EXISTENCE AND UNIQUENESS

Consider
$$y' = f(x,y), \quad y(x_0) = y_0 \quad (1)$$

where f is continuously differentiable near x_0, y_0.

How do we show that there exists (at least locally) a unique solution of (1)?

Construct it.

<u>Prop</u> (1) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\,dx \quad (2)$$

<u>Proof</u> Any continuous solution of (2) is continuously differentiable b/c of the integral on the R.H.S.

$\Rightarrow$ by differentiation of (2) or integration of (1) the two are equivalent.

Why is (2) better than (1)? Because we can iterate it:

Choose $y_0(x) \equiv y_0$. Let

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1(x))\, dx$$

etc.

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1}(x))\, dx.$$

This process is well-defined, because it involves <u>integration</u> of continuous functions.

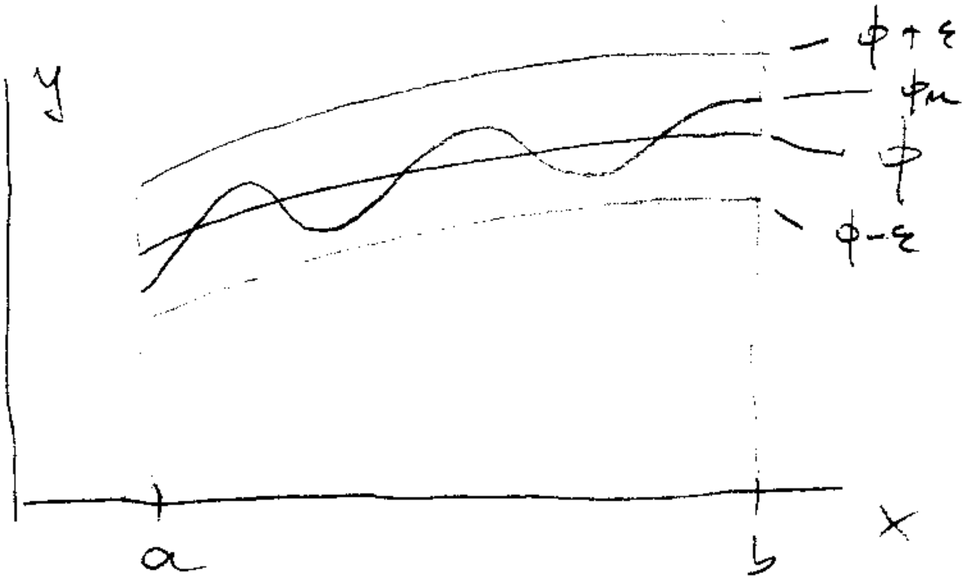
Does this process converge? How do functions converge anyway? Can we interchange integration and the limit process?

TOOL: Uniform convergence of functions

$\{\phi_n\}$ is said to converge uniformly to ϕ on $[a,b]$ if for $\forall \varepsilon>0$ $\exists N$ s.t.

$$|\phi_n(x) - \phi(x)| < \varepsilon$$

for $\forall n \geq N$ and all $x \in [a,b]$,



THM 1: $\{\phi_n\}$ converges uniformly if it uniformly satisfies the Cauchy property: for $\forall \varepsilon>0$ $\exists N$ s.t.

$$|\phi_n(x) - \phi_m(x)| < \varepsilon$$

for $\forall n, m \geq N$ and all $x \in [a,b]$

PROOF: Numerical Cauchy sequences converge.

THM 2: Let all ϕ_n be continuous on $[a,b]$, and let $\phi_n \to \phi$ uniformly. Then ϕ is continuous.

PROOF: Let $\varepsilon > 0$. By uniform convergence, we can choose N s.t.

$$|\phi(x) - \phi_n(x)| < \frac{\varepsilon}{4}$$

for $n > N$, and so

$$|\phi(y) - \phi_n(y) - (\phi(x) - \phi_n(x))| < \frac{\varepsilon}{2}$$

for $\forall x, y \in [a,b]$.

Since ϕ_n are continuous, for $\forall n > N$ and $\forall x \in [a,b]$, $\exists \delta = \delta(n,x)$ s.t.

$$|\phi_n(y) - \phi_n(x)| < \frac{\varepsilon}{2}$$

if $|y-x| < \delta$, $y \in [a,b]$.

Thus given $\varepsilon > 0$, $\exists \delta = \delta(x)$ s.t.

$$|\phi(x) - \phi(y)| \le |\phi(y) - \phi_n(y) - (\phi(x) - \phi_n(x))| + |\phi_n(y) - \phi_n(x)| < \varepsilon$$

if $|x-y| < \delta$, which is continuity

THM 3: Let ϕ_n be continuous on $[a,b]$, $\phi_n \to \phi$ uniformly. Then for $\forall \alpha, \beta \in [a,b]$, $\beta > \alpha$,

$$\int_\alpha^\beta \phi_n(x)\,dx \to \int_\alpha^\beta \phi(x)\,dx$$

PROOF: Given $\varepsilon > 0$, uniform convergence implies $\exists N$ s.t.

$$|\phi_n(x) - \phi(x)| < \frac{\varepsilon}{b-a}, \quad n \geq N$$

$$\Rightarrow \left| \int_\alpha^\beta (\phi(x) - \phi_n(x))\,dx \right| \leq$$

$$\leq \int_\alpha^\beta |\phi(x) - \phi_n(x)|\,dx < \frac{\varepsilon\,(\beta-\alpha)}{b-a} \leq \varepsilon$$

Remark If $\alpha > \beta$, just note

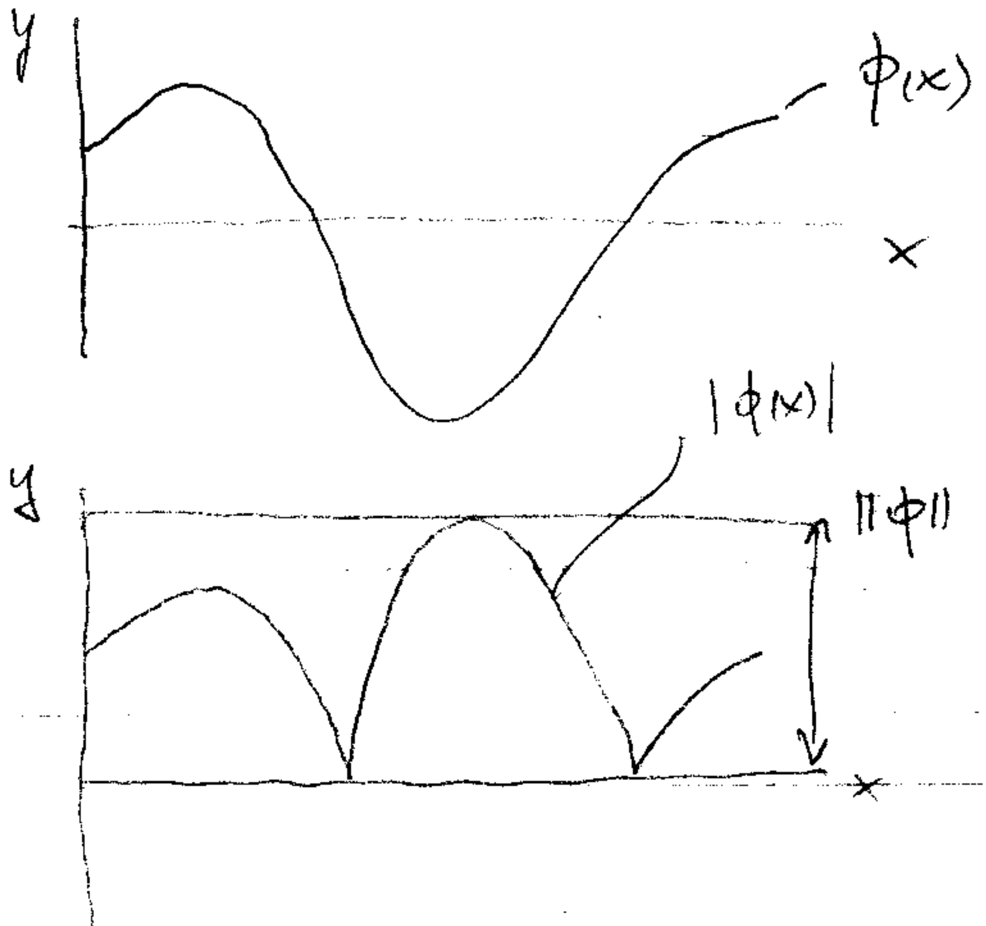
$$\int_\alpha^\beta = -\int_\beta^\alpha, \text{ and this}$$

shows that the theorem holds in this case, too.

FUNCTIONAL NORM

Measures the magnitude of a function

$$\|\phi\| = \sup \{ |\phi(x)| \mid x \in [a,b] \}$$



<u>Remark</u>: If ϕ is continuous $\sup = \max$.

<u>PROPERTIES</u>: $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$

$\|\alpha \phi\| = |\alpha| \, \|\phi\|$

$\|\phi\| = 0 \iff \phi = 0$

EQUIVALENT NORMS

sometimes are more convenient.

Let $\rho(x) > 0$ for $\forall x \in [a, b]$ and continuous.

$\|\phi\|_\rho = \sup \{ |\phi(x)|\rho(x) \mid x \in [a,b] \}$

still satisfies the three properties of the norm.

Also let $r = \inf \{\rho(x) \mid x \in [a,b]\}$

$R = \sup \{\rho(x) \mid x \in [a,b]\}$

Then

THM 4 $\quad r \|\phi\| \leq \|\phi\|_\rho \leq R \|\phi\|.$

DISTANCE BETWEEN TWO FUNCTIONS

Given $\rho(x) > 0$, continuous, then

$\mathrm{Dist}(\phi, \psi) = \|\phi - \psi\|_\rho$

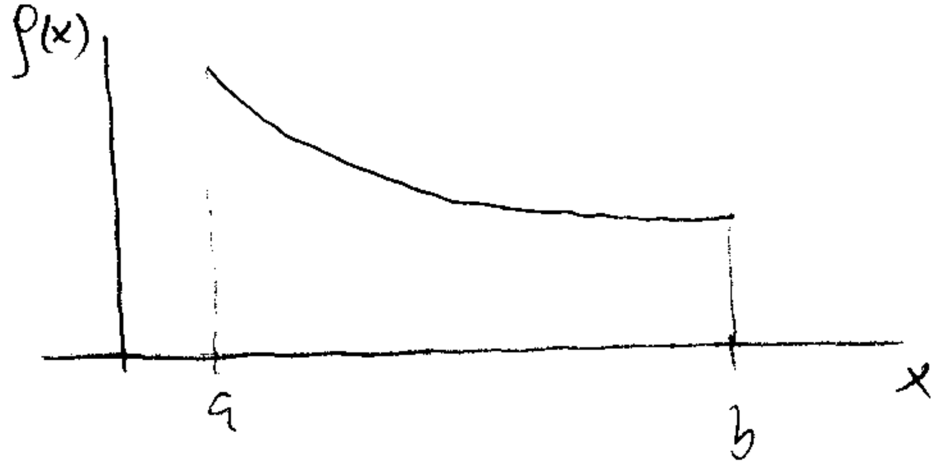
The space of functions equipped

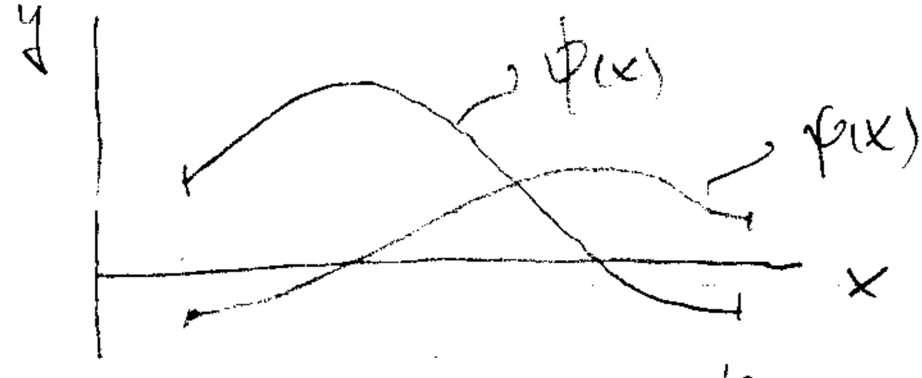
with this distance forms a metric space:

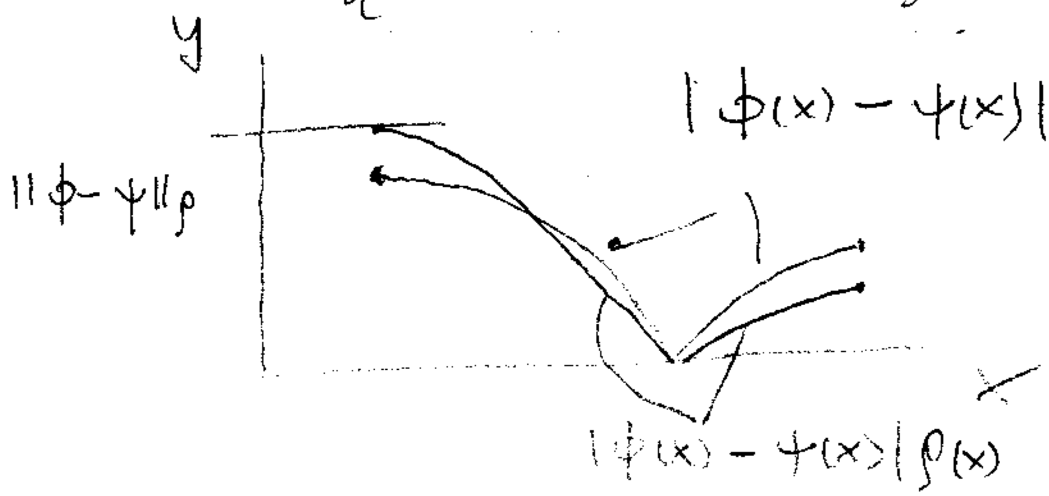
$$\|\phi-\psi\|_\rho \geq 0, \; = 0 \text{ iff } \phi = \psi$$

$$\|\phi-\psi\|_\rho = \|\psi-\phi\|_\rho$$

$$\|\phi-\psi\|_\rho \leq \|\phi-\theta\|_\rho + \|\theta-\psi\|_\rho$$







THM 5: $\|\phi_n - \phi\|_\rho \to 0$ for $\forall \rho$ iff

$\phi_n \to \phi$ uniformly on $[a,b]$

PROOF $\|\phi_n - \phi\| \to 0$ (for $\rho = 1$) is exactly uniform convergence. The rest follows from THM 4.

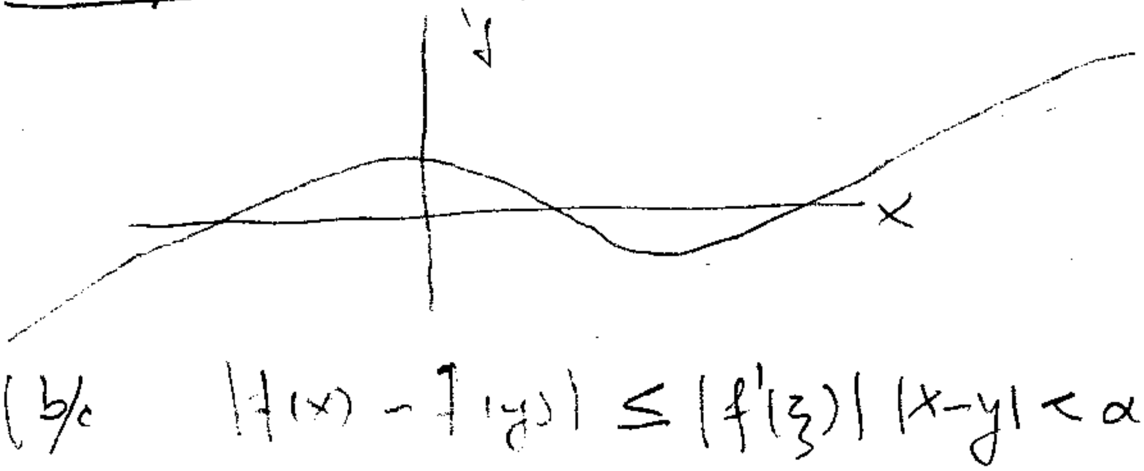
CONTRACTION PRINCIPLE

Let M be a metric space with distance d. A map $A: M \to M$ is a contraction if $\exists \alpha$, $0 < \alpha < 1$

$$d(A\phi, A\psi) \le \alpha\, d(\phi, \psi)$$

for $\forall \phi, \psi \in M$.

Example $M = \mathbb{R}$, $A = f(x)$ s.t. $|f'(x)| \le \alpha < 1$



(b/c $|f(x) - f(y)| \le |f'(\xi)|\,|x-y| < \alpha |x-y|$)

$\bar{\phi} \in M$ is a <u>fixed point</u> of A if $A\bar{\phi} = \bar{\phi}$.

<u>Contraction mapping theorem</u>: Let M be complete, and let $A: M \to M$ be a contraction. Then A has a unique fixed point $\bar{\phi}$ in M. Moreover, starting with any $\phi_0 \in M$, the sequence $\{\phi_n\}$ given by $\phi_{n+1} = A\phi_n$ converges to $\bar{\phi}$.

<u>PROOF</u> Choose any $\phi_0 \in M$ and let

$$\phi_{n+1} = A\phi_n \ , \ n = 0, 1, 2, \ldots$$

Then

$$d(\phi_{n+1}, \phi_n) = d(A\phi_n, A\phi_{n-1}) \leq \alpha\, d(\phi_n, \phi_{n-1})$$
$$\leq \alpha^2 d(\phi_{n-1}, \phi_{n-2}) \leq \cdots \leq \alpha^n d(\phi_1, \phi_0)$$

and if $m > n$

$$d(\phi_m, \phi_n) \leq d(\phi_m, \phi_{m-1}) + \cdots + d(\phi_{n+1}, \phi_n)$$
$$< \alpha^n [\alpha^{m-n} + \cdots + \alpha + 1]\, d(\phi_1, \phi_0)$$

$$d(\phi_m, \phi_n) \le \alpha^n \frac{1-\alpha^{m+1-n}}{1-\alpha} d(\phi_1,\phi_0) < \frac{\alpha^n}{1-\alpha} d(\phi_1,\phi_0)$$

(Recall $0<\alpha<1$!)

$\Rightarrow \{\phi_n\}$ is a Cauchy sequence in the distance $d(\cdot,\cdot)$. Since M is complete, there exists a limit

$$\lim_{n\to\infty} \phi_n = \bar\phi = \lim_{n\to\infty} \phi_{n+1} =$$

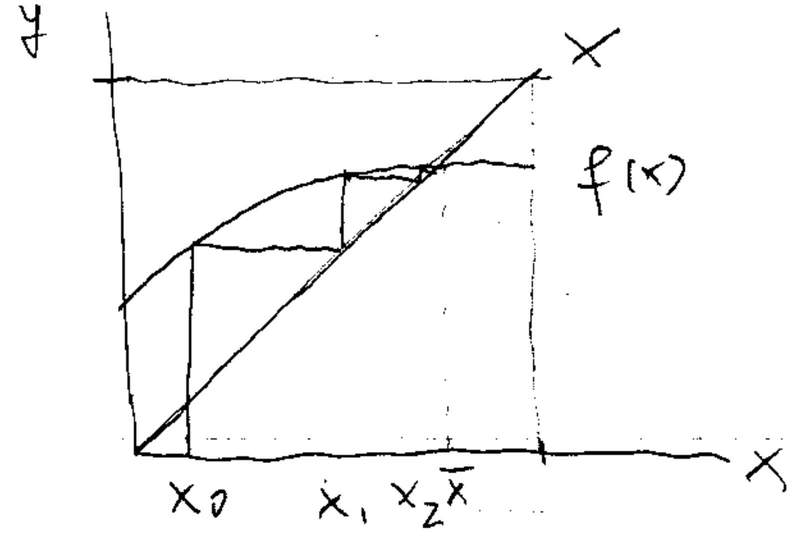
$$= \underbrace{\lim_{n\to\infty} A\phi_n = A \lim_{n\to\infty} \phi_n}_{\text{because } A \text{ is continuous } (\delta=\varepsilon \text{ will do for that})} = A\bar\phi$$

Suppose there was more than one fixed point: $\bar\phi$ and $\bar\psi$. Then

$$d(\bar\phi,\bar\psi) = d(A\bar\phi, A\bar\psi) \le \alpha\, d(\bar\phi,\bar\psi)$$

$$\Rightarrow d(\bar\phi,\bar\psi) = 0 \Rightarrow \bar\phi = \bar\psi.$$

EXAMPLE: $f:[0,1]\to[0,1]$ $|f'(x)|\le\alpha<1$



Iteration of the map f.

BACK TO EXISTENCE AND UNIQUENESS:

$$y' = f(x,y) \qquad y(x_0) = y_0, \qquad (1)$$

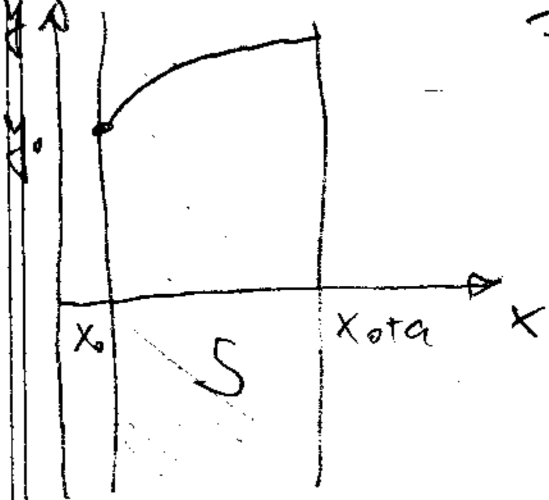
with $f(x,y)$ continuously differentiable near (x_0,y_0), is equivalent to the integral equation

$$(2) \qquad y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))\,d\xi \equiv$$

$$\equiv (Ty)(x, x_0, y_0)$$

(Often we will just write $(Ty)(x)$)

THM 6 Let $f(x,y)$ be continuously differentiable on $S = \{(x,y) \mid x_0 \le x \le x_0+a,\ -\infty < y < \infty\}$, and let $\left|\frac{\partial f(x,y)}{\partial y}\right| \le L$ on S.
Then the initial-value problem (1) has a unique solution $y(x)$, which exists in the entire interval $x_0 \le x \le x_0+a$



PROOF. Let M be the metric space of all continuous functions with the distance $\|\phi - \psi\|_\rho$, where $\rho = e^{-\alpha x}$, $\alpha > 0$.
(We will determine α later.). M is complete by THM's 2 and 5.

Consider the map T given by (2),

1.) $T: M \to M$ because the integral of a continuous function is continuous

2.) Let us show that for some appropriate α, T is a contraction.

$$|(Ty_1)(x) - (Ty_2)(x)| = \left| \int_{x_0}^{x} [f(\xi, y_1(\xi)) - f(\xi, y_2(\xi))] d\xi \right|$$

$$\leq \int_{x_0}^{x} |f(\xi, y_1(\xi)) - f(\xi, y_2(\xi))| \, d\xi \leq$$

$$\leq \int_{x_0}^{x} \left| \frac{\partial f}{\partial y}(\xi, y_{\text{INTERMEDIATE}}(\xi)) \right| |y_1(\xi) - y_2(\xi)| \, d\xi$$

$$\leq L \int_{x_0}^{x} |y_1(\xi) - y_2(\xi)| \, d\xi =$$

$$= L \int_{x_0}^{x} |y_1(\xi) - y_2(\xi)| e^{-\alpha\xi} e^{\alpha\xi} \, d\xi \leq$$

$$\leq L \|y_1 - y_2\|_\rho \int_{x_0}^{x} e^{\alpha\xi} \, d\xi =$$

$$= \frac{L}{\alpha} \|y_1 - y_2\|_\rho \left(e^{\alpha x} - e^{\alpha x_0} \right) \leq$$

$$\leq \frac{L}{\alpha} \|y_1 - y_2\|_\rho \, e^{\alpha x}$$

$$\Rightarrow |(Ty_1)(x) - (Ty_2)(x)| e^{-\alpha x} \leq \frac{L}{\alpha} \|y_1 - y_2\|_\rho$$

$$\Rightarrow \|Ty_1 - Ty_2\|_\rho \leq \frac{L}{\alpha} \|y_1 - y_2\|_\rho$$

Choose α such that $\frac{L}{\alpha} < 1$ e.g. $\alpha = 2L$. Then T is a contraction, and the theorem is proven.

REMARK:
An identical theorem with an almost identical proof holds for $S = \{(x,y) \mid x_0 - a < x < a,\ -\infty < y < \infty\}$.

(For instance, you can apply the previous result if you transform: $z(x) = y(2x_0 - x)$, $g(x,y) = -f(2x_0 - x, y)$. Geometrically this amounts to a reflection across the line $x = x_0$. You get $z' = g(x,z)$, $x_0 \le x \le x_0 + a$, $z(x_0) = y_0$)

THE EXISTENCE AND UNIQUENESS THEOREM:

Let $R = \{(x,y) \mid x_0 - a_L \le x \le x_0 + a_R,\ |y - y_0| < b\}$ and let $f(x,y)$ be continuously differentiable on R. Let $A = \max_{(x,y)\in R} |f(x,y)|$ and let $\alpha_L = \min\{a_L, \frac{b}{A}\}$

$$\alpha_R = \min\{a_R, \frac{b}{A}\}.$$

Then there exists a unique solution of the initial-value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \qquad (1)$$

at least for $x_0 - \alpha_L \le x \le x_0 + \alpha_R$.

<u>Proof</u> Let us extend $f(x,y)$ continuously onto the strip

$$S = \{(x,y) \mid x_0 - a_L \le x \le x_0 + a_R, -\infty < y < \infty\}$$

by

$$F(x,y) = \begin{cases} f(x, y_0 - b) & \text{if } y < y_0 - b \\ f(x,y) & \text{in } R \\ f(x, y_0 + b) & \text{if } y > y_0 + b \end{cases}$$

Now, this function is continuous in S but not necessarily continuously differentiable.

However, a careful look at the proof of Thm 6 shows that we only need the <u>Lipschitz property</u>:

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$$

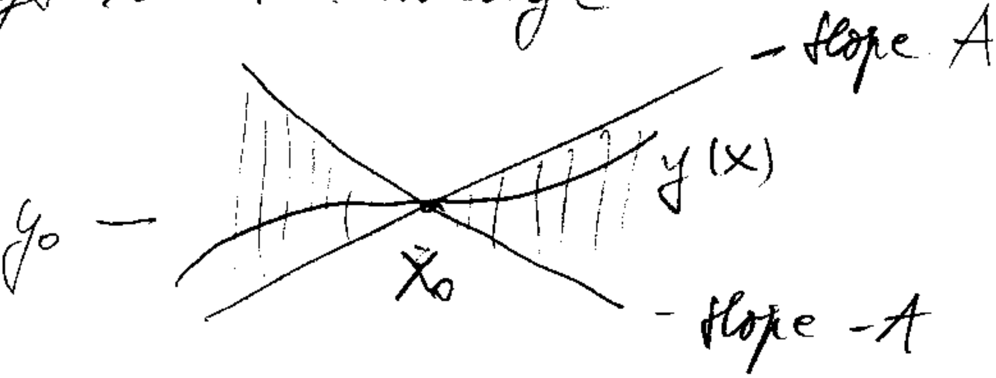
which is here clearly preserved with $L = \max_R \left|\frac{\partial f}{\partial y}(x,y)\right|$.

Therefore, a unique solution $y(x)$ of

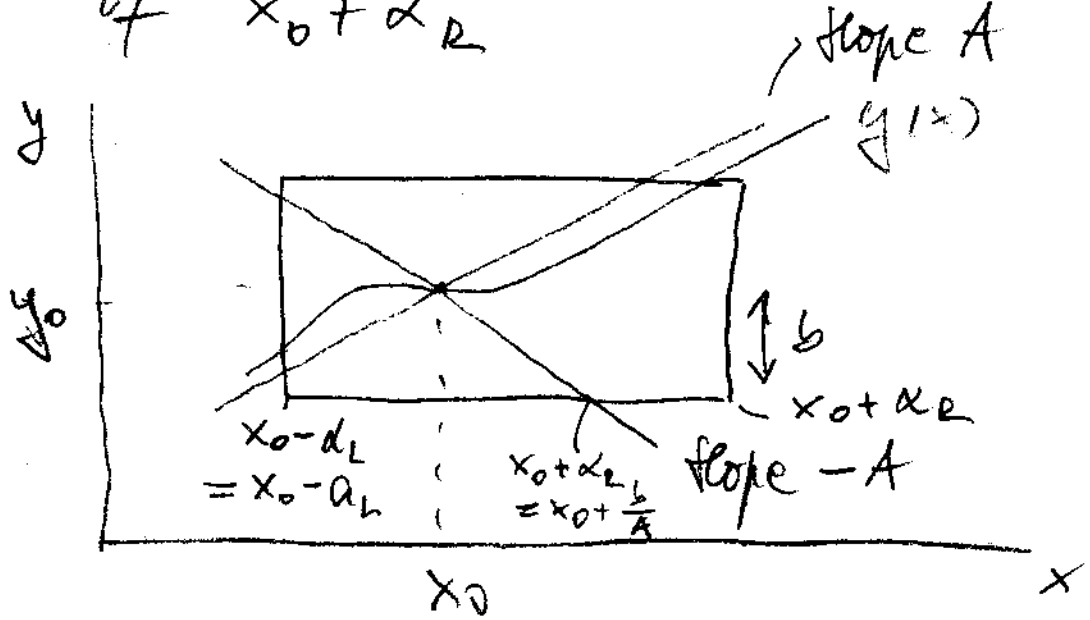
$$y' = F(x,y) \quad , \quad y(x_0) = y_0$$

exists in S. For as long as this solution stays in R, it is also a solution of (1).

Now since $\sup\{|F(x,y)|\} = \sup\{|f(x,y)|\} = A$, we have $|y'| \le A$ so the solution $y(x)$ stays in the wedge



Therefore $y(x)$ can only leave R to the left of $x_0 - \alpha_L$ and to the right of $x_0 + \alpha_R$



<u>Remark</u>: By the discussion in the proofs of the above theorem and THM6, we see that it is sufficient for

$$y' = f(x,y), \quad y(x_0) = y_0$$

to have a unique solution near (x_0, y_0) if $f(x,y)$ is continuous in x and <u>Lipschitz continuous</u> in y:

$|f(x,y_1) - f(x,y_2)| < L|y_1 - y_2|$ for some $L > 0$, near (x_0, y_0).

CONTINUOUS DEPENDENCE ON PARAMETERS AND INITIAL CONDITIONS:

THM: Let $f(x,y,\mu)$ be continuous near $(x_0,y_0,\bar{\mu})$ and have continuous $\frac{\partial f}{\partial y}(x,y,\mu)$. Then the unique solution of

$$y' = f(x,y,\mu) \qquad y(x_0) = y_0$$

is continuous in μ, x_0 and y_0.

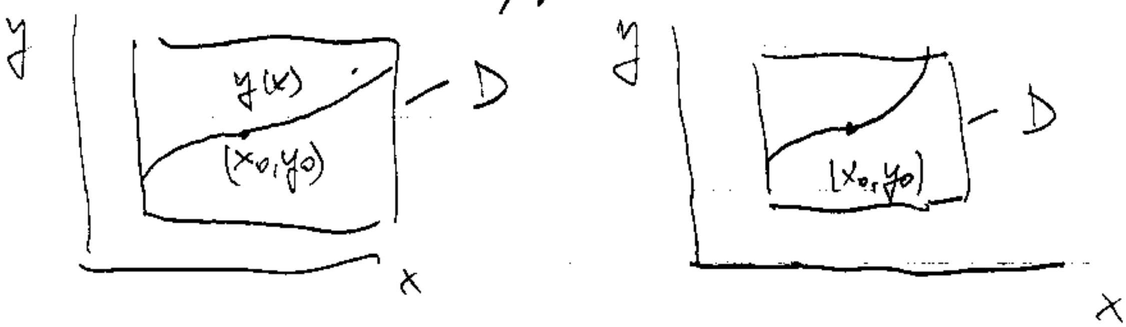
The proof proceeds as above, except that we must use the norm

$$\|y\|_\rho = \sup\{y(x,x_0,y_0,\mu)\,\rho(x)\,|\,x,x_0,y_0,\mu\}$$

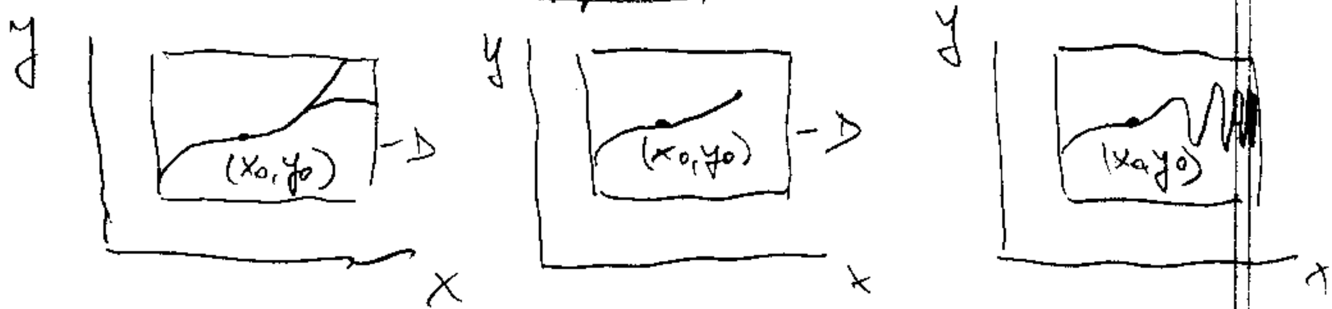
EXTENSION OF SOLUTIONS

THM: Let $D \subset \mathbb{R}^2$ be compact (= closed and bounded; closed = containing its boundary pts) and let $f(x,y)$ be continuously differentiable in D. Then there exists a unique solution $y(x)$ of $y' = f(x,y)$, $y(x_0) = y_0$, $(x_0,y_0) \in D$ and can be continued all the way to the boundary of D in a unique fashion

What can happen:



What cannot happen:



PROOF Arnold (better proof in the Springer edition), Coddington & Levinson

DIFFERENTIABILITY:

THM If $f(x, y, \mu)$ is C^r near (x_0, y_0, μ_0), then $y(x, x_0, y_0, \mu)$ is C^r.

PROOF: Incomprehensible in Arnold, OK in Coddington & Levinson. Rather hard.