

Could have more than one:



We must rule this out.

Rule out:



corresponds to a closed orbit in the plane

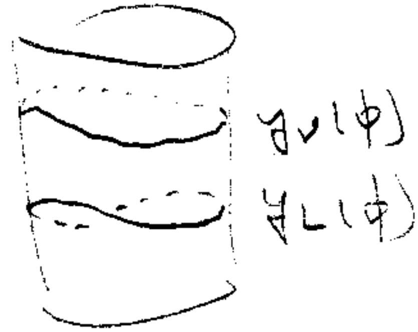
By index theory: there should be an equilibrium

Bendixson criterion:

$\nabla \cdot (R.H.S) = -\alpha < 0$

$\Rightarrow$ no closed orbits

Rule out:



Clearly $y_U(\phi) > y_L(\phi)$, for $\forall \phi$,

Show contradiction by using energy

$$E = \frac{1}{2} y^2 - \cos\phi$$

After one circuit around $y_{U,L}(t)$, $\Delta E = 0$

$$0 = \Delta E = \int_0^{2\pi} \frac{dE}{d\phi}\, d\phi$$

But $$\frac{dE}{d\phi} = \frac{\frac{1}{2} y\, dy + \sin\phi\, d\phi}{d\phi} =$$
$$= \frac{1}{2} y \frac{dy}{d\phi} + \sin\phi$$

and $$\frac{dy}{d\phi} = \frac{\dot{y}}{\dot{\phi}} = \frac{I - \sin\phi - \alpha y}{y}$$

$$\Rightarrow \quad \frac{dE}{d\phi} = I - \alpha y$$

$$\Rightarrow \quad 0 = \Delta E = \int_0^{2\pi} (I - \alpha y)\, d\phi$$

for any periodic $y(\phi)$

$$\Rightarrow \int_0^{2\pi} y(\phi)\, d\phi = \frac{2\pi I}{\alpha}$$

But since $y_U(\phi) > y_L(\phi)$

$\int_0^{2\pi} y_U(\phi)\, d\phi > \int_0^{2\pi} y_L(\phi)\, d\phi$ – contradiction

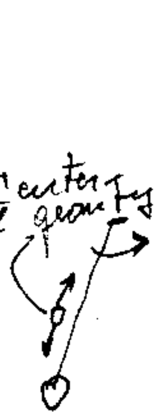
$\Rightarrow$ Only one periodic orbit.

More examples:

$$\ddot{x} + (1+\varepsilon f(t))\sin x = 0$$

(CHILD ON A SWING, parametric resonance)

Center of gravity



$f(t)$ periodic : $f(t+T) = f(t)$

$x=0$ becomes unstable near

$$T = \frac{1}{4n} \quad \text{for } \varepsilon > 0$$

$$\ddot{x} + (1+\omega^2 \sin\omega t)\sin x = 0$$

(shaking pendulum)

If $\omega \gg 1$, $x=\pi$ becomes stable

REMARK: Usually, you can only compute Floquet multipliers numerically.