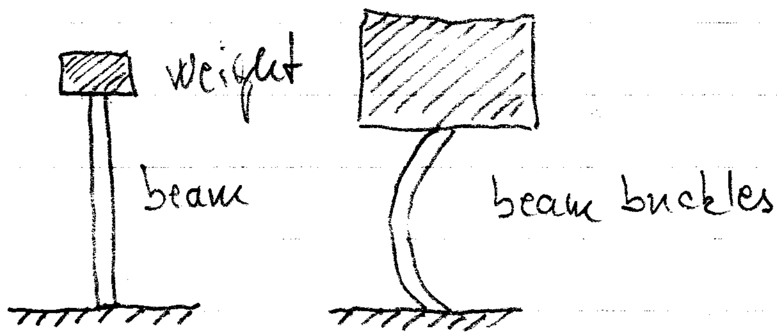
LOCAL BIFURCATIONS

Dependence of dynamics on parameters:
qualitative changes as they are varied
qualitative changes = bifurcations
parameter values = bifurcation points

EXAMPLE: Buckling of a beam

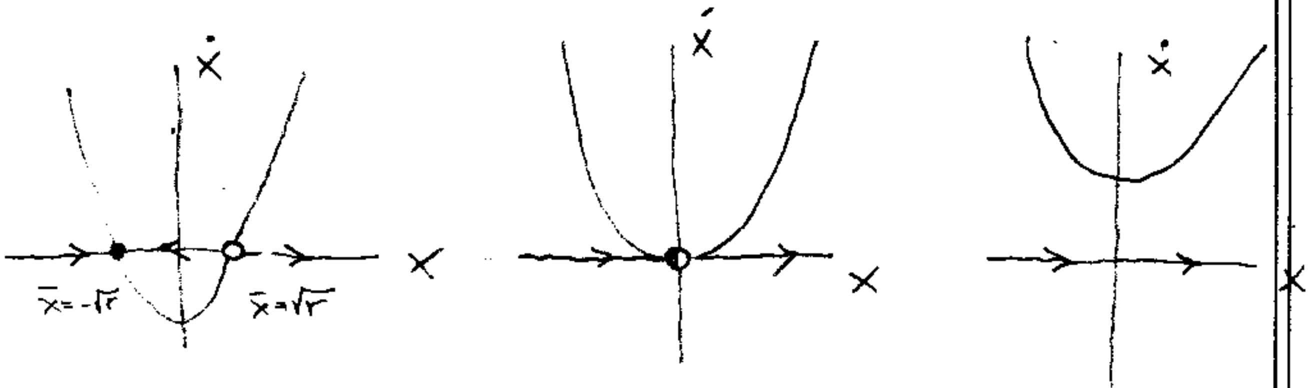


weight = control parameter
deflection of the beam from the vertical =
= dynamical variable x.

SADDLE-NODE BIFURCATION

Basic mechanism by which equilibria are created and destroyed

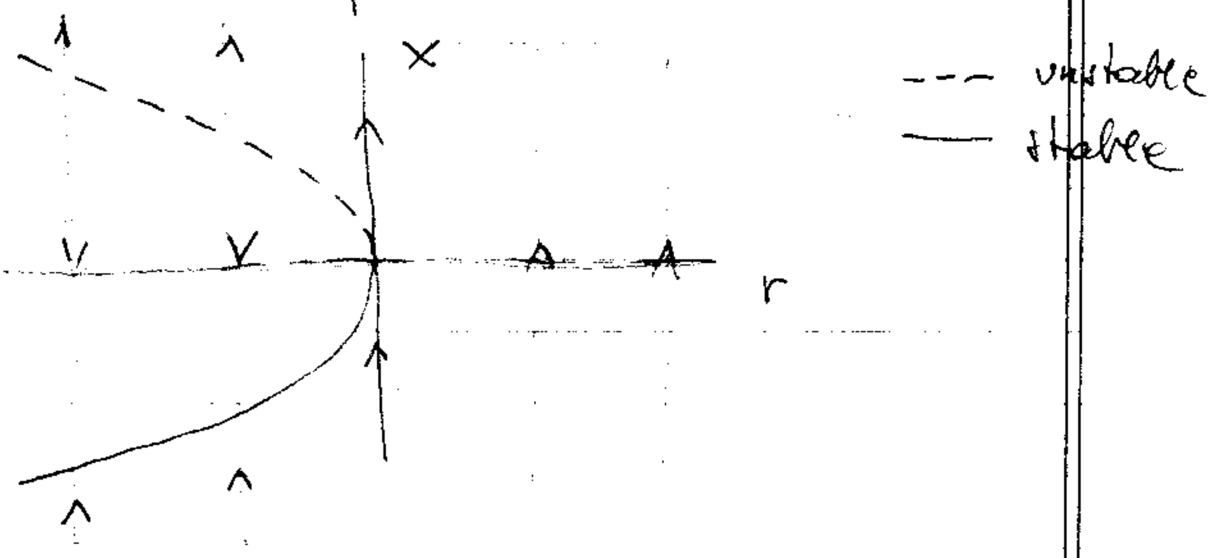
PROTOTYPE $\dot{x} = r + x^2$



$r<0$ $\qquad$ $r=0$ $\qquad$ $r>0$

bifurcation at $r=0$

Bifurcation diagram:



Stability analysis: equilibria $\bar{x}(r) = \pm\sqrt{-r}$

$$f(x) = r + x^2, \quad f'(x) = 2x$$

$\bar{x}(r) = +\sqrt{-r} \rightarrow f'(\bar{x}) > 0$ – unstable

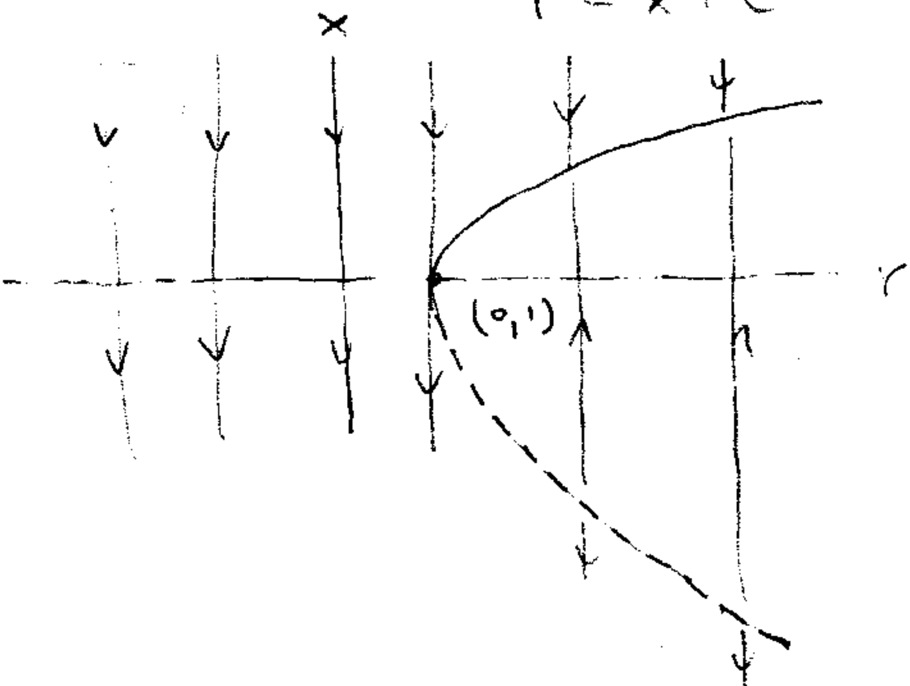
$\bar{x}(r) = -\sqrt{r} \Rightarrow f'(\bar{x}) < 0$ – stable

$r=0$ – bifurcation point $\Rightarrow f'(\bar{x}) = 0$

EXAMPLE $\dot{x} = r - x - e^{-x}$

EQUILIBRIA: $r - x - e^{-x} = 0$

$$r = x + e^{-x}$$



$$\frac{dr}{dx} = 1 - e^{-x} = 0$$
$$x = 0$$
$$r = 1$$

NORMAL FORM

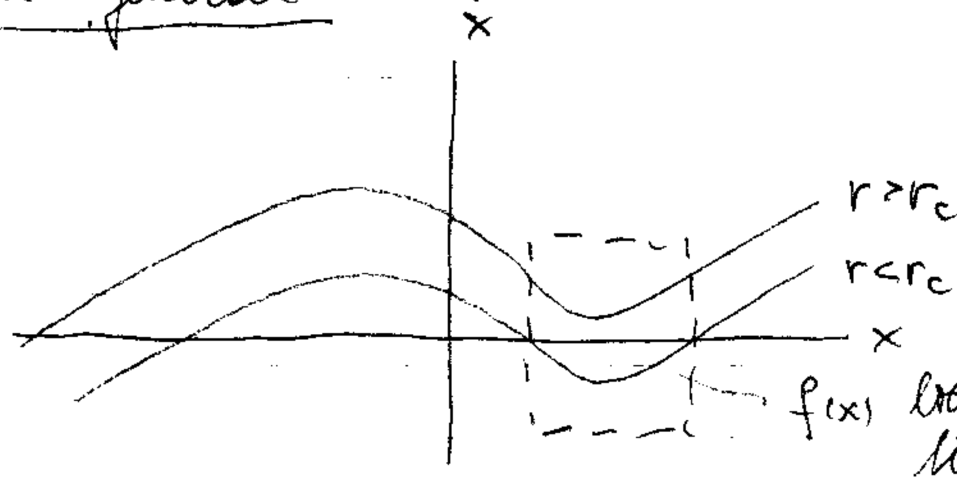
Consider $\dot{x} = r - x - e^{-x}$

$$\dot{x} = r - x - 1 + x - \frac{x^2}{2} + \dots$$
$$= (r-1) - \frac{x^2}{2} + \dots$$

By rescaling x and shifting $r - 1 \mapsto r$

$$\Rightarrow \dot{x} = r - x^2$$

In general:



$f(x)$ look like like a parabola here

$\dot{x} = f(x, r)$; bifurcation $x = \bar{x}$, $r = r_c$

$$\dot{x} = f(\bar{x}, r_c) + (x - \bar{x}) \frac{\partial f}{\partial x}(\bar{x}, r_c) + (r - r_c)\frac{\partial f}{\partial r}(\bar{x}, r_c) +$$
$$+ \frac{1}{2}(x - \bar{x})^2 \frac{\partial^2 f}{\partial x}(\bar{x}, r_c) + \dots$$

$f(\bar{x}, r_c) = 0$ b/c $\bar{x}$ is an equilibrium

$\frac{\partial f}{\partial x}(\bar{x}, r_c) = 0$ b/c of tangency condition

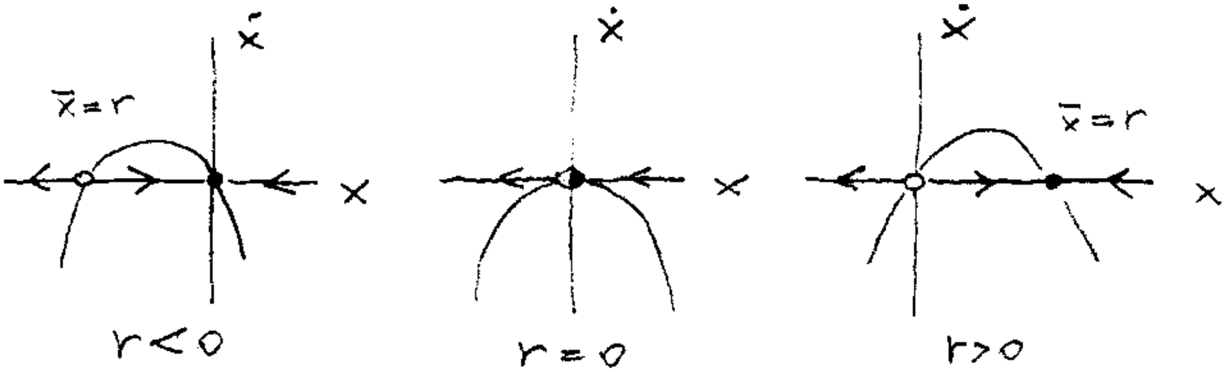
$$\Rightarrow \quad \dot{x} = a(r - r_c) + b(x - \bar{x})^2 + \dots$$

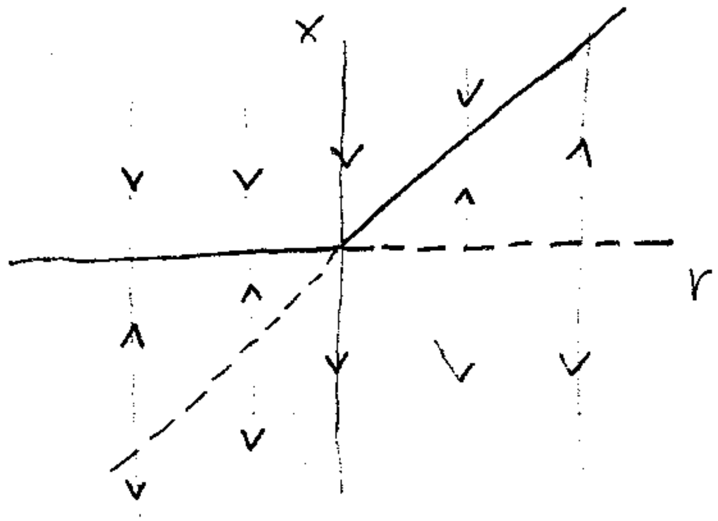
$$a = \frac{\partial f}{\partial r}(\bar{x}, r_c), \quad b = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x}, r_c)$$

TRANSCRITICAL BIFURCATION

Two equilibria collide and exchange stability

Normal form: $\dot{x} = rx - x^2$





EXAMPLE: $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

$x=0$ is an equilibrium for $\forall x$

$$1-e^{-bx} = 1-\left[1-bx+\tfrac{1}{2}b^2x^2+\mathcal{O}(x^3)\right]$$
$$= bx - \tfrac{1}{2}b^2x^2 + \mathcal{O}(x^3)$$

$$\Rightarrow \dot{x} = x - a\left(bx - \tfrac{1}{2}b^2x^2\right) + \mathcal{O}(x^3)$$

$$= (1-ab)x + \left(\tfrac{1}{2}ab^2\right)x^2 + \mathcal{O}(x^3)$$

When $ab=1$ → TRANSCRITICAL BIFURCATION

The other equilibrium is given by $1-ab+\left(\tfrac{1}{2}ab^2\right)x \approx 0$

$$\bar{x} \approx \frac{2(ab-1)}{ab^2} + \mathcal{O}\left((ab-1)^2\right)$$

EXAMPLE: $\dot{x} = r\ln x + x - 1$

$x=1$ is always an equilibrium

$\Rightarrow$ let $u = x-1$ $\quad \dot{u} = \dot{x}$

$$\dot{u} = r\ln(1+u) + u \simeq r\left[u - \tfrac{1}{2}u^2 + \mathcal{O}(u^3)\right] + u$$

$$= (r+1)u - \tfrac{1}{2}ru^2 + \mathcal{O}(u^3)$$

For normal form, scale out the coefficient in front of u^2

Let $u = av$

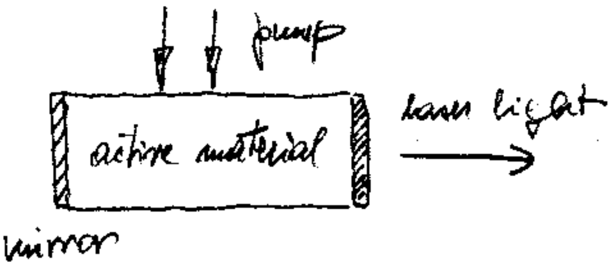
$$\dot{v} = (r+1)v - \left(\tfrac{1}{2}ra\right)v^2 + \mathcal{O}(v^3)$$

Choose $a = 2/r$ $\Rightarrow$ $\underbrace{\dot{v} = (r+1)v - v^2 + \mathcal{O}(v^3)}$

TRANSCRITICAL BIFURCATION AT $r=-1$

In original variables $v = \frac{u}{a} = \frac{1}{2}r(x-1)$

EXAMPLE: LASER THRESHOLD



$n(t)$ — # of photons in the laser field

$$\dot{n} = \text{gain} - \text{loss} = GnN - kn$$

gain — from stimulated emission – occurs via random encounters between photons and excited atoms

$N(t)$ — # of excited atoms

$G > 0$ - gain coefficient

$k > 0$ - rate constant

$\tau = \frac{1}{k} > 0$ - typical lifetime of a photon in the laser.

N_0 — # of excited atoms in the absence of laser action

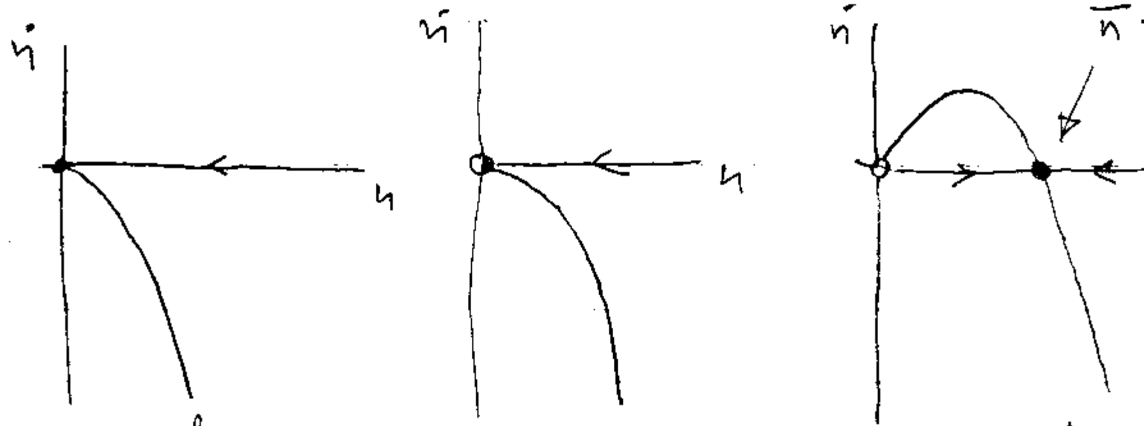
ASSUME: $N(t) = N_0 - \alpha n$

(After an excited atom emits a photon it drops down to a lower energy level and it is no longer excited $\Rightarrow N$ decreases by the # of photons)

$\underline{\alpha > 0}$ — rate at which atoms drop back to their ground states

$$\Rightarrow \quad \dot{n} = Gn(N_0 - \alpha n) - kn =$$
$$= (GN_0 - k)n - (\alpha G)n^2$$

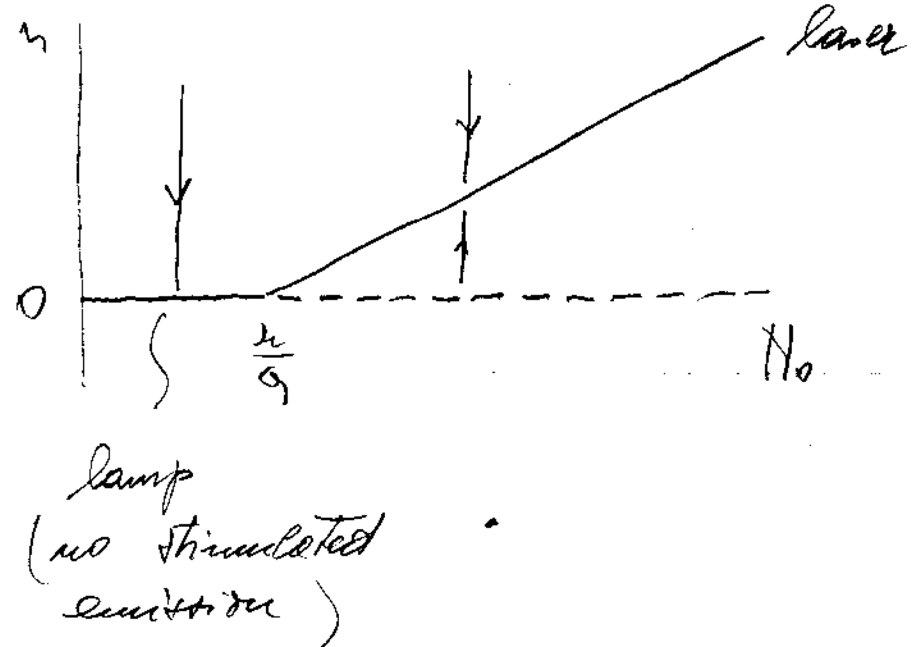
$\bar{n} = \frac{GN_0 - k}{\alpha G} > 0$



$N_0 < \frac{k}{G}$ $\qquad$ $N_0 = \frac{k}{G}$ $\qquad$ $N_0 > \frac{k}{G}$

$\bar{n}$ - spontaneous laser action

$N_0 = \frac{k}{G}$ - laser threshold



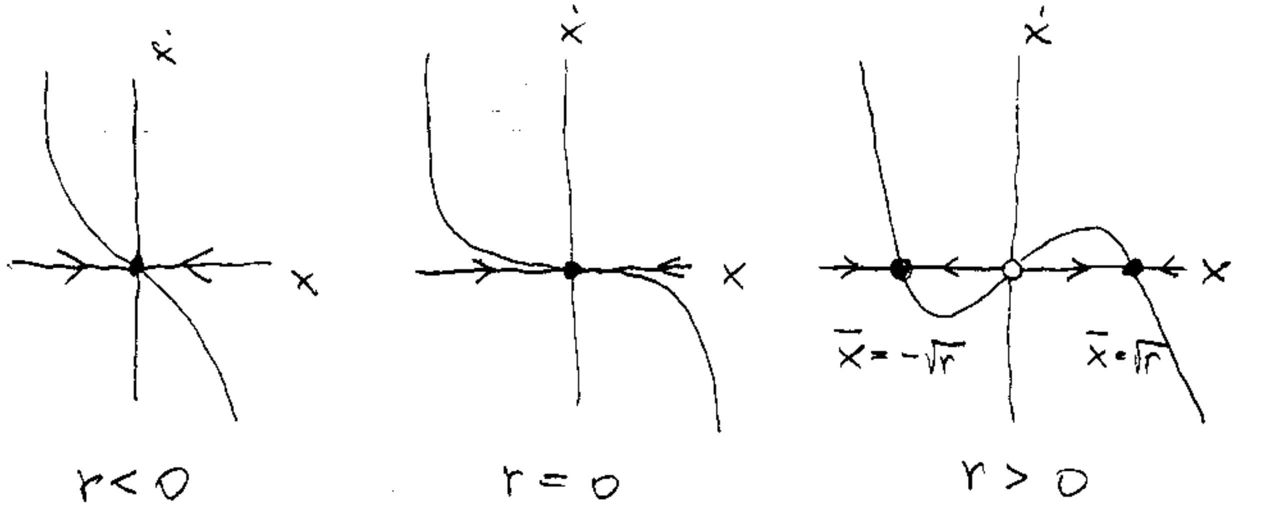
PITCHFORK BIFURCATION

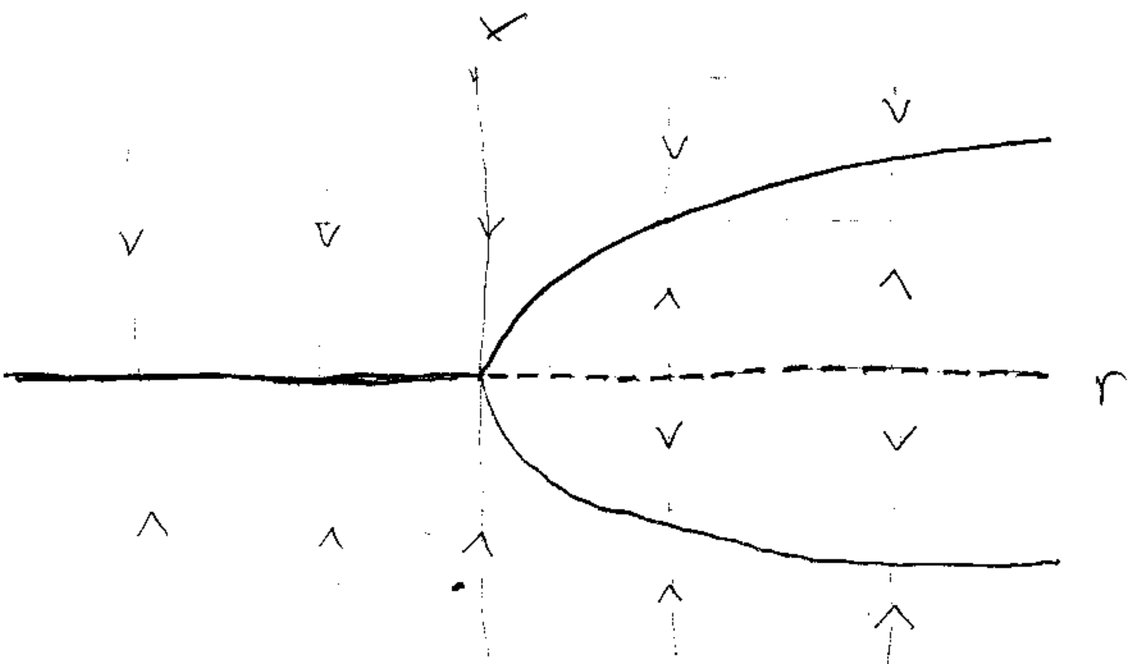
Problems with symmetry $x \mapsto -x$.

EXAMPLE Beam buckling left or right

SUPERCRITICAL PITCHFORK BIFURCATION

$$\dot{x} = rx - x^3 \quad (\text{invariant under } x \to -x)$$



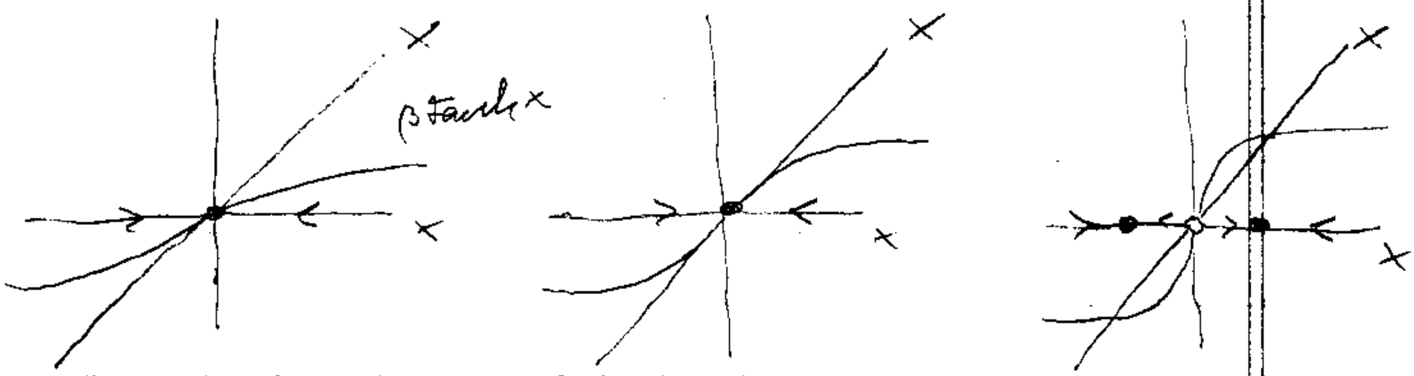


EXAMPLE: $\dot{x} = -x + \beta \tanh x$

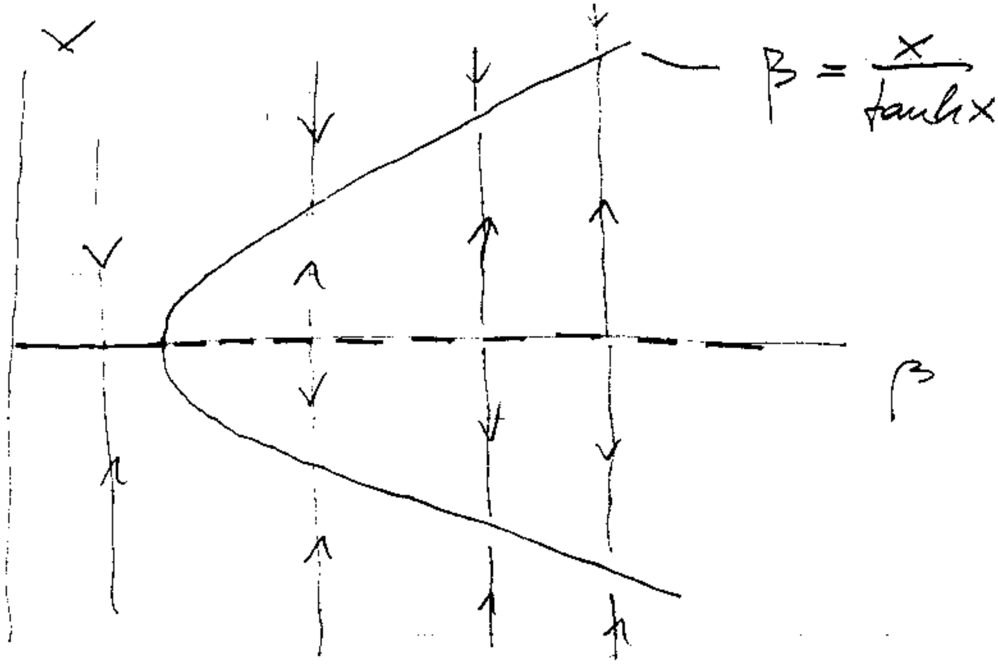
(used in models of magnets and neural networks)

$x = \bar{x} = 0$ is always a solution

Equilibrium points: $x = \beta \tanh x$

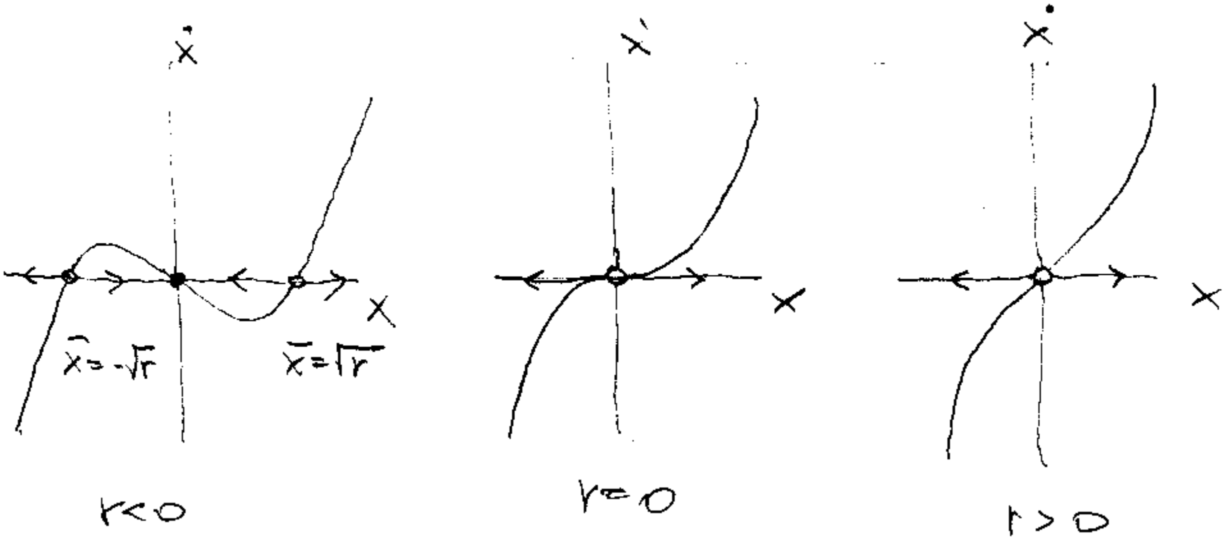


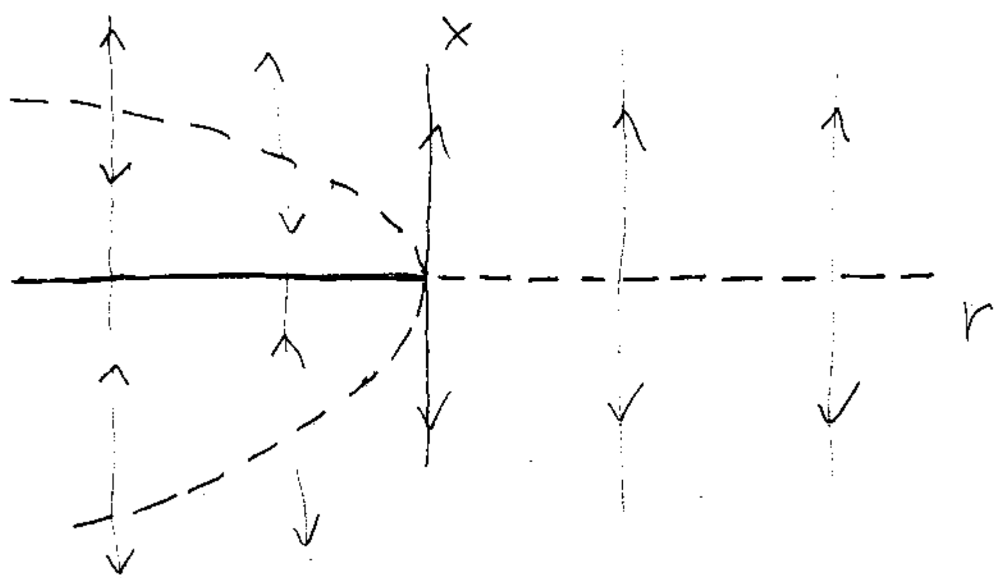
$\beta < 1$ $\qquad\qquad$ $\beta = 1$



SUBCRITICAL PITCHFORK BIFURCATION

$$\dot{x} = rx + x^3$$

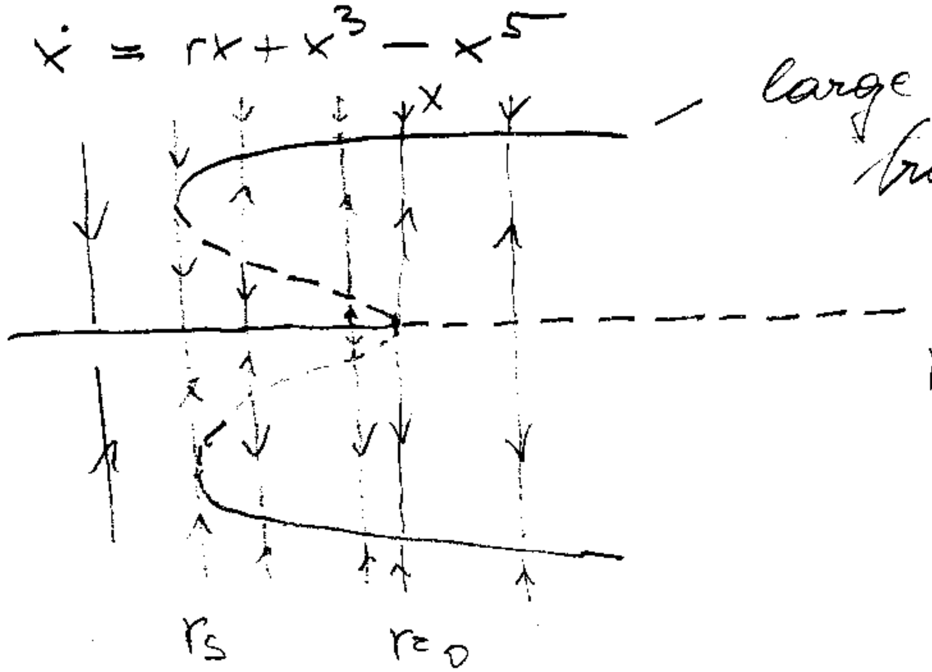




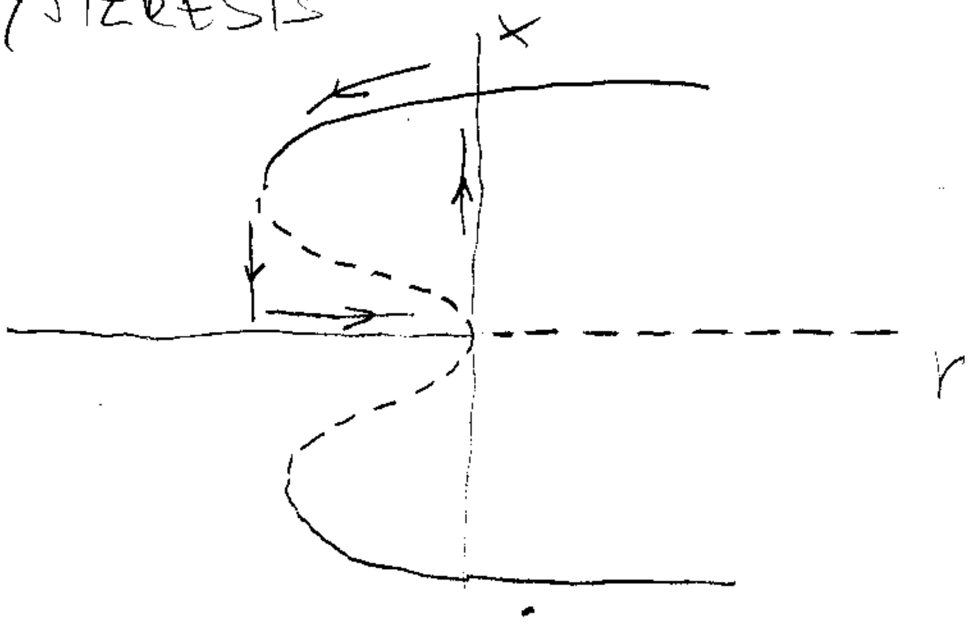
(Solutions for $r>0$ blow up in finite time!)

The system is stabilized by the influence of higher-order terms. Symmetry $x \mapsto -x$ forces the first such term to be x^5:

$$\dot{x} = rx + x^3 - x^5$$



HYSTERESIS



NOTE: To find equilibria of

$$\dot{x} = f(x, r)$$

we seek for solutions of the equation

$$f(x, r) = 0 .$$

Let $x = \bar{x}(r)$ be such a solution

If $\frac{\partial f}{\partial x}(\bar{x}(r_0), r_0) \neq 0$, the implicit function theorem guarantees that there is a unique solution $\bar{x}(r)$ for all r close to r_0.

- Therefore, there can be no bifurcation at such point.

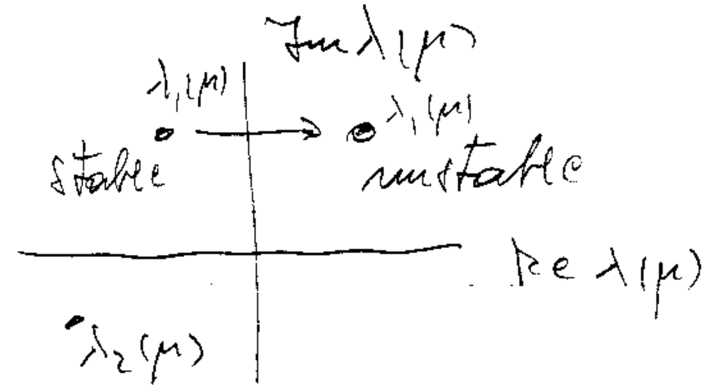
$\Rightarrow$ A necessary condition for $\bar{x}(r)$ to have a bifurcation point at $r = r_0$ is that

$$\frac{\partial f}{\partial x}(\bar{x}(r_0), r_0) = 0$$

BIFURCATIONS IN $\mathbb{R}^2$

Consider $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}^2$ (1)

How can an equilibrium pt. $\bar{x}(\mu)$ of (1) lose its stability? One of the eigenvalues of $\frac{\partial f}{\partial x}(\bar{x}(\mu), \mu)$ crosses the imaginary λ-axis

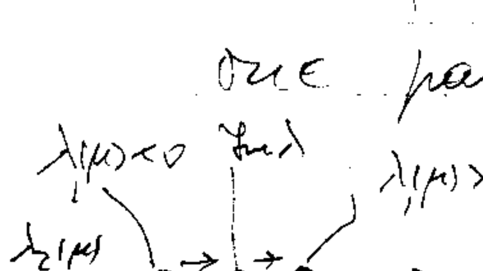


Now if f is real, so is $\frac{\partial f}{\partial x}(\bar{x}(\mu), \mu)$

$\Rightarrow$ 2 cases:

CASE 1: Both λ_1 and λ_2 are real.

Loss of stability occurs if one passes through $\lambda = 0$



It can be shown that in an appropriate coordinate system, we have a saddle-node, transcritical or pitchfork bifurcation in one dimension, and nothing changes in the second dimension

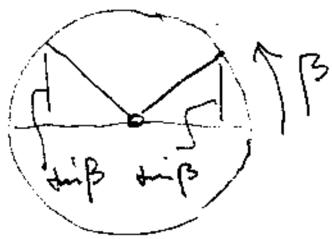
EXAMPLE:

Pendulum with a constant torque:

$$\dot{x} = y \qquad \dot{y} = -\sin x + \beta - \alpha y \;, \quad \alpha, \beta \geq 0$$

EQUILIBRIA: $\bar{x} = \arcsin\beta, \; \pi - \arcsin\beta$

$\bar{y} = 0$



Points merge in a saddle-node bifurcation when $\beta = 1$

LINEARIZATION MATRIX: $A(x,y) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix}$

$$A(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}$$

$$A(\bar{x},\bar{y}) = \begin{bmatrix} 0 & 1 \\ \mp\sqrt{1-\beta^2} & -\alpha \end{bmatrix}$$

Eigenvalues:

$$\lambda^2 + \alpha\lambda \pm \sqrt{1-\beta^2} = 0$$

$x = \arcsin\beta \qquad \lambda^2 + \alpha\lambda + \sqrt{1-\beta^2} = 0$

$$\lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \sqrt{1-\beta^2}}$$

$\beta < 1$ but close to 1 : $\lambda_{1,2} < 0$

$x = \pi - \arcsin\beta \qquad \lambda^2 + \alpha\lambda - \sqrt{1-\beta^2} = 0$

$$\lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 + \sqrt{1-\beta^2}}$$

$\lambda_1 > 0$, $\lambda_2 < 0$ saddle

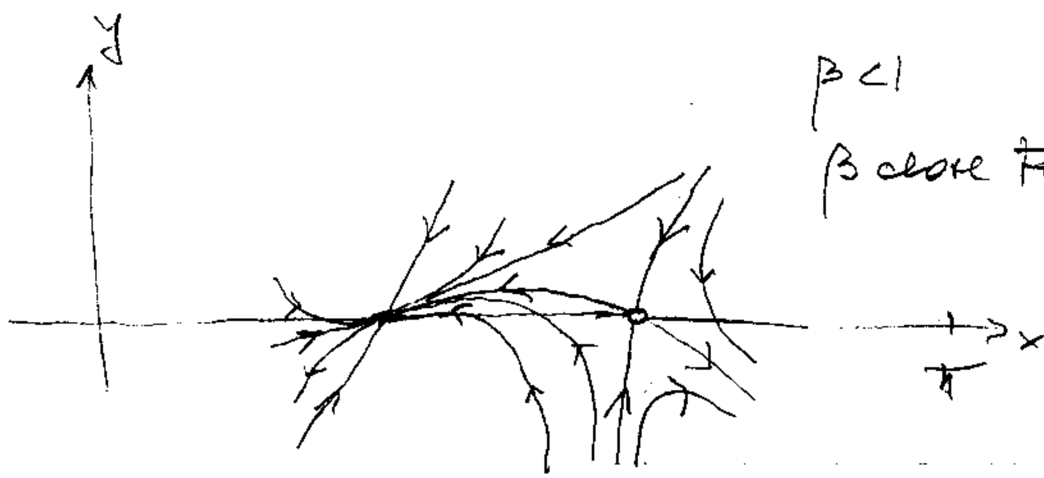
In both cases, eigenvectors are

$e_{1,2} = (1, -\lambda_{1,2})$

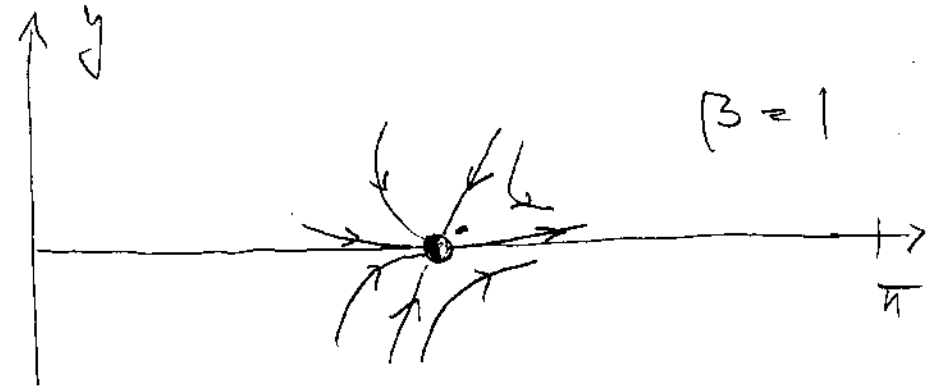
$\beta < 1$

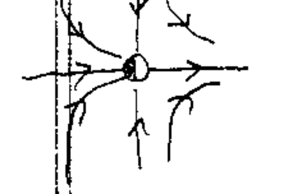
β close to 1

Schematic:



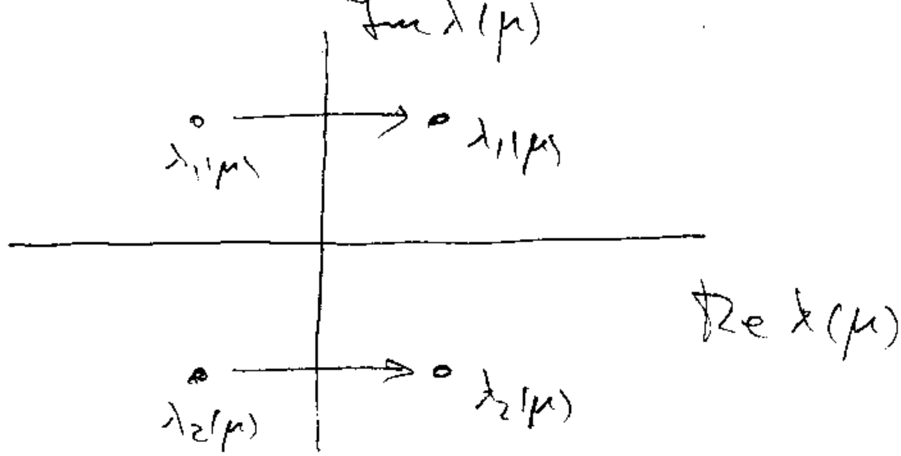
$\beta = 1$





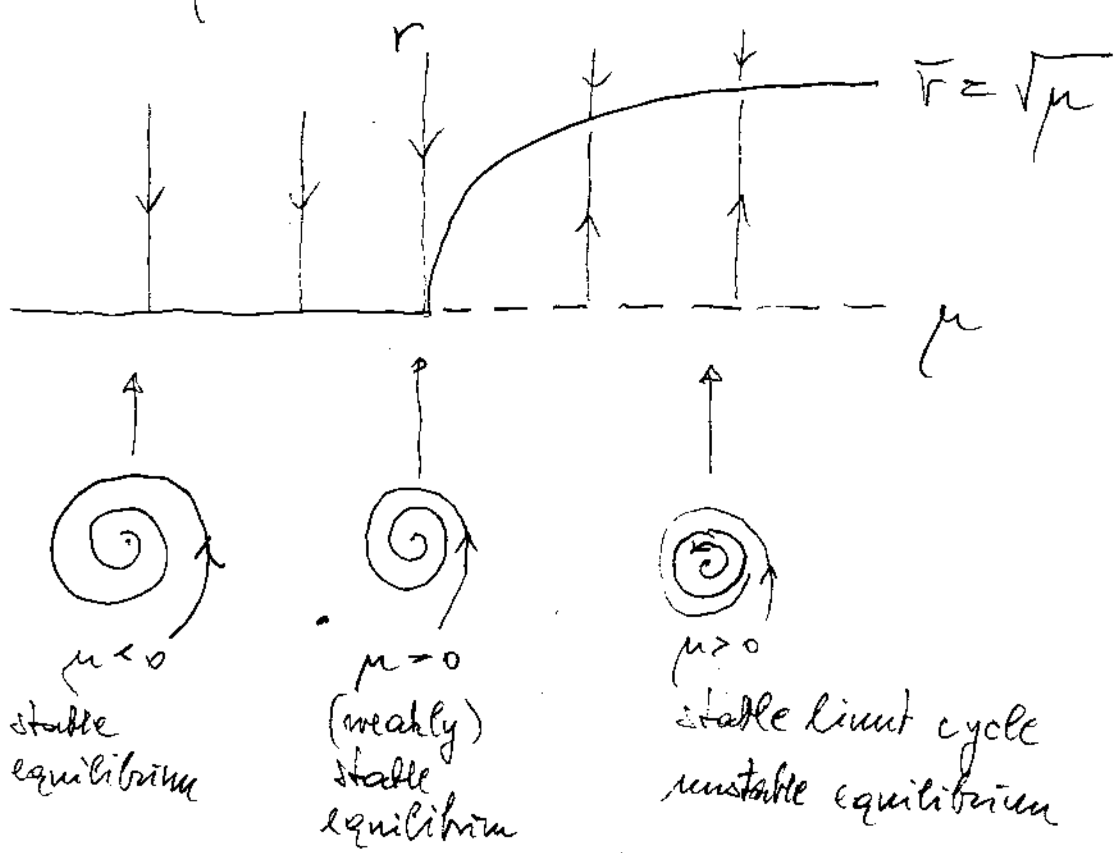
CASE 2 $\lambda_1 = \alpha + i\omega \qquad \lambda_2 = \alpha - i\omega$

Loss of stability occurs when α passes through 0



SUPERCRITICAL HOPF BIFURCATION

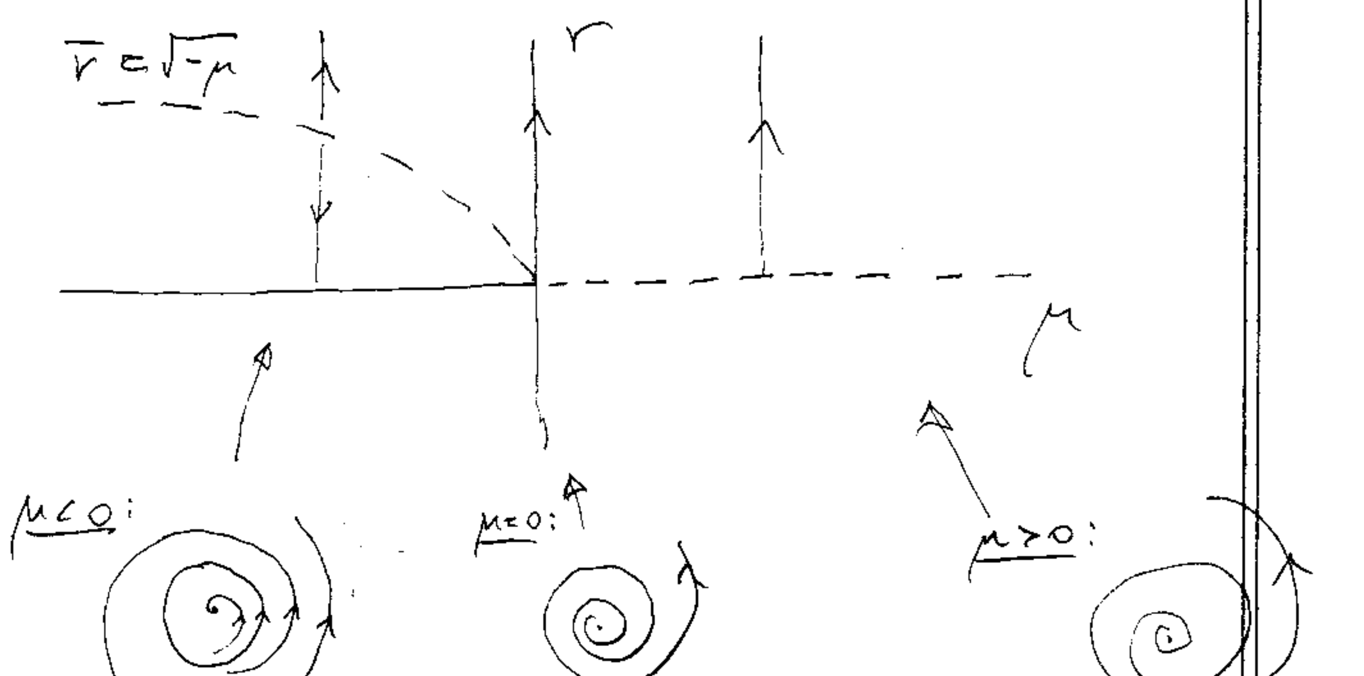
$$\dot{r} = \mu r - r^3 \qquad \dot{\theta} = \omega + b r^2$$



Besides the loss of stability, a limit cycle is created.

SUBCRITICAL HOPF BIFURCATION

$$\dot r = \mu r + r^3 \qquad \dot\theta = \omega + b r^2$$



Stable equilibrium
unstable limit cycle

(weakly)
unstable
equilibrium

unstable
equilibrium

EIGENVALUES: $x = r\cos\theta \quad y = r\sin\theta$

$$\dot x = \dot r\cos\theta - r\sin\theta\,\dot\theta = (\mu r \pm r^3)\cos\theta - r(\omega + b r^2)\sin\theta$$

$$= (\mu - (x^2+y^2))x - (\omega + b(x^2+y^2))y = \mu x - \omega y + O(r^3)$$

likewise: $\dot y = \omega x + \mu y + O(r^3)$

Jacobian at $(0,0)$: $A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$, eigenvalues $\lambda = \mu \pm i\omega$

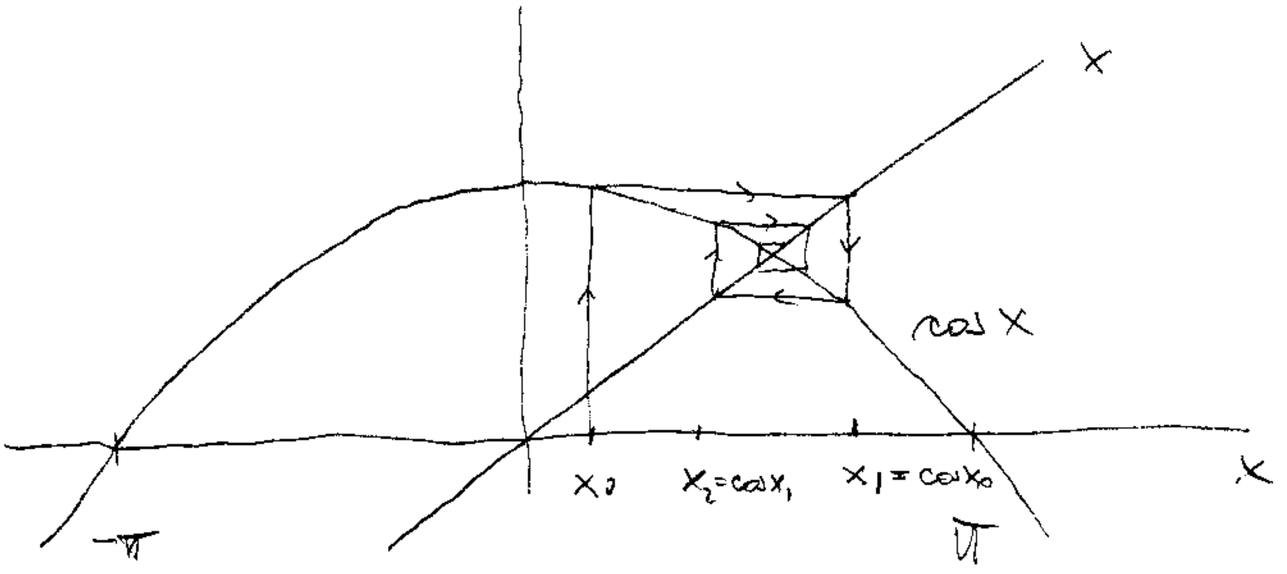
ITERATED MAPS AND CHAOTIC DYNAMICS

Systems with discrete time. Also known as difference equations, recursion relations, maps.

EXAMPLE Choose x_0. Let $x_1 = \cos x_0$

$x_2 = \cos x_1$, etc.

$$x_{n+1} = \cos x_n$$



Applications:

1.) Numerical iterations – finding zeros of functions
2.) Tools for analyzing differential equations: Poincaré maps etc.
3.) Models of natural phenomena with discrete time: computers, finance, biology, bouncing ball, …
4.) Simple examples of chaotic dynamics

<u>Fixed points</u> : Consider $x_{n+1} = f(x_n)$
$\bar{x}$ is a <u>fixed point</u> of f if $f(\bar{x}) = \bar{x}$.

<u>Linear Stability</u>: Let $\bar{x}$ be a fixed point of f
Let $x_n = \bar{x} + \eta_n$

$$x_{n+1} = \bar{x} + \eta_{n+1} = f(\bar{x} + \eta_n) =$$
$$= \underbrace{f(\bar{x})}_{=\bar{x}} + f'(\bar{x})\eta_n + O(\eta_n^2)$$
$$\Rightarrow \eta_{n+1} = f'(\bar{x})\eta_n + O(\eta_n^2)$$

<u>CLAIM</u>: If $|f'(\bar{x})| \neq 1$, we can neglect the $O(\eta_n^2)$ for small enough η_n.

$f'(\bar{x}) \equiv \lambda$ — eigenvalue or <u>multiplier</u>

$$\eta_1 = \lambda\eta_0, \quad \eta_2 = \lambda\eta_1 = \lambda^2\eta_0, \ldots, \eta_n = \lambda^n\eta_0, \ldots$$

$|\lambda| < 1 \Rightarrow \eta_n \to 0 \Rightarrow \bar{x}$ is stable
$|\lambda| > 1 \Rightarrow \eta_n \to \infty \Rightarrow \bar{x}$ is unstable
$|\lambda| = 1 \Rightarrow$ must consider $O(\eta_n^2)$ terms

EXAMPLES: 1) Logistic map

$$x_{n+1} = r x_n (1 - x_n) \qquad x_n \geq 0$$

Fixed points: $x = r x (1-x)$

$$\bar{x} = 0 \quad \text{or}$$

$$1 = r - r\bar{x}$$

$$\bar{x} = \frac{r-1}{r} \quad \Rightarrow \quad r \geq 1$$

Multiplier: $\lambda = r(1-\bar{x}) - r\bar{x} = r(1-2\bar{x})$

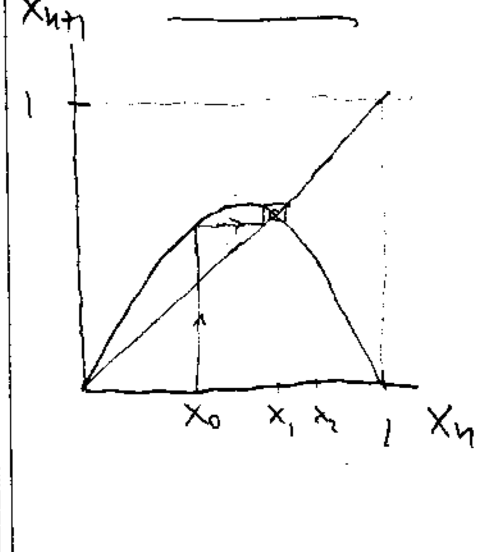
$$\lambda = r\left(1 - 2\frac{r-1}{r}\right) = r - 2r + 2$$

$$= 2 - r$$

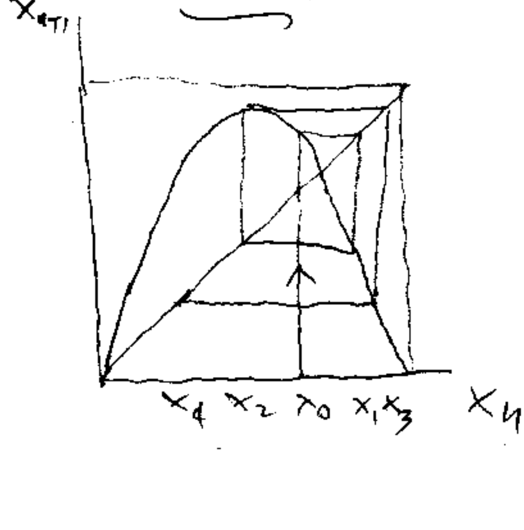
$$1 < r < 3 \quad \Rightarrow \quad |\lambda| < 1 \quad \text{stable}$$

$$r > 3 \quad \Rightarrow \quad |\lambda| > 1 \quad \text{unstable}$$

$1 < r < 3$:



$r > 3$:



2.) $x_{n+1} = x_n^2$

<u>FIXED POINTS</u>: $x = x^2 \Rightarrow x = 0, 1$

<u>Multiplier</u>: $\lambda = 2x$ $\quad x=0 \Rightarrow \lambda = 0$ stable

$x = 1 \Rightarrow \lambda = 2$ unstable

$x=0$ is in fact <u>superstable</u> because $\lambda = 0$

$x_0, \; x_1 = x_0^2, \; x_2 = x_1^2 = (x_0^2)^2 = x_0^{2^2}$

$x_3 = x_2^2 = \left(x_0^{2^2}\right)^2 = x_0^{2^3}$ etc.

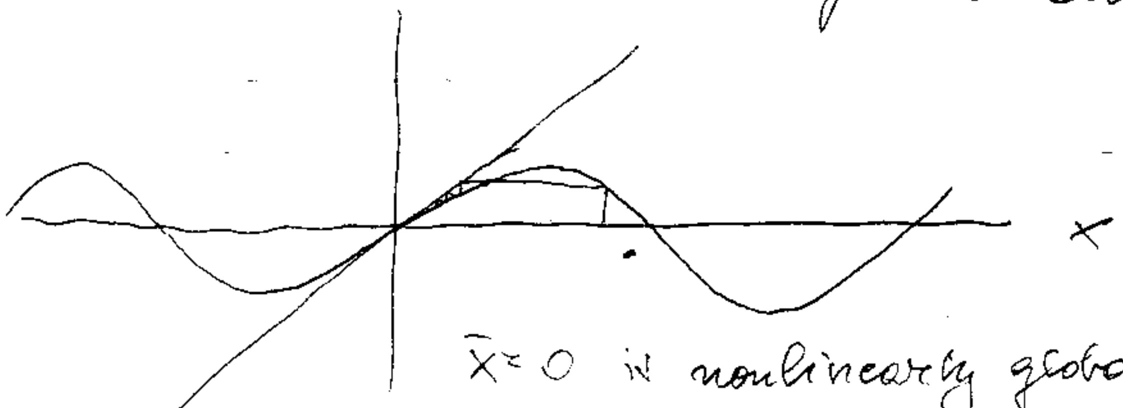
$\ldots$

$x_{n+1} = x_0^{2^n}$

This is <u>superconvergence</u>

3.) $x_{n+1} = \sin x_n$ $\quad$ Fixed point $\bar{x} = 0$

Multiplier: $\lambda = \cos \bar{x} = 1$ — marginal case



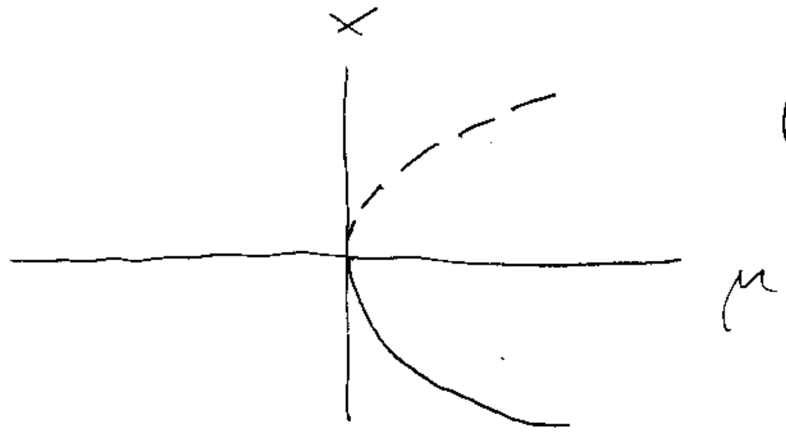
$\bar{x} = 0$ is nonlinearly globally stable.

BIFURCATIONS OF MAPS

$x_{n+1} = x_n + \mu + x_n^2$ – SADDLE-NODE

FIXED POINTS: $\bar{x} = \pm\sqrt{\mu}$

MULTIPLIERS: $\lambda = 1 + 2\bar{x} = 1 \pm 2\sqrt{\mu}$

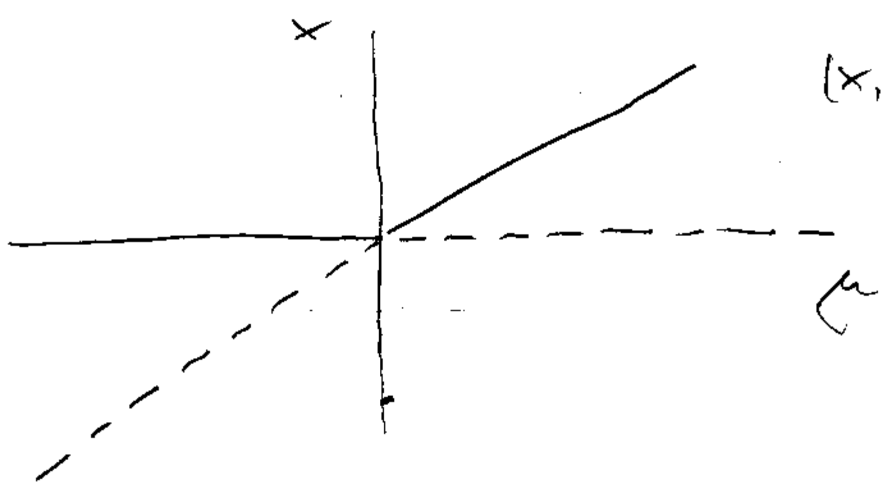


(x, μ) - small

$x_{n+1} = x_n + \mu x_n - x_n^2$ - TRANSCRITICAL

FIXED POINTS: $\bar{x} = 0, \mu$

MULTIPLIERS: $\lambda = 1 + \mu - 2\bar{x} = 1 \pm \mu$

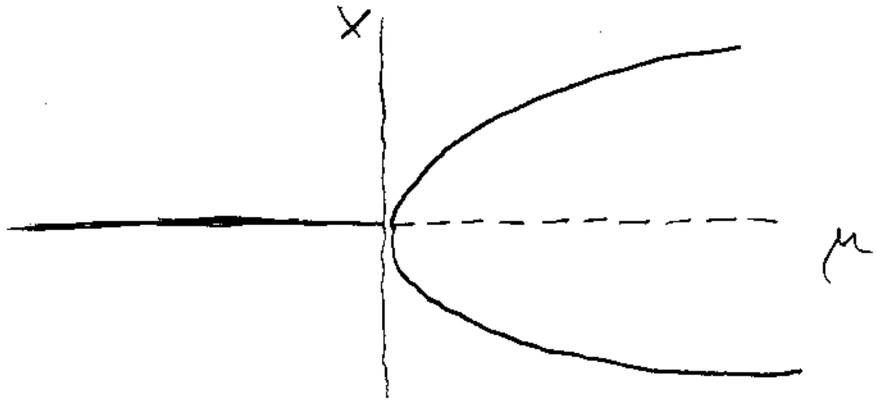


(x, μ) - small

$x_{n+1} = x_n + \mu x_n - x_n^3$ – SUPERCRITICAL PITCHFORK

FIXED POINTS: $\bar{x} = 0, \pm\sqrt{\mu}$

MULTIPLIERS: $\lambda = 1+\mu-3\bar{x}^2$: $1+\mu$, $1-2\mu$

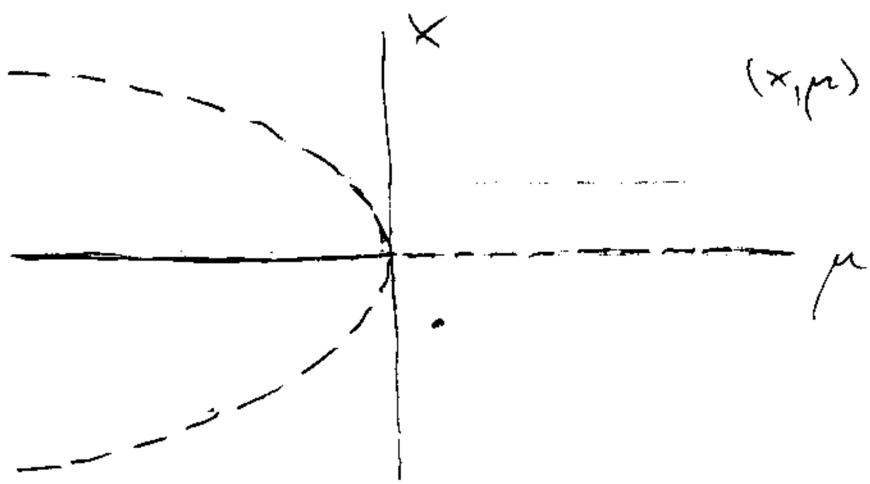


(x,μ) – small

$x_{n+1} = x_n + \mu x_n + x_n^3$ – SUBCRITICAL PITCHFORK

FIXED POINTS: $\bar{x} = 0, \pm\sqrt{-\mu}$

MULTIPLIERS: $\lambda = 1+\mu+3\bar{x}^2$: $1+\mu$, $1-2\mu$



(x,μ) – small

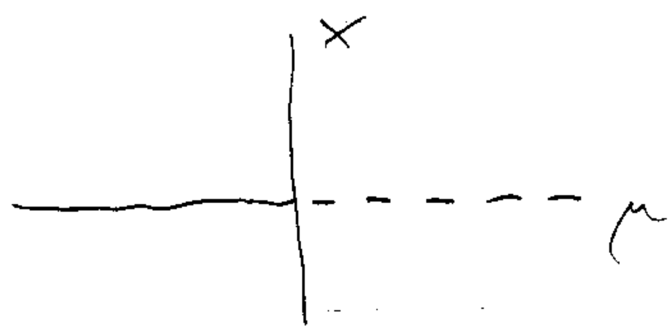
PERIOD-DOUBLING BIFURCATION

$$x_{n+1} = -(1+\mu)x_n + x_n^2 \equiv f(x_n, \mu)$$

CONSIDER JUST THE FIXED POINT $\bar{x} = 0$

MULTIPLIER: $\lambda = -(1+\mu)$

$\mu < 0 \Rightarrow \lambda > -1$ - stable
$\mu = 0 \Rightarrow \lambda = -1$ - neutral
$\mu > 0 \Rightarrow \lambda < -1$ - unstable



No other fixed point bifurcates off of $\bar{x} = 0$ at $\mu = 0$. What is going on?

Consider the second iterate

$$x_{n+1} = f(f(x_n,\mu),\mu) =$$
$$= -(1+\mu) f(x_n,\mu) + f^2(x_n,\mu)$$

$$x_{n+1} = -(1+\mu)\left(-(1+\mu)x_n + x_n^2\right) + \left(-(1+\mu)x_n + x_n^2\right)^2 =$$

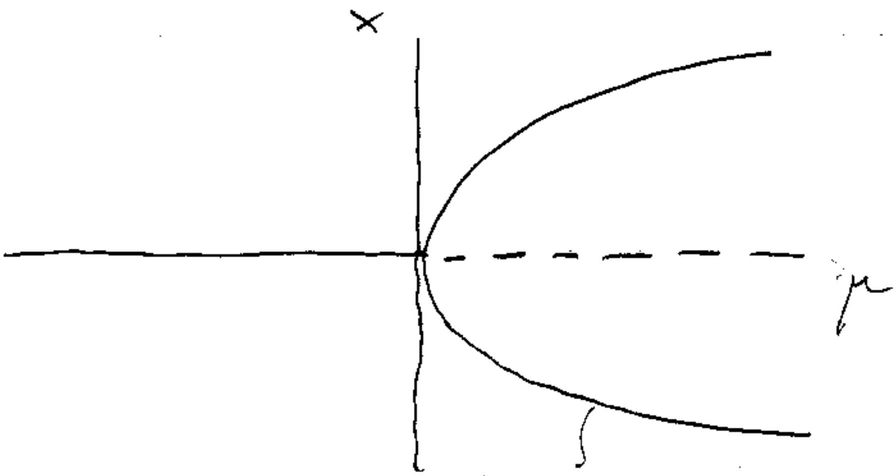
$$= (1+\mu)^2 x_n - (1+\mu)x_n^2 + (1+\mu)^2 x_n^2 - 2(1+\mu)x_n^3 + x_n^4 =$$

$$= (1+2\mu)x_n - 2x_n^3 + O(\mu x_n^2) + O(x_n^4)$$

$$x_{n+1} = (1+2\mu)x_n - 2x_n^3$$ – SUPERCRITICAL PITCHFORK

FIXED POINTS: $\bar{x} = 0, \pm\sqrt{\mu}$

MULTIPLIER: $\lambda = 1+2\mu - 6\bar{x}^2$: $1+2\mu$, $1-4\mu$

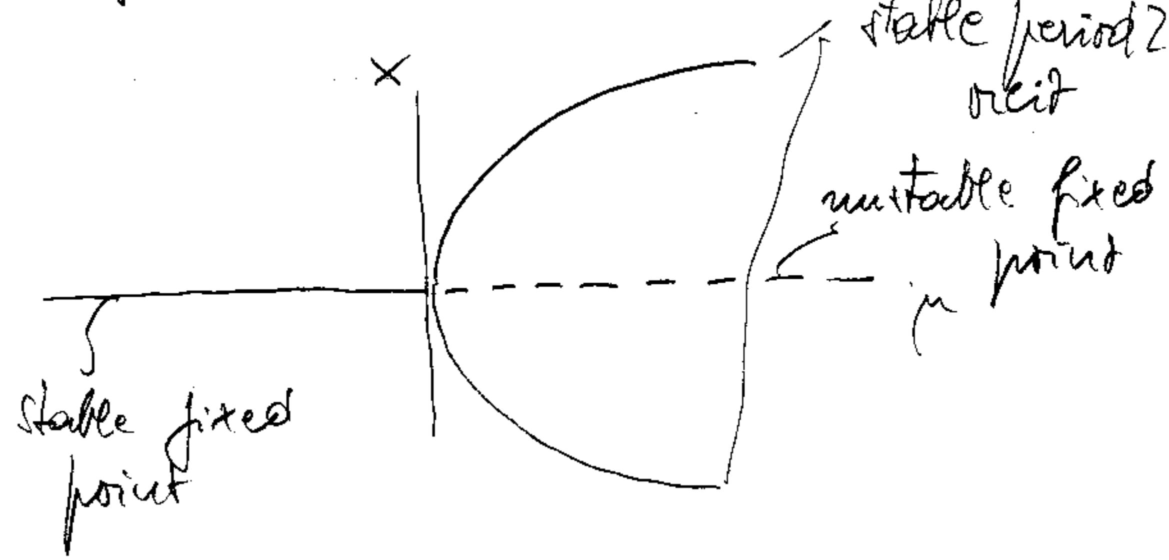


(x, μ) – small

stable fixed points for $f^2 = f \circ f$
not fixed points for f

$\Rightarrow \quad \bar{x} = \pm\sqrt{\mu}$ are <u>period 2</u> points

for $x_{n+1} = -(1+\mu)x_n + x_n^2$



$f: \sqrt{\mu} \to -\sqrt{\mu}$ and $f: -\sqrt{\mu} \to \sqrt{\mu}$

<u>EXAMPLE</u>

Logistic map $x_{n+1} = r x_n (1 - x_n)$

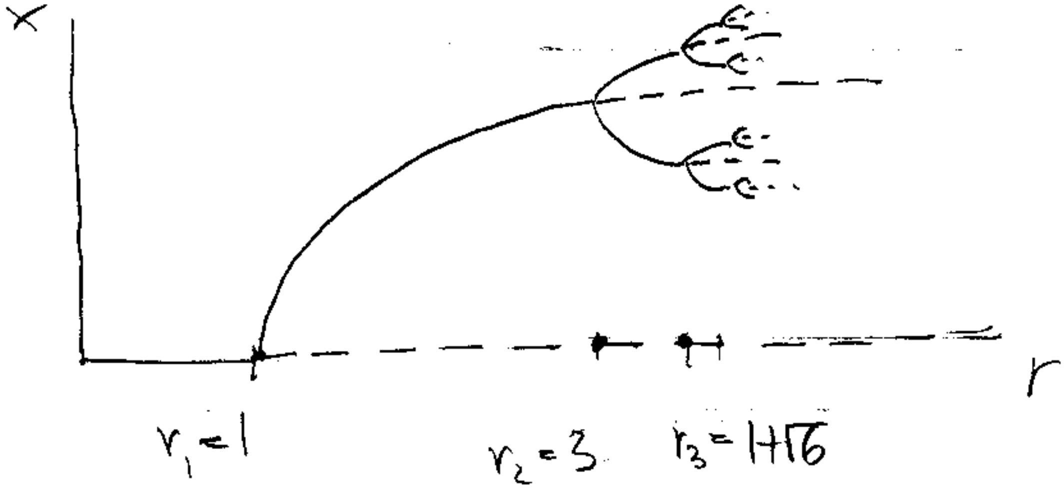
Consider the FIXED POINT $\bar{x} = \frac{r-1}{r}$

MULTIPLIER $\lambda = 2 - r$

PERIOD DOUBLING AT $r = 3$.

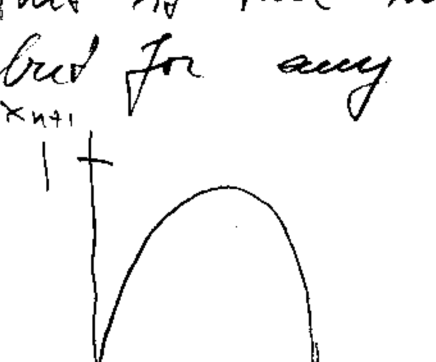
PERIOD-DOUBLING CASCADES AND FEIGENBAUM'S SEQUENCE

One can show the following bifurcation diagram for the logistic map $x_{n+1} = r x_n (1 - x_n)$. Every bifurcation is a period-doubling



$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

This is true not only for the logistic map but for any unimodal map:



$f: [0,1] \to [0,1]$

One maximum

$f(0) = 0$, $f(1) = 0$

δ is the same for all such maps: δ is a universal constant.

CHAOTIC DYNAMICS

Binary shift map : $x_{n+1} = 2x_n \pmod 1$

$x_n \in [0,1]$

Binary representation of x_n

$x_n = 0.s_1 s_2 s_3 s_4 \cdots\cdots, \quad \forall s_j \in \{0,1\}$

$x_{n+1} = 0.s_2 s_3 s_4 s_5 \cdots\cdots$

FIXED POINTS

$\bar{x} = 0.0000\cdots\cdots = 0$

$\bar{x} = 0.11111\cdots\cdots = 1$

PERIODIC ORBITS

Let σ_p be any sequence of 0's and 1's of length p. Then

$x_p = 0.\sigma_p \sigma_p \sigma_p \cdots$

is a periodic orbit with period p.

There are countably many such periodic orbits.

APERIODIC ORBITS

$x = 0.s_1 s_2 s_3$ — aperiodic sequence

There are uncountably many such aperiodic orbits.

INSTABILITY

Let $x^{(1)} = 0.s_1^{(1)} s_2^{(1)} s_3^{(1)} \ldots$
$x^{(2)} = 0.s_1^{(2)} s_2^{(2)} s_3^{(2)} \ldots$

The distance between x_1 and x_2 is clearly $|x^{(1)} - x^{(2)}|$

Let $x^{(1)} = 0.s_1 s_2 s_3 \ldots s_{m-1} s_m^{(1)} s_{m+1}^{(1)} s_{m+2}^{(1)} \ldots$
$x^{(2)} = 0.s_1 s_2 s_3 \ldots s_{m-1} s_m^{(2)} s_{m+1}^{(2)} s_{m+2}^{(2)} \ldots$

where $s_m^{(1)} \neq s_m^{(2)}$ and $s_{m+1}^{(1)} \neq s_{m+1}^{(2)}$. Then

$$\frac{1}{2^m} \leq |x^{(1)} - x^{(2)}| \leq \frac{1}{2^{m-1}}$$

For the n-th images, $n < m$,

$|x_n^{(1)} - x_n^{(2)}|$ satisfies

$$\frac{1}{2^{m-n}} \le |x^{(1)} - x^{(2)}| \le \frac{1}{2^{m-n-1}}$$

$\Rightarrow$ The distance increases as

$$2^n \, |x^{(1)} - x^{(2)}| .$$

$\Rightarrow$ All orbits are exponentially unstable, including all the periodic orbits.

SENSITIVE DEPENDENCE ON INITIAL CONDITIONS

Choose x with finite precision:

$$x = 0.s_1 s_2 \dots s_m$$

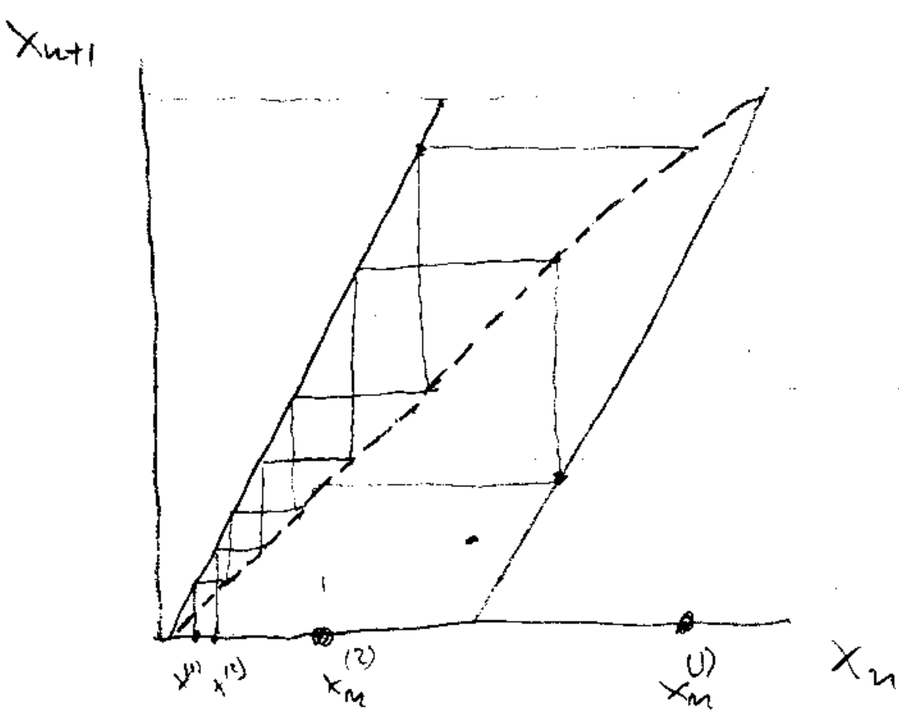
Digits higher than s_m are unknown, so x can be anywhere in the interval

$$0.s_1 \dots s_m 0000\dots \le x \le 0.s_1 \dots s_m 111111$$

The width of this interval is
$\Delta = \frac{1}{2^m}$.

For the n-th image $n < m$, the width of the imprecision interval increases to
$2^n \Delta = \frac{1}{2^{m-n}}$.

After m iterations, all information about x is lost;
$\Rightarrow$ Orbits exhibit sensitive dependence on initial conditions



DENSE ORBIT

$\{x_n\}$ is dense if for $\forall \varepsilon$, $\forall p \in [0,1]$, $\exists n$ such that $|x_n - p| < \varepsilon$.

In other words, $\{x_n\}$ eventually gets arbitrarily close to any point p in $[0,1]$.

Construct a dense orbit for $x_{n+1} = 2x_n \pmod 1$:

Concatenate all possible finite sequences of length m, $m = 1, 2, 3, \ldots$. Let this be x_1. Then some x_n matches the first N digits of any given p.

LOGISTIC MAP WITH $r=4$

$$x_{n+1} = 4x_n(1-x_n)$$

Prop: Let $\{\theta_n\}$ be an orbit for the binary shift map $\theta_{n+1} = 2\theta_n \pmod 1$. Let $\{x_n\}$ be defined by $x_n = \sin^2(\pi\theta_n)$

Then $x_{n+1} = 4x_n(1-x_n)$.

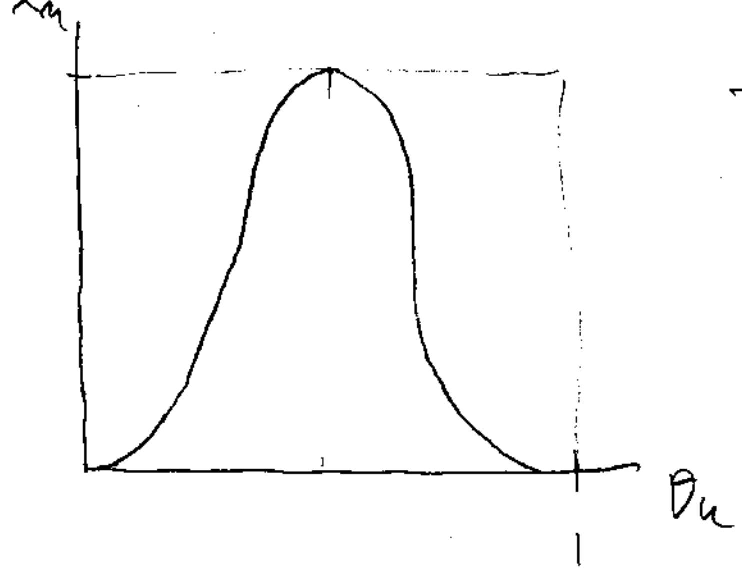
PROOF Let $x_n = \sin^2(\pi\theta_n)$. Then

$$x_{n+1} = \sin^2(2\pi\theta_n \pmod{2\pi})$$ and

$$4x_n(1-x_n) = 4\sin^2(\pi\theta_n)(1-\sin^2(\pi\theta_n))$$

$$= 4\sin^2(\pi\theta_n)\cos^2(\pi\theta_n) =$$

$$= \sin^2(2\pi\theta_n \pmod{2\pi}) = x_{n+1}$$



The correspondence is $2 \to 1$

$\theta_n,\ \pi-\theta_n \to x_n$

$\Rightarrow$ Each orbit $\{\theta_n\}$ corresponds to a unique orbit $\{x_n\}$. Since $\sin^2(\ )$ is continuous, statements about instability and sensitive dependence on initial conditions are true about $x_{n+1} = 4x_n(1-x_n)$. Since $\sin^2(\ )$ is onto $[0,1]$, $x_{n+1} = 4x_n(1-x_n)$ has a dense orbit.