AUTONOMOUS EQUATIONS

(1) $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, no t in f

$\mathbb{R}^n$ – phase space

x – phase point

$\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^n \times \{t\}$ – extended phase space

$f(x)$ – vector field

$\bar{x}$ s.t. $f(\bar{x}) = 0$ – Equilibrium point

Let $\phi(t, t_0)$ be the solution of (1) with
$\phi(t_0, t_0) = x_0$

map: $x_0 \longmapsto \phi(t, t_0)$ – flow of the vector field f

$\{x = \phi(t, t_0)\}$ – trajectory, orbit

$\{(\phi(t, t_0), t)\}$ – integral curve

Autonomous versus non-autonomous

non-autonomous $\dot{x} = F(x,t)$

autonomous $\begin{cases} \dot{x} = F(x,\varphi) \\ \dot{\varphi} = 1 \end{cases}$

Prop Let $x(t)$ be a solution of $\dot{x} = f(x)$. Then $x(t+c)$ is also a solution for any constant c.

Proof $\dot{x}(t+c) = \dfrac{dx(t+c)}{d(t+c)} = f(x(t+c))$

(Time shifts of a solution on a trajectory are all solutions on the same trajectory. If your environment does not depend on time, you will do the same thing tomorrow in the same way as today!)

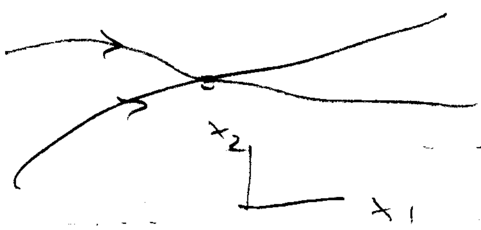
Example $\dot{x} = y$ $\dot{y} = -x$

$$(x,y) = (a\cos(t+c), a\sin(t+c))$$

For a fixed a, we get an infinity of solutions, but just one trajectory

$$x^2+y^2=a^2$$

THM Through any point of $\mathbb{R}^n$ passes at most one trajectory of $\dot{x} = f(x)$



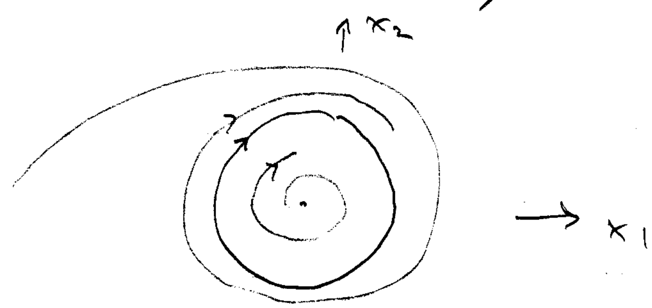
⇐ CANNOT HAPPEN

Proof Let $x^{(1)}(t_1) = x^{(2)}(t_2)$

Then $x(t) = x^{(1)}(t + t_1 - t_2)$

satisfies $x(t_2) = x^{(2)}(t_2)$.
By uniqueness $x(t) = x^{(2)}(t)$ so that $x^{(1)}(t)$ is just a time shift of $x^{(2)}(t) \Rightarrow$ they have the same trajectory.

However a trajectory can self intersect in a certain sense! It can close up:



We get a periodic solution.

THM Let $x(t)$ solve $\dot{x} = f(x)$. If for $t_1 \neq t_2$, $x(t_1) = x(t_2)$, then the solution is defined for all t and either

(i) $x(t) = \bar{x}$, equilibrium point, or

(ii) $x(t)$ is periodic with period $T > 0$.

Proof By the previous THM and uniqueness

$x(t) = x(t + t_1 - t_2)$ for every applicable

$\Rightarrow x(t)$ is defined for $\forall t \in \mathbb{R}$.

Let $\Pi = \{$all periods of $x(t)\}$.

$t_1 - t_2 \in \Pi$, and if $T_1, T_2 \in \Pi$ then $T_1 + T_2 \in \Pi$.

Also if $\{T_n\} \subset \Pi$ and $T_n \to \bar{T}$ then $\bar{T} \in \Pi$ b/c $x(t)$ is continuous

$\Rightarrow \Pi$ is closed.

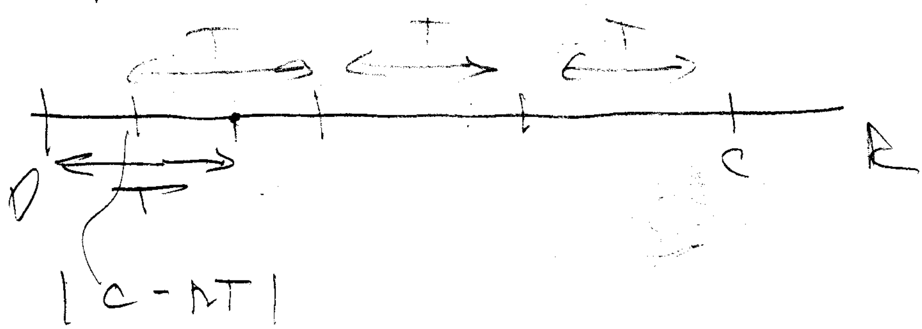
Two cases

(i) $\Pi = \mathbb{R} \Rightarrow$ equilibrium: $x(t) = \bar{x}$

(ii) $\exists$ minimal positive number $T \in \Pi$.
$\Rightarrow$ periodic solution

b/c (i) If there is no minimal positive $T \in \Pi$, then for $\forall \varepsilon$ $\exists c \in \Pi$ s.t. $0 < c < \varepsilon$.
Given any $r \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ s.t. $|r - mc| \le \varepsilon \Rightarrow$ since Π is closed $\Pi = \mathbb{R}$

(ii) Let T be the minimal positive number in Π. $\Rightarrow$ $\forall c$ in Π is mT, $m \in \mathbb{Z}$.
If it weren't, $\exists n \in \mathbb{Z}$ s.t. $|c - nT| < T$



But, $c - nT \in \Pi$ so T wouldn't be minimal.

$\Rightarrow \Pi = \{nT, n \in \mathbb{Z}\}$

EXAMPLE: $\dot{x} = y + x(1 - x^2 - y^2)$

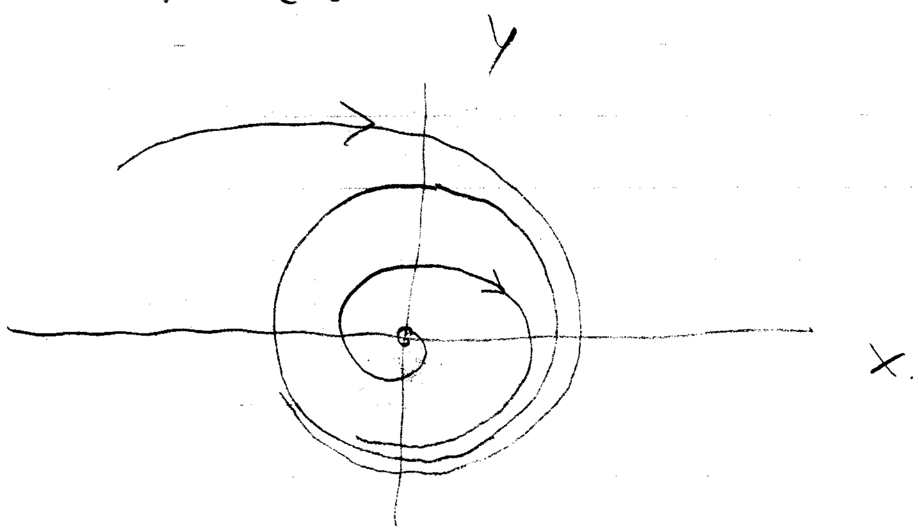
$\dot{y} = -x + y(1 - x^2 - y^2)$

Use $x = r\cos\theta$, $y = r\sin\theta$

$\rightarrow \dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}$, $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2}$

$\dot{r} = r(1 - r^2)$, $\dot{\theta} = -1$

$r(t) = \dfrac{1}{\sqrt{1 + c e^{-2t}}}$ $\theta(t) = -t + t_0$



Stable periodic solution at $r = 1$

– attracts all other solutions

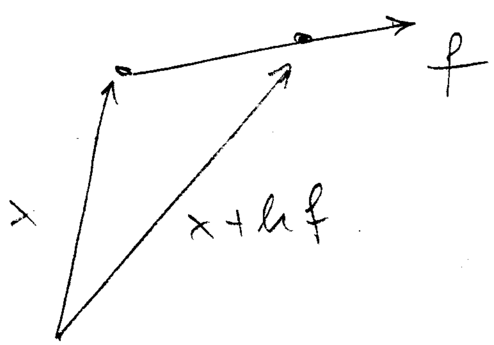
$\lim_{t \to \infty} r(t) = 1$

Directional derivatives and first integrals

$x \in \mathbb{R}^n$, f - vector in $\mathbb{R}^n$ attached at x

$F(x)$: function $F : \mathbb{R}^n \to \mathbb{R}$

Directional derivative:



$$(L_f F)\,x = \lim_{h \to 0} \frac{F(x+hf) - F(x)}{h}$$

$$= f \cdot \frac{\partial F}{\partial x}$$

If $f(x)$ is a vector field, then for $\forall x$

$$(L_f F)(x) = f(x) \cdot \frac{\partial F}{\partial x}(x)$$

Consider

$$\dot{x} = f(x). \qquad (1)$$

$F : \mathbb{R}^n \to \mathbb{R}$ is a <u>1st integral of (1)</u>

if $(L_f F)\,x = 0$

Prop Let $F(x)$ be a 1^{st} integral of $\dot{x} = f(x)$, and $x = x(t)$ a solution. Then

$$F(x(t)) = \text{const}$$

Proof; $\frac{dF(x(t))}{dt} = \frac{\partial F}{\partial x}(x(t)) \cdot \dot{x}(t) =$

$$= \frac{\partial F}{\partial x}(x(t)) f(x(t)) = (L_f F)(x(t)) = 0$$

Prop. Every solution of $\dot{x} = f(x)$ belongs to precisely one level set of $F(x)$.

EXAMPLES: 1.) $\ddot{x} = f(x)$, $E = \frac{1}{2}\dot{x}^2 - \int f(x)\,dx$

2.) $H = H(p,q)$ $\quad \dot{p} = -\frac{\partial H}{\partial q}$, $\dot{q} = \frac{\partial H}{\partial p}$

H = integral of motion b/c

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} = \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} = 0$$

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3.) $\vec{r} = (x, y, z) \quad \vec{p} = (p_x, p_y, p_z)$

$$H = |\vec{p}|^2 + U(r), \quad r = \sqrt{x^2+y^2+z^2} = |\vec{r}|$$

$$\dot{\vec{p}} = -\frac{\partial U}{\partial r}\frac{\vec{r}}{r}, \quad \dot{\vec{r}} = \vec{p}$$

$\vec{r} \times \vec{p} = $ CONST (3 integrals of motion)

b/c

$$(\vec{r}\times\vec{p})^{\cdot} = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}} =$$

$$= \underbrace{\dot{\vec{r}}\times\dot{\vec{r}}}_{0} - \frac{\partial U}{\partial r}\underbrace{\vec{r}\times\vec{r}}_{0} = 0$$

$\vec{L} = \vec{r}\times\vec{p}$ — angular momentum.

4.) For what values of k does

$\dot{x} = y \quad \dot{y} = kx$

have a 1st integral?

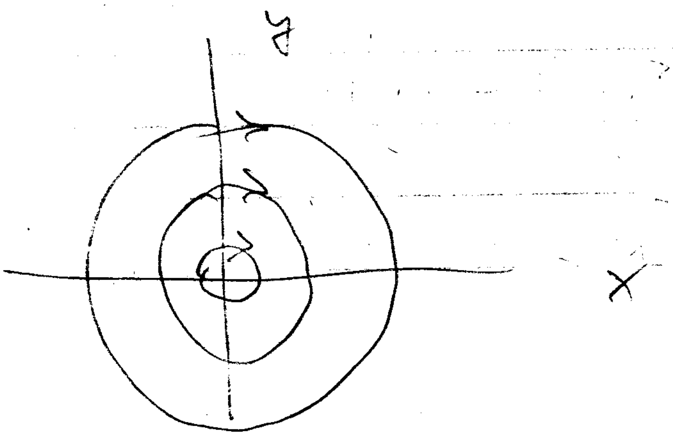
$F(x,y) = $ CONST

$F_x\dot{x} + F_y\dot{y} = F_x y + F_y kx = 0$

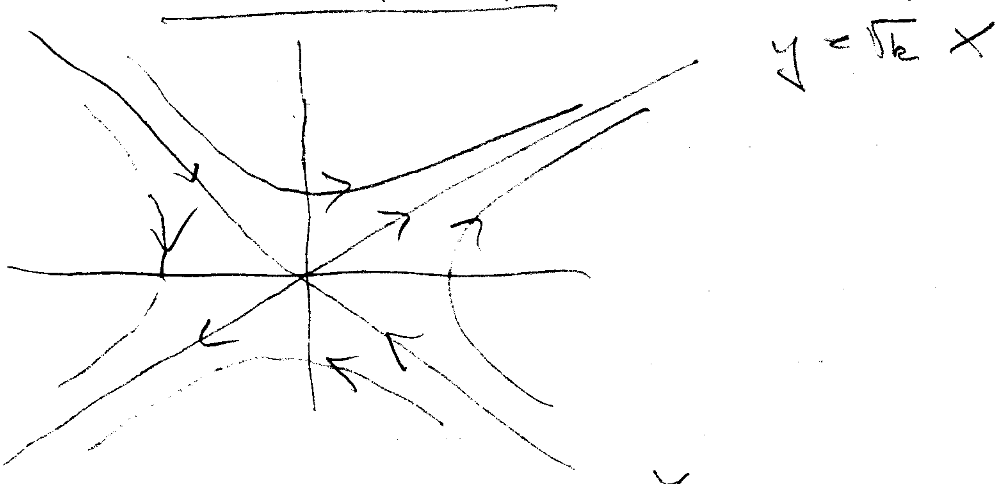
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{kx}{y} \Rightarrow y\,dy = kx\,dx$$

$$F(x,y) = \frac{1}{2}y^2 - \frac{1}{2}kx^2 = \text{CONST}, \quad \forall k$$

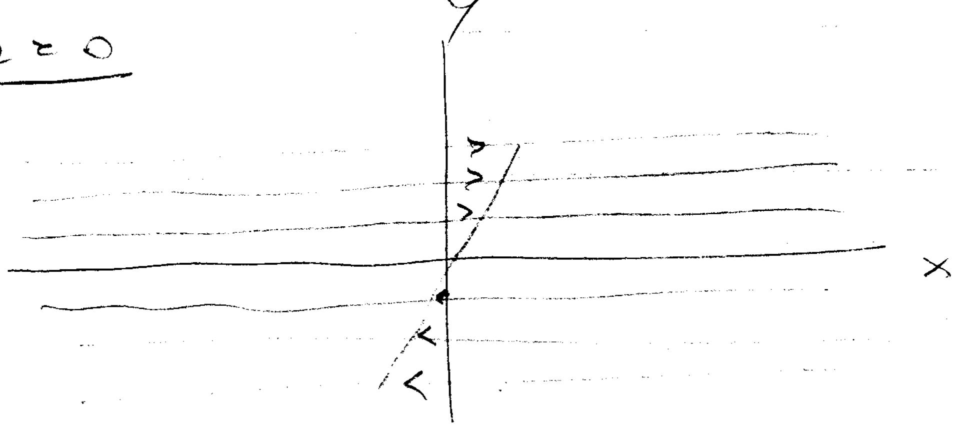
<u>$k<0$ center</u>



<u>$k>0$ saddle</u>



<u>$k=0$</u>



Conservative systems with one degree of freedom

(# of degrees of freedom = # of q's = # of (p,q)'s)

Newton's equation $\ddot{x} = f(x)$

$T = \frac{1}{2}\dot{x}^2$, $V(x) = -\int f(x)\,dx$

kinetic energy, potential energy

$L = \frac{1}{2}\dot{x}^2 - V(x)$ – Lagrangian

momentum $p = \dot{x}$

$H = \frac{p^2}{2} + V(x)$ – Hamiltonian

$\frac{dH}{dt}(x(t), p(t)) = 0$

$H = E = \text{const}$ – energy.

Hamiltons equations:

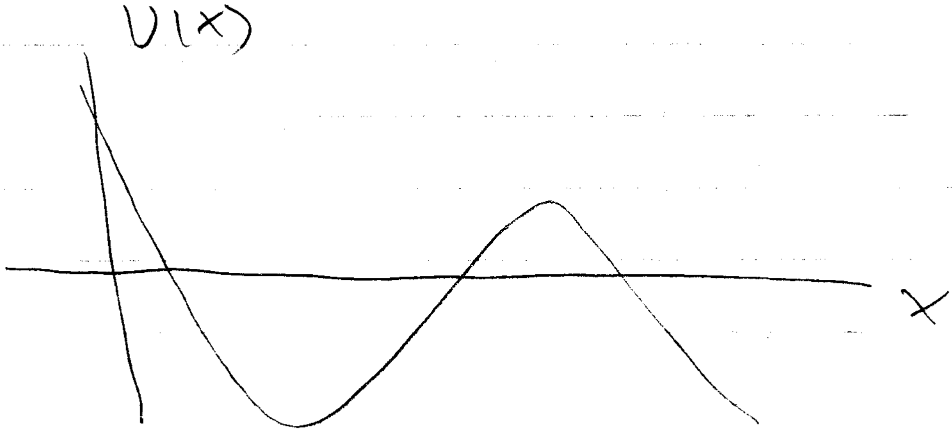
$\dot{x} = p \qquad \dot{p} = -V'(x) = f(x)$

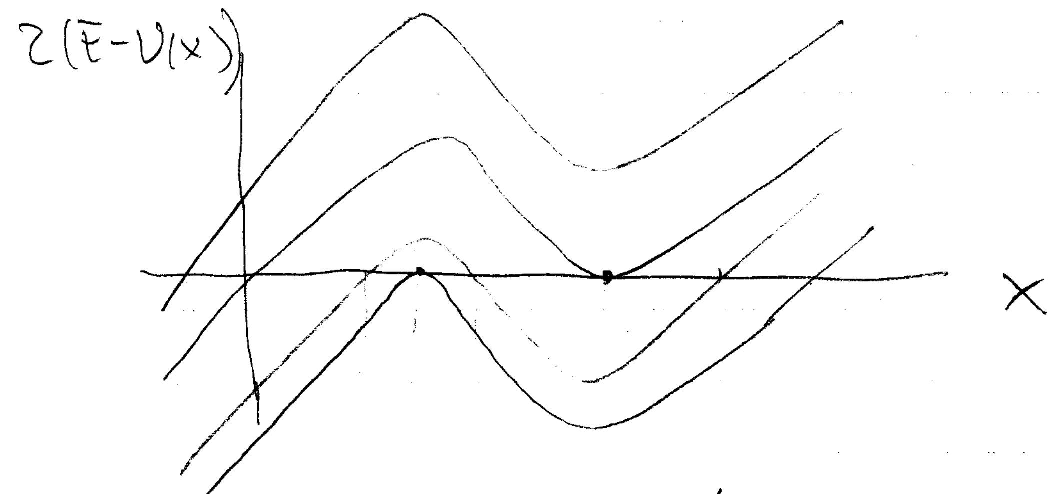
Prop Equilibria can only be on the x-axis and are extrema of $V(x)$, i.e. $V'(x) = 0$

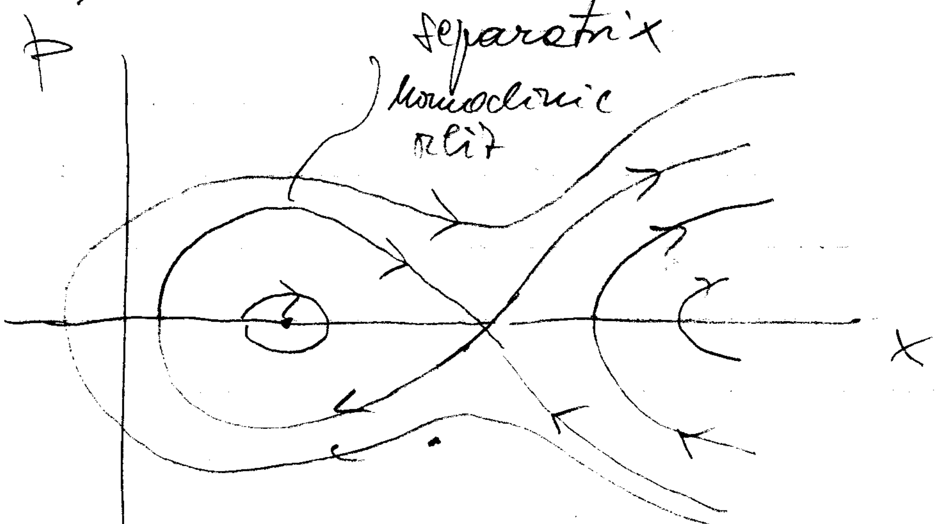
Trajectories are level curves of H:

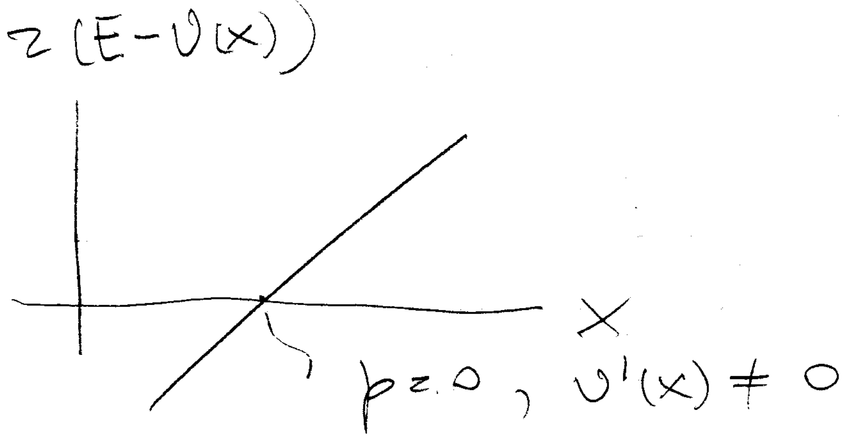
$$H(x,p) = E = \text{const}$$

$$p = \pm\sqrt{2(E - U(x))}$$

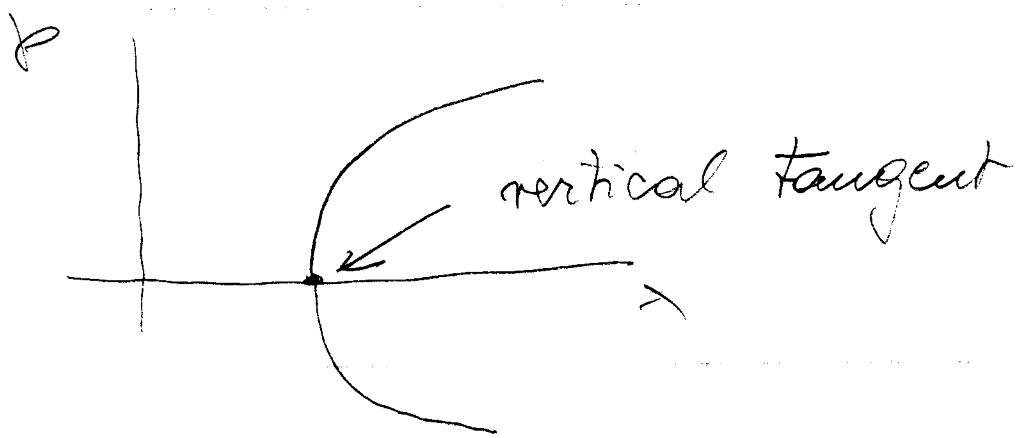




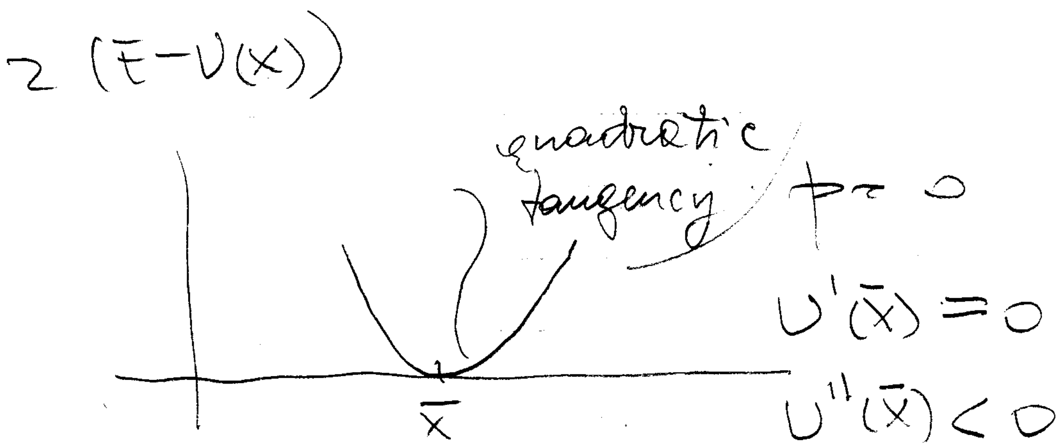




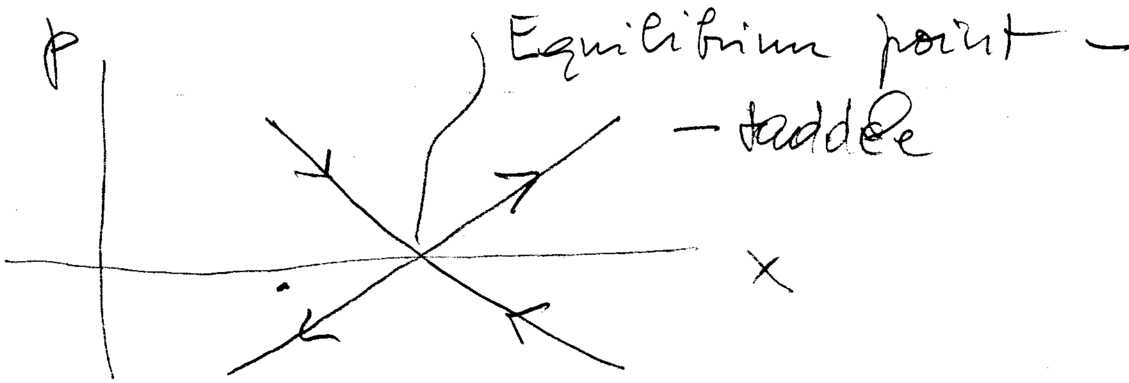
$2(E-V(x))$
x
$p=0, \ V'(x) \neq 0$



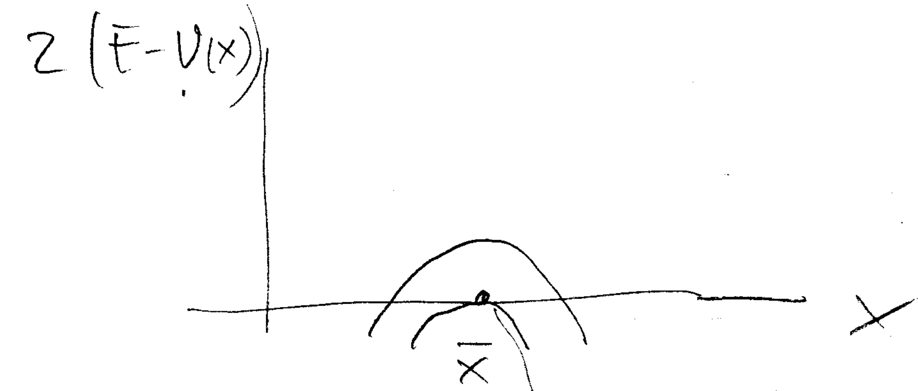
p
vertical tangent
x



$2(E-V(x))$
quadratic tangency
$p=0$
$V'(\bar{x})=0$
$V''(\bar{x})<0$
$\bar{x}$



p
Equilibrium point – saddle
x

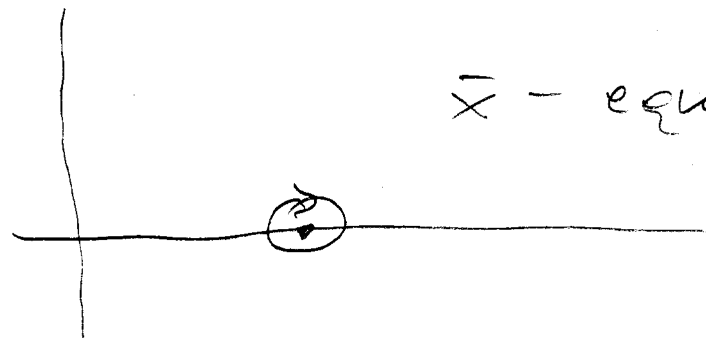


$p=0$

$U'(\bar{x})=0$

$U''(\bar{x})>0$

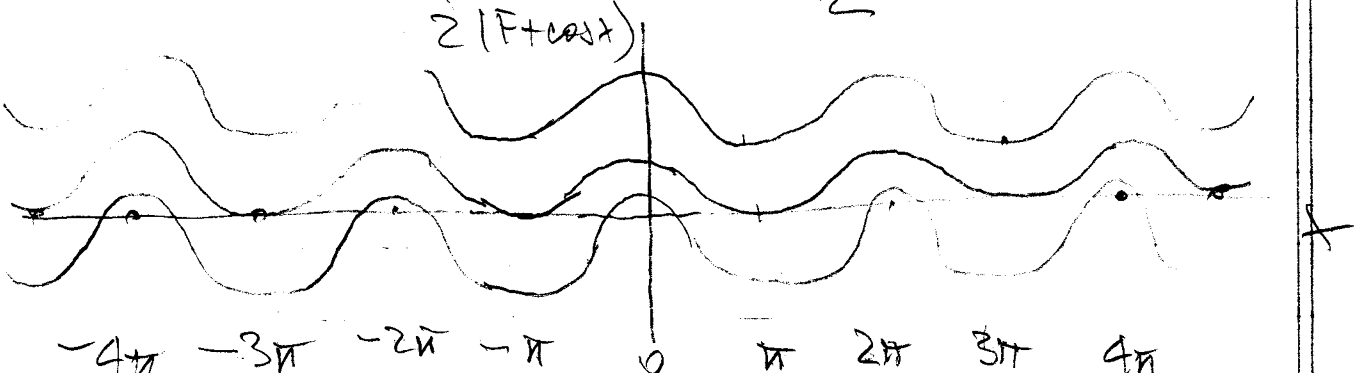
quadratic tangency

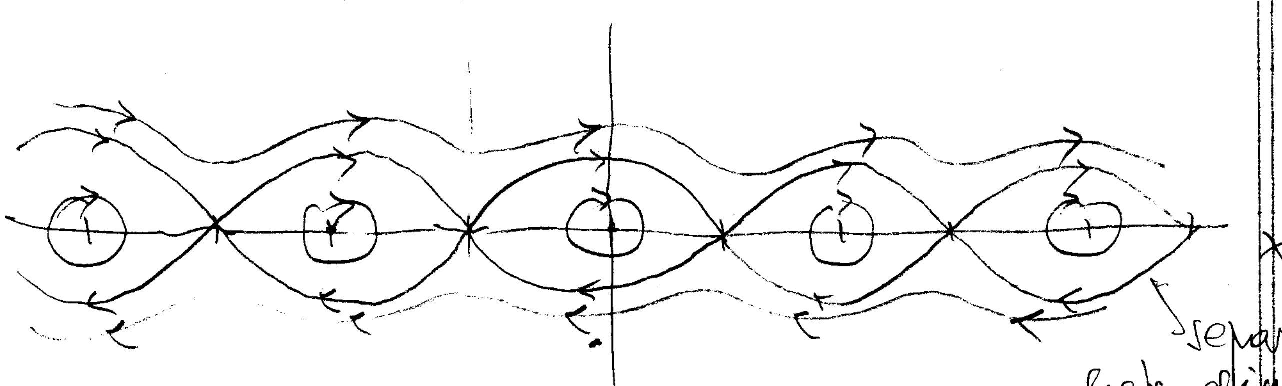


$\bar{x}$ – equilibrium point
center

EXAMPLE: PENDULUM $\ddot{x}+\sin x=0$

$H=\frac{p^2}{2}-\cos x$

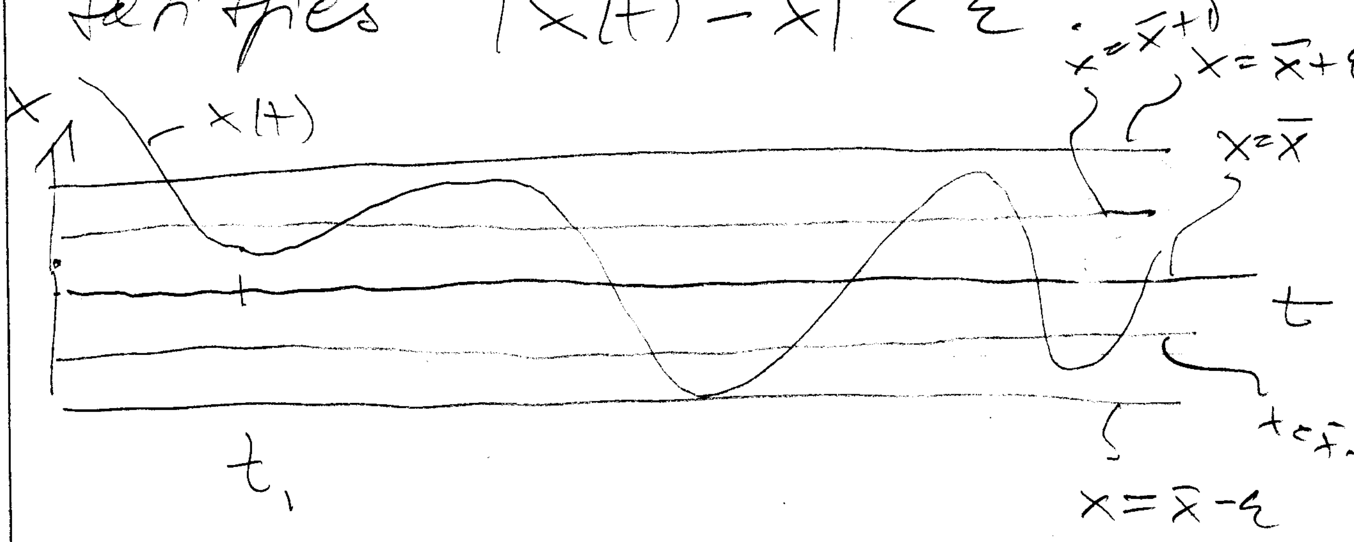




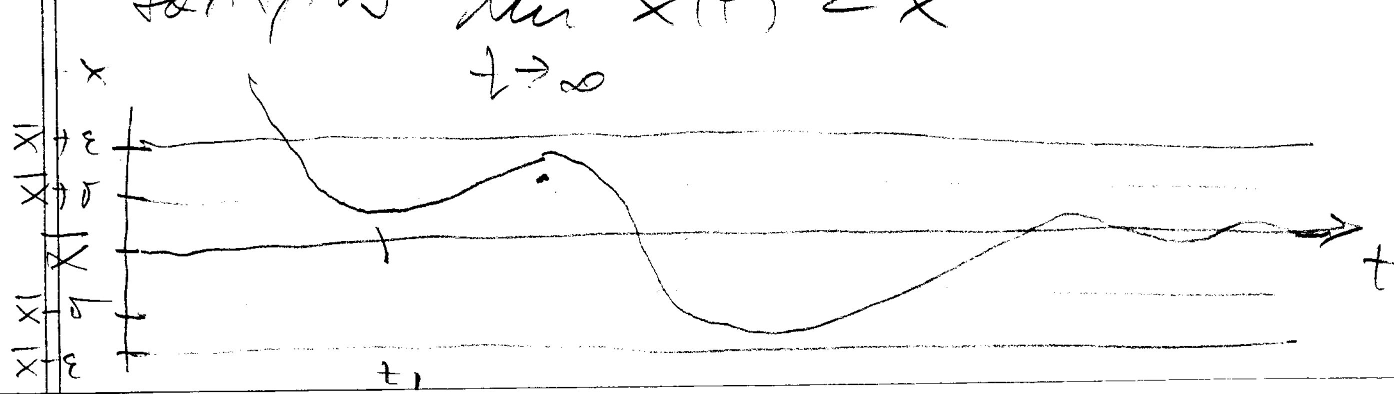
separatrix
heteroclinic orbit

STABILITY

An equilibrium point $\bar{x}$ of $\dot{x} = f(x)$ is *stable* if for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall$ solution $x(t)$ with $|x(t_1) - \bar{x}| < \delta$ for some t_1 exists for $\forall\, t \geq t_1$ and satisfies $|x(t) - \bar{x}| < \varepsilon$.



$\bar{x}$ is *asymptotically stable* if it is stable and $\forall\, x(t)$ s.t. $|x(t_1) - \bar{x}| < \delta$ for some t_1, satisfies $\lim_{t\to\infty} x(t) = \bar{x}$



$\bar{x}$ is <u>unstable</u> if it is not stable.

<u>EXAMPLES:</u>

2×2 systems $\dot{x} = Ax$, A - constant

asymptotically stable - sink, spiral sink

just stable - center

unstable - source, spiral source, saddle.

Definitions of stability and asymptotic stability for any other solution $\phi(t)$ of $\dot{x} = f(x)$ or even $\dot{x} = F(x,t)$ are very similar.

<u>Examples</u> 1.) $\dot{x} = -x + \sin t$

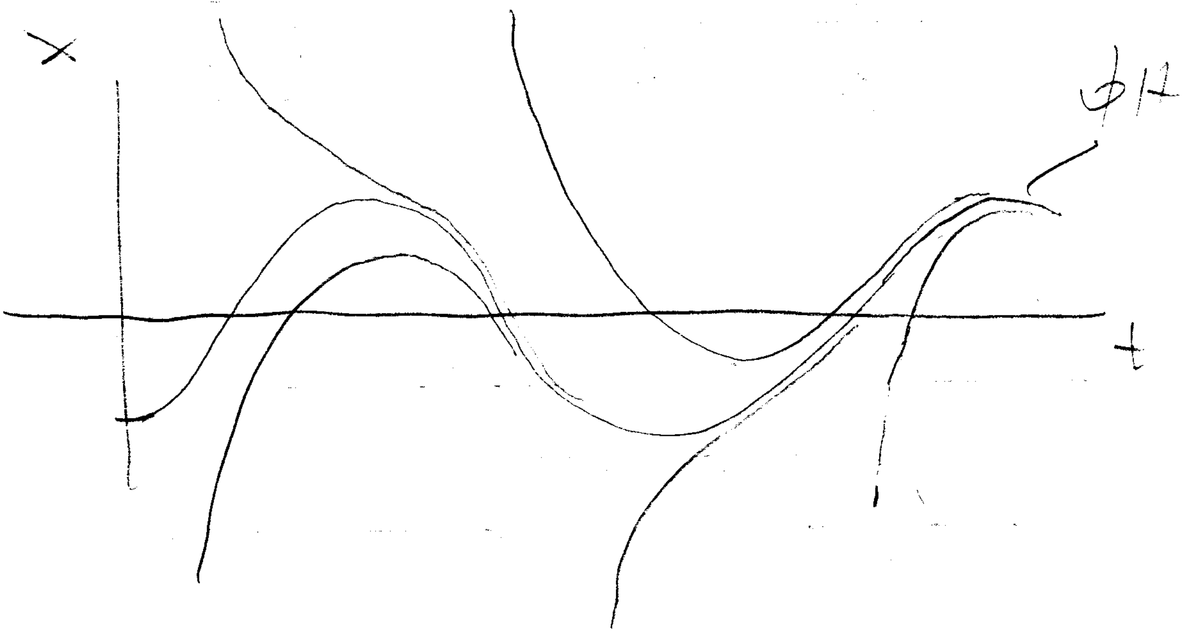
PERIODIC SOLUTION

$$\phi(t) = \frac{1}{2}(\sin t - \cos t) \tag{1}$$

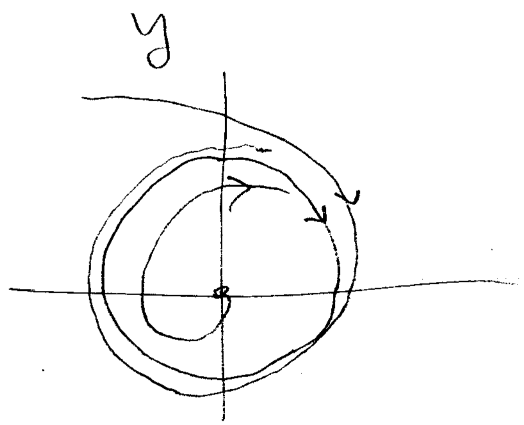
general solution

$$x(t,c) = \frac{1}{2}(\sin t - \cos t) + c\,e^{-t}$$

(1) is asymptotically stable.



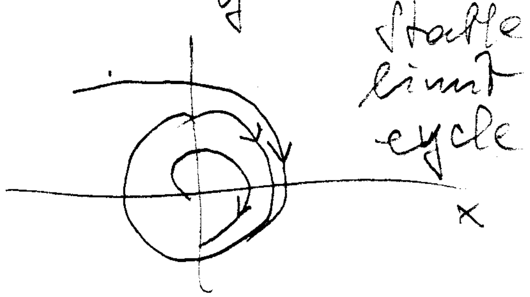
$\dot{r} = r(1-r^2) \qquad \dot{\theta} = -1$

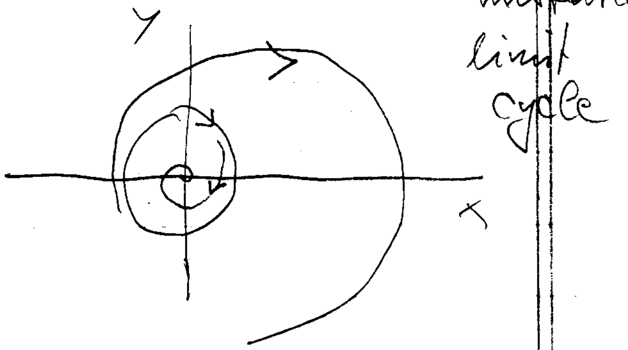


Asymptotically stable periodic orbit

<u>Def</u> A periodic orbit which is asymptotically stable for $\dot{x} = f(x)$ is called a <u>stable limit cycle</u>.

If it is asymptotically stable for $\dot{x} = -f(x)$ $(t \to -t)$ it is an <u>unstable limit cycle</u>.





EXAMPLE $\quad x = (x_1, x_2, x_3)$

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin t$$

Does this equation have a stable limit cycle?

Solution

Homogeneous system

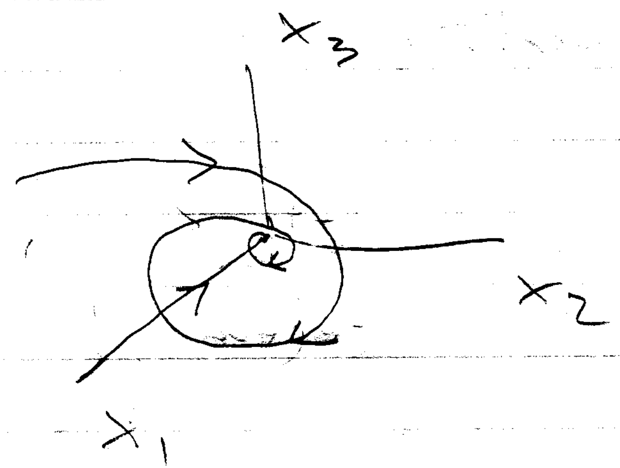
$$\det \begin{bmatrix} -2-\lambda & 0 & 0 \\ 0 & -1-\lambda & 2 \\ 0 & -2 & -1-\lambda \end{bmatrix} = -(2+\lambda)\left[(1+\lambda)^2 + 4\right] \neq 0$$

$$\lambda_1 = -2, \quad \lambda_{2,3} = -1 \pm 2i$$

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \xi_{2,3} = \begin{pmatrix} 0 \\ 1 \\ \pm i \end{pmatrix}$$

$$x_c(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} e^{-t}$$

$$+ c_3 \begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix} e^{-t}$$



$\sin t$ is not in resonance with $x_c(t)$, and only occurs in the 1st equation, so try

$$x_p(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (A \sin t + B \cos t)$$

$$\dot{x}_1 + 2x_1 = A \cos t - B \sin t + 2A \sin t + 2B \cos t = \sin t$$

$$A + 2B = 0 \Rightarrow A = -2B$$

$$2A - B = 1$$

$$-5B = 1 \Rightarrow A = -\tfrac{1}{5}, \quad B = \tfrac{2}{5}$$

Periodic solution

$$x_p(t) = \frac{1}{5}\begin{pmatrix} 2\cos t - \sin t \\ 0 \\ 0 \end{pmatrix}$$

General solution

$$x(t) = c_1 \begin{pmatrix} e^{-2t} \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-t}\cos t \\ -e^{-t}\sin t \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{-t}\sin t \\ e^{-t}\cos t \end{pmatrix}$$

$$+ \frac{1}{5}\begin{pmatrix} 2\cos t - \sin t \\ 0 \\ 0 \end{pmatrix}$$

$$x(t \to \infty) \longrightarrow \frac{1}{5}\begin{pmatrix} 2\cos t - \sin t \\ 0 \\ 0 \end{pmatrix} = x_p(t)$$

$\Rightarrow$ $x_p(t)$ is a stable limit cycle.

Linearization

Let $\phi(t)$ be a solution of $\dot{x} = f(x)$. Look for the behavior close to $\phi(t)$: $x(t) = \phi(t) + u(t)$

Assume $|u(t)| \ll 1$

$$\dot{\phi}(t) + \dot{u}(t) = f(\phi(t)) + \frac{\partial f}{\partial x}(\phi(t)) \cdot u(t) + O(|u|^2)$$

$$\Rightarrow \dot{u}(t) = \frac{\partial f}{\partial x}(\phi(t))\, u(t) + O(|u|^2)$$

<u>Linearization</u> of $\dot{x} = f(x)$ around $\phi(t)$ is

$$\dot{u}(t) = \frac{\partial f}{\partial x}(\phi(t)) \cdot u(t)$$

- linear equation, non-constant coefficients in general
- should help us find stability of $\phi(t)$.

Special case   Equilibrium point
$$\phi(t) = \bar{x}$$

Linearization
$$\dot{u} = \underbrace{\frac{\partial f}{\partial x}(\bar{x})}_{\text{constant matrix}} u$$

Many systems are of the form
$$\dot{x} = Ax + g(x),$$
$$g(x) = O(|x|^2)$$
$\left(g(x) = O(|x|^2) \text{ as } x \to 0 \text{ if } |g(x)| < C|x|^2 \right)$

Linearization $\quad \dot{u} = Au$

EXAMPLE: $\quad \dot{y} + y = \sin t$

$Y_p(t) = \frac{1}{2}(\sin t - \cos t)$

Linearization: $\quad \dot{u} = -u$

$u = k_0 e^{-t}$ – indicates stability of $Y_p(t)$

EXTENDED EXAMPLE

SMALL OSCILLATIONS

Let us have N particles with positions $\vec{r}_i$. Then

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\vec{r}}_i|^2$$

Replace $\vec{r} = (\vec{r}_1, \ldots, \vec{r}_N)$ by

$$q = (q_1, \ldots, q_n) : \vec{r} = \vec{r}(q)$$

$$\Rightarrow \quad \dot{\vec{r}} = \frac{\partial \vec{r}}{\partial q}(q) \cdot \dot{q}$$

$$\Rightarrow \sum_i^N m_i |\dot{\vec{r}}_i|^2 = \sum_{i,j=1}^{n} a_{ij}(q) \dot{q}_i \dot{q}_j$$

$$T = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(q) \dot{q}_i \dot{q}_j$$

If $V = V(q)$ then

$$L = \frac{1}{2} \sum_{i} a_{ij}(q) \dot{q}_i \dot{q}_j - V(q)$$

<u>THM</u>. The point $q = q_0$, $\dot{q} = \dot{q}_0$ will be an equilibrium of the mechanical system iff $\dot{q}_0 = 0$ and q_0 is a critical point of the potential energy, i.e.

$$\left.\frac{\partial U}{\partial q}\right|_{q_0} = 0$$

<u>PF</u> $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

$$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial q} - \frac{\partial U}{\partial q}$$

For an equilibrium, we must clearly have $\dot{q} = \dot{q}_0 = 0$.

Also then

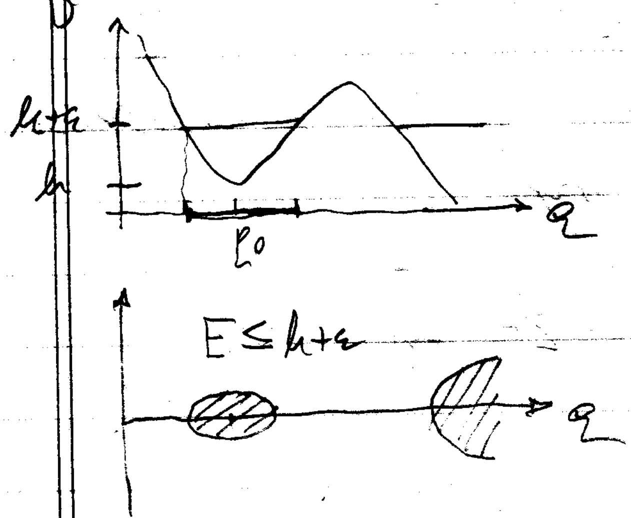
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = -\frac{\partial U}{\partial q} = 0 \quad \text{iff}$$

$$\left.\frac{\partial U}{\partial q}\right|_{q_0} = 0$$

Stability of equilibrium positions

THM. If the point q_0 is a strict local minimum of the potential energy U, then the equilibrium $q = q_0$ is stable.

PROOF: Let $U(q_0) = h$. For small enough $\varepsilon > 0$, the connected component of $\{q \mid U(q) \le h+\varepsilon\}$ containing q_0 will be an arbitrarily small neighborhood of q_0. Also, the component of the corresponding region in the phase space p, q $\{p, q \mid E(p,q) \le h+\varepsilon$



(here $p = \frac{\partial T}{\partial \dot q}$ is the momentum and $E = T+U$ is the total energy) will be an arbitrarily small neighborhood of $p=0$, $q = q_0$

But the region $\{p, q \mid E \le h+\varepsilon\}$ is invariant under the phase flow by conservation of energy. Therefore, for initial conditions $p(0)$ $q(0)$ close enough to $(0, q_0)$, every trajectory $(p(t), q(t))$ is close to $(0, q_0)$

Linearization of a Lagrangian system

Choose $q_0 = 0$ (no loss of generality)

THM. In order to linearize the Lagrangian system with

$$L = \frac{1}{2} \sum_{ij} a_{ij}(q)\dot{q}_i\dot{q}_j - U(q)$$

in a neighborhood of the equilibrium $q=0$, replace T by

$$T_2 = \frac{1}{2} \sum a_{ij}(0)\dot{q}_i\dot{q}_j$$

and U by

$$U_2 = \frac{1}{2} \sum \frac{\partial^2 U}{\partial q_i \partial q_j}(0)\, q_i q_j$$

Proof. Reduce the Lagrangian system to the form $\dot{x} = f(x)$ by using the canonical variables p and q

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p} \quad H(p,q) = T+U$$

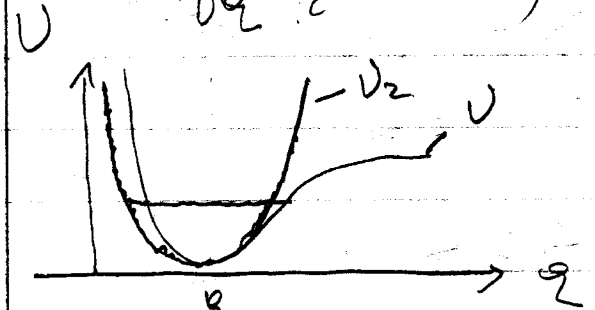
Since $p=q=0$ at the equilibrium, the Taylor expansions on the R.H.S begin by linear terms in p and q, which come from the quadratic terms H_2 of the expansion for $H(p,q)$. But $H_2 = T_2 + U_2 \Rightarrow$ corresponding Lagrangian is $L_2 = T_2 - U_2$, (b/c $H_2 = T_2(p) + U_2(q)$.)

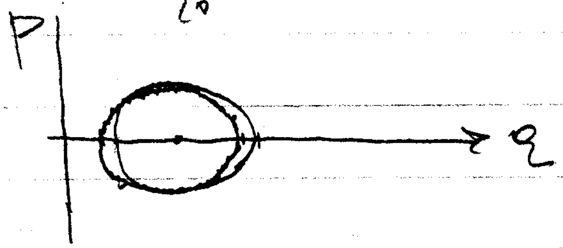
EXAMPLE One degree of freedom

$$T = \frac{1}{2} a(q)\dot{q}^2, \quad V = V(q)$$

Let $q = q_0$ be a stable equil. pt.:

$$\frac{\partial V}{\partial q}(q_0) = 0, \quad \frac{\partial^2 V}{\partial q^2}(q_0) > 0$$





Near q_0, for initial conditions close to $q = q_0$, solutions are periodic with period τ, which depends on the I.C.'s.

PROP $\tau \to \tau_0 = \frac{2\pi}{\omega_0}$, $\omega_0^2 = \frac{b}{a}$

$$b = \frac{\partial^2 V}{\partial q^2}(q_0), \quad a = a(q_0),$$

as the amplitudes of the oscillations decrease.

PROOF. Let $q_0 = 0$. $T_2 = \frac{1}{2} a\dot{q}^2$, $V_2 = \frac{1}{2} b q^2$

Lagrange's equation is

$$\ddot{q} = -\omega_0^2 q$$

solutions

$$q = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

period $\tau_0 = \frac{2\pi}{\omega_0}$. The rest follows from

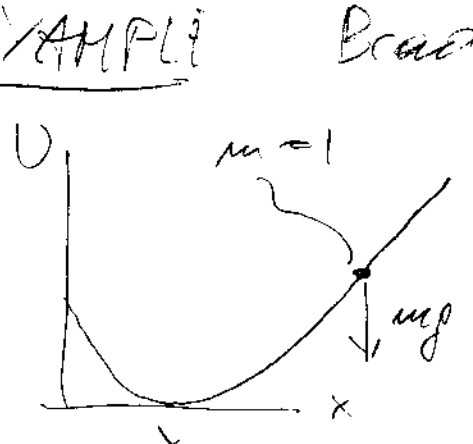
THM. Let $f(x) = Ax + \mathcal{O}(|x|^2)$ and consider $\dot{x} = f(x)$, $\dot{y} = Ax$. (Both have an equilibrium at $x = 0$.) Let $x(0) = y(0) = x_0$. Then for $\forall T > 0$ and $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $|x(0)| = |y(0)| < \delta$, then $|x(t) - y(t)| < \varepsilon$ for $\forall t$ with $0 < t < T$.

Small Oscillations

Motions in a linearized system with $L_2 = T_2 - V_2$ are called small oscillations near an equilibrium $q = q_0$.

For a one-dimensional problem
T_0 = period
ω_0 = frequency

EXAMPLE Bead on a wire:



(Let $g = 1$.)

$U = mgy = U(x)$

$T = \frac{1}{2} m v^2 = \frac{1}{2}\left[1 + \left(\frac{\partial U}{\partial x}\right)^2\right]\dot{x}^2$

$\Rightarrow$ Let $q = x - x_0$

$\Rightarrow T_2 = \frac{1}{2}\dot{q}^2$, $U_2 = \frac{1}{2}\omega^2 q^2$, $\omega^2 = \frac{\partial^2 U}{\partial x^2}(x_0)$

GENERAL PROBLEM OF SMALL OSCILLATIONS

Let $T = \frac{1}{2}(A\dot{q}, \dot{q})$, $U = \frac{1}{2}(Bq, q)$

$q, \dot{q} \in \mathbb{R}^n$. Let $T > 0$ for $\forall \dot{q}$
(positive definite)

Lagrange's equations are

$$A\ddot{q} + Bq = 0,$$

where A and B are symmetric and A is positive definite.

Assume $q(t) = e^{i\omega t}\xi$

$$\Rightarrow \quad (B - \omega^2 A)\xi = 0 \qquad (1)$$

From (1) we find n (not necessarily distinct) eigenvalues $\lambda_k = \omega_k^2$

From linear algebra, we get

<u>THM</u> There exist n linearly independent eigenvectors $\xi_1, \dots, \xi_n$. Moreover $(A\xi_j, \xi_k) = 0$ for $j \neq k$. Finally, all eigenvalues λ_k are real.

$$\Rightarrow \underline{q}(t) = \operatorname{Re} \sum_{k=1}^{n} \underbrace{c_k e^{i\omega_k t} \xi_k}_{\text{characteristic oscillations}}, \quad c_k \in \mathbb{C}$$

<u>Remark</u>: $\lambda_k = \omega_k^2 > 0 \Rightarrow \operatorname{Re}(c_k e^{i\omega_k t} \xi_k)$ contains $\cos \omega_k t$ and $\sin \omega_k t$ – oscillation

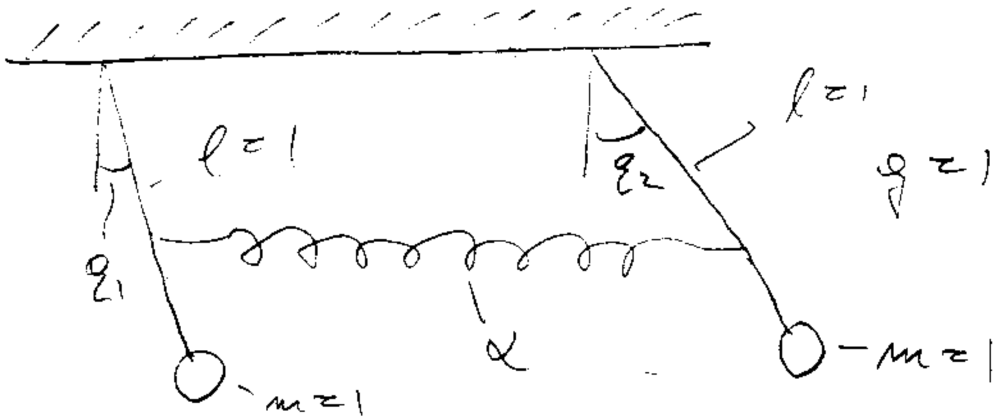
$\lambda_k = \omega_k^2 < 0 \Rightarrow \omega_k = i k_k$

$\operatorname{Re}(c_k e^{i\omega_k t} \xi_k)$ contains $e^{-k_k t}$, $e^{k_k t}$ – instability

$\lambda_k = 0 \Rightarrow \operatorname{Re}(c_k e^{i\omega_k t} \xi_k)$ must be replaced by

$\operatorname{Re}((c_{k1} + c_{k2} t) \xi_k)$; $c_{k1}, c_{k2} \in \mathbb{R}$

EXAMPLE



$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)$$

$$V = \frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 + \frac{1}{2}\alpha (q_1 - q_2)^2$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1+\alpha & -\alpha \\ -\alpha & 1+\alpha \end{bmatrix}$$

$$B - \lambda A = \begin{bmatrix} 1+\alpha-\lambda & -\alpha \\ -\alpha & 1+\alpha-\lambda \end{bmatrix}$$

$$\det(B-\lambda A) = (1+\alpha-\lambda)^2 - \alpha^2 = 0$$

$$1+\alpha-\lambda = \pm\alpha$$

$$\lambda_1 = 1, \quad \lambda_2 = 1+2\alpha$$

$$\omega_1 = 1 \qquad \omega_2 = \sqrt{1+2\alpha}$$

$\lambda_1 = 1$

$$(B-\lambda_1 A)\zeta = \begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = 0$$

$$\zeta^{(1)} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 + 2\alpha$$

$$(B - \lambda_2 A)\,\xi^{(2)} = \begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

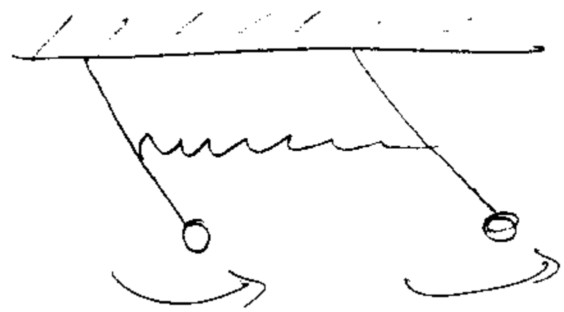
$$\xi^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = (A_1 \cos t + B_1 \sin t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_2 \cos\sqrt{1+2\alpha}\, t + B_2 \sin\sqrt{1+2\alpha}\, t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Characteristic oscillations

1.) $A_2 = B_2 = 0$

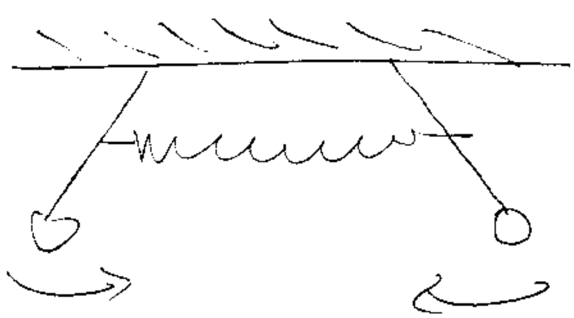
spring has no effect



in-phase oscillations $\omega = 1$

2.) $A_1 = B_1 = 0$

spring makes $\omega_2 > 1$.



opposite phase $\omega_2 > 1$

BEATS Let $\alpha \ll 1$

Let $q_1(0) = q_2(0) = 0$

$\dot{q}_1(0) = v, \quad \dot{q}_2(0) = 0$

$$\Rightarrow \left.\begin{array}{l} A_1 + A_2 = 0 \\ A_1 - A_2 = 0 \end{array}\right\} \quad A_1 = A_2 = 0$$

$$\left.\begin{array}{l} B_1 + \omega B_2 = v \\ B_1 - \omega B_2 = 0 \end{array}\right\} \quad \begin{array}{l} B_2 = \frac{v}{2\omega} \\ B_1 = \frac{v}{2} \end{array}$$

$$q(t) = \frac{v}{2}\begin{pmatrix} \sin t + \frac{1}{\omega}\sin\omega t \\ \sin t - \frac{1}{\omega}\sin\omega t \end{pmatrix} = \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

Since $\alpha \ll 1$, $\omega = \sqrt{1+2\alpha} \approx 1+\alpha$

and neglecting $v\left(1-\frac{1}{\omega}\right)\sin\omega t \approx v\alpha\sin\omega t$:

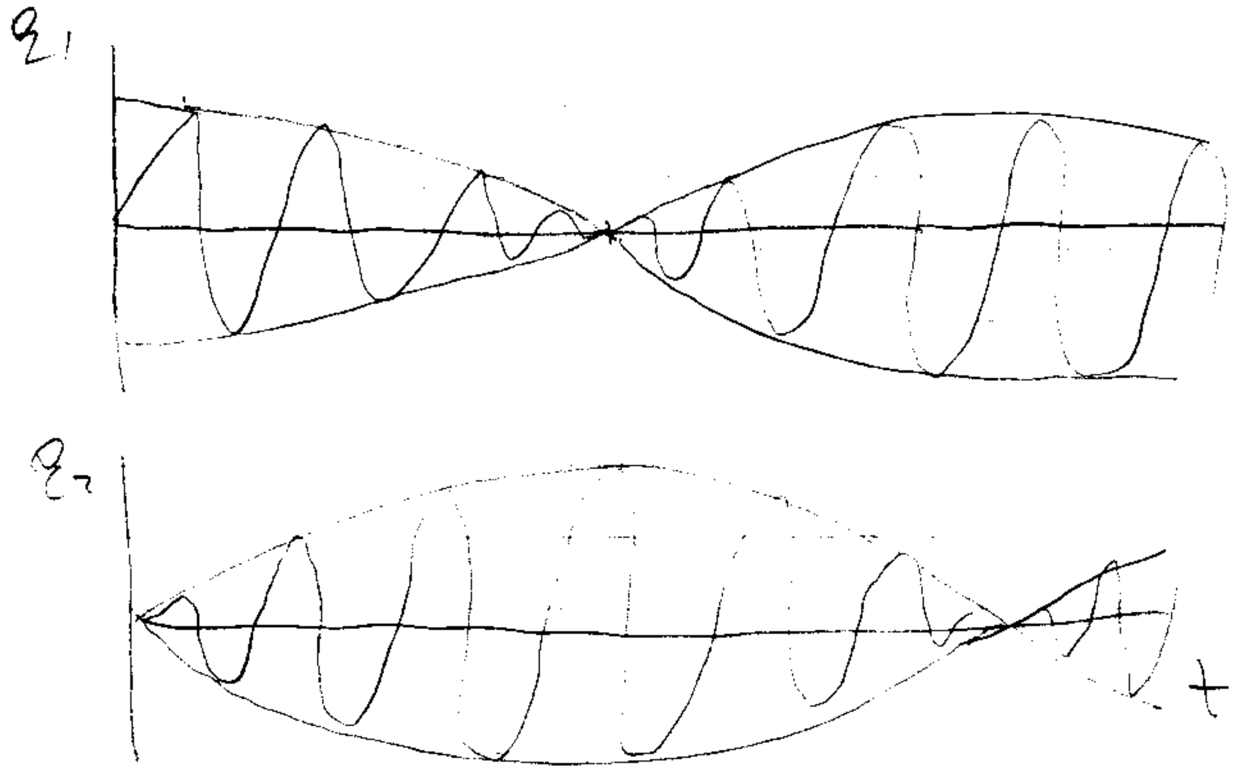
$$q_1(t) \approx \frac{v}{2}(\sin t + \sin\omega t) = v\cos\varepsilon t \sin\omega' t$$

$$q_2(t) \approx \frac{v}{2}(\sin t - \sin\omega t) = -v\cos\omega' t \sin\varepsilon t$$

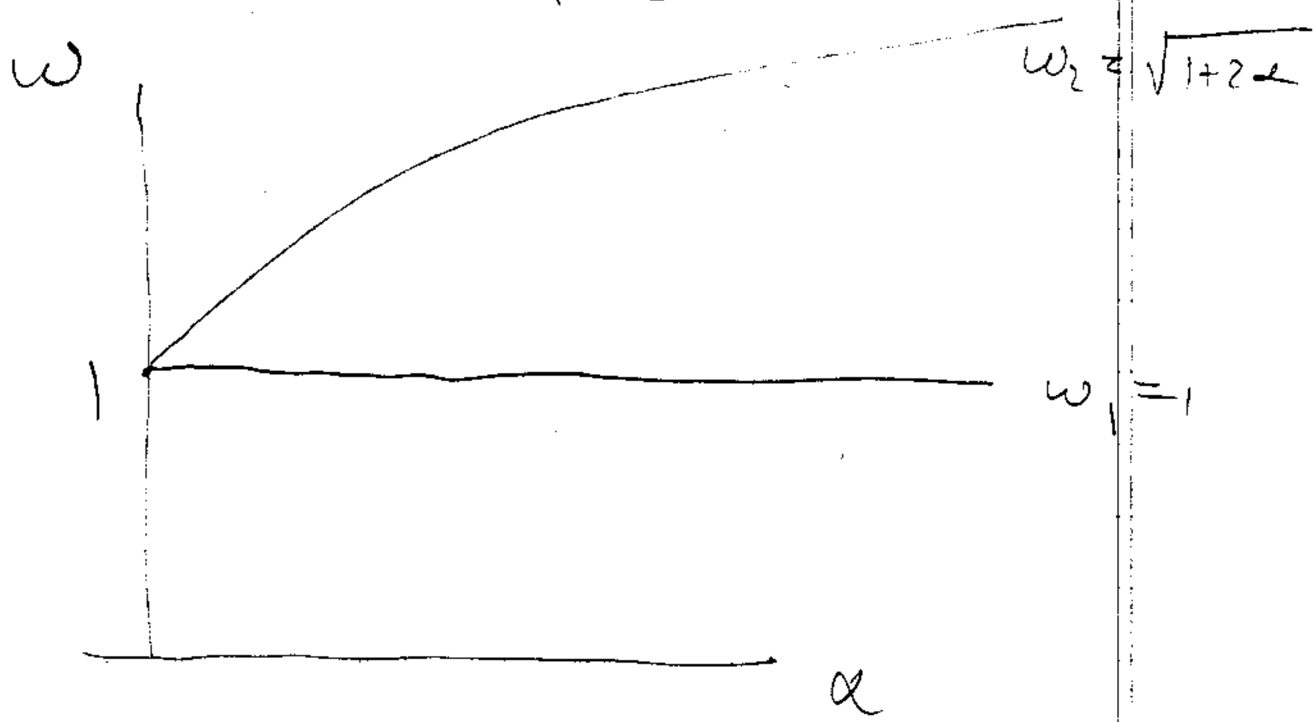
$$\varepsilon = \frac{\omega-1}{2} \approx \frac{\alpha}{2}, \quad \omega' = \frac{\omega+1}{2} \approx 1$$

$q_1(t)$ – fast oscillations with slowly-varying amplitude $v\cos\varepsilon t$

$q_2(t)$ – — " — $v\sin\varepsilon t$



Dependence of frequencies on α



Phase space geometry of two-degree-of-freedom small oscillations

$$\ddot{q}_1 + \omega_1^2 q_1 = 0$$
$$\ddot{q}_2 + \omega_2^2 q_2 = 0$$

$$\dot{q}_1 = \omega_1 p_1 \quad \dot{p}_1 = -\omega_1 q_1 \quad \text{in } \mathbb{R}^2_1$$
$$\dot{q}_2 = \omega_2 p_2 \quad \dot{p}_2 = -\omega_2 q_2 \quad \text{in } \mathbb{R}^2_2 \qquad (1)$$

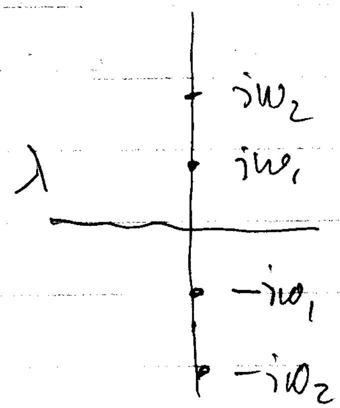
Trajectories in $\mathbb{R}^2_j$, $j=1,2$ are circles:

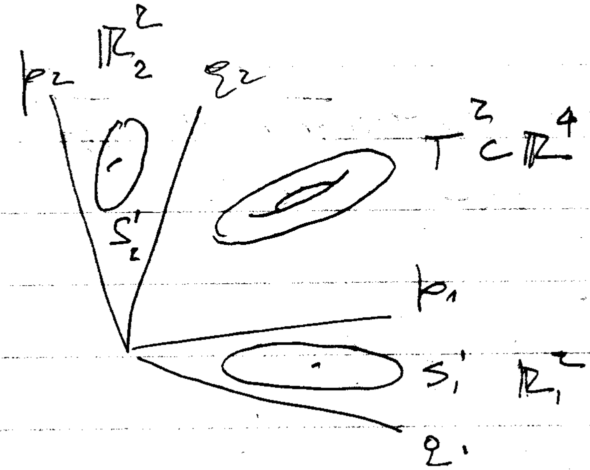
$$S^1_j = \{(q_j, p_j) \in \mathbb{R}^2_j \mid q_j^2 + p_j^2 = c_j > 0\}$$

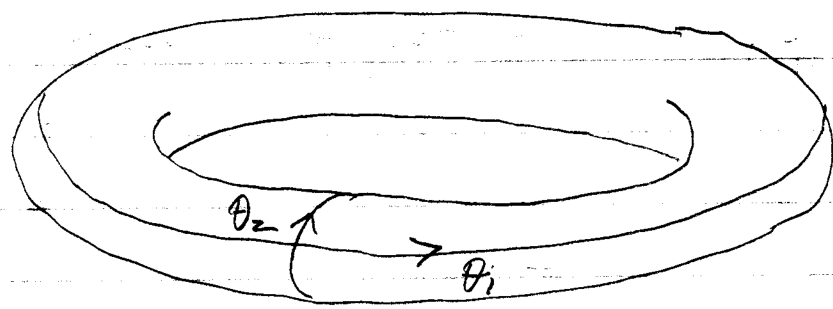
Phase flows: rotations through angles $\omega_1 t$, $\omega_2 t$.

In $\mathbb{R}^4$, every trajectory lies on a two-dimensional torus, T^2

$$T^2 = S^1 \times S^1 = \{(q_1, p_1, q_2, p_2) \in \mathbb{R}^4 \mid q_1^2 + p_1^2 = c_1,\ q_2^2 + p_2^2 = c_2\}$$



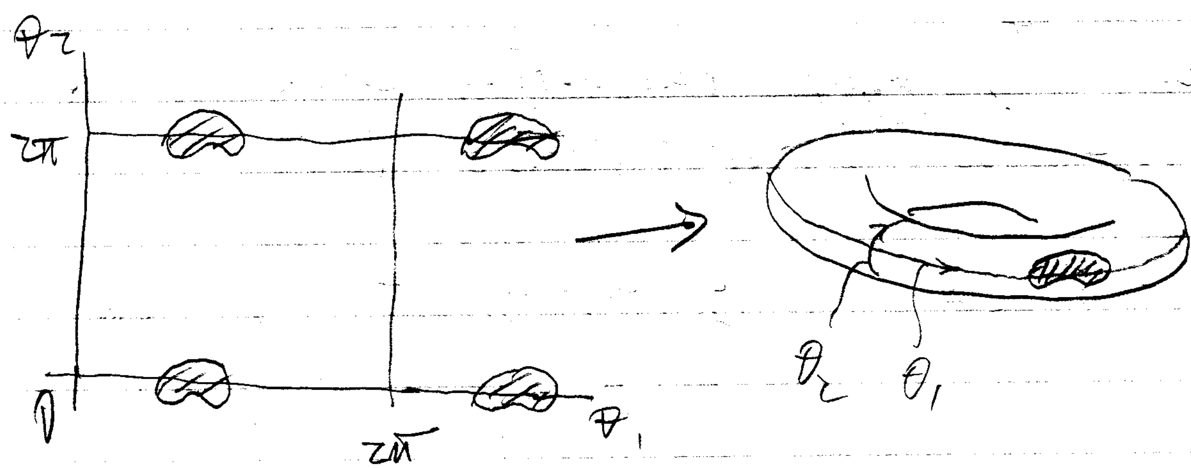


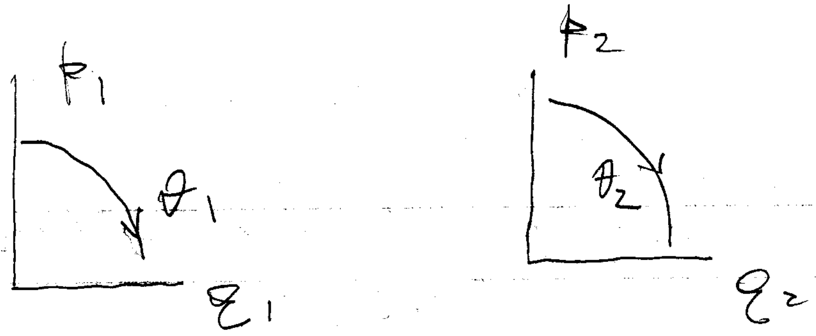


$\theta_1 \bmod 2\pi$ – longitude

$\theta_2 \bmod 2\pi$ – latitude

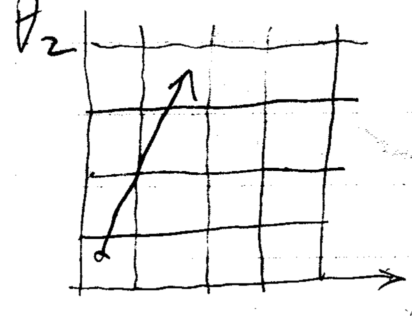
Square $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_2 \leq 2\pi$ – map of T^2

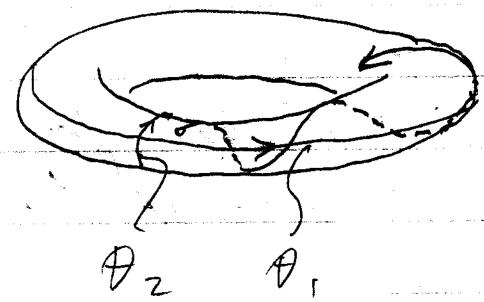




$$\underbrace{\dot{\theta}_1 = \omega_1 \qquad \dot{\theta}_2 = \omega_2}$$

The differential equation for the trajectories of the flow (1) on T^2 — "winding" around the torus





straight line — orbit

Orbits of $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ on the torus

ω_1, ω_2 are rationally independent

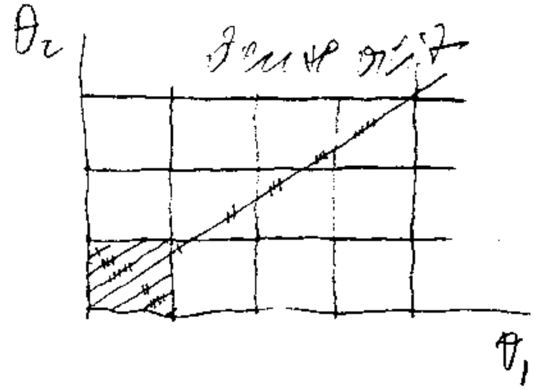
if $k_1\omega_1 + k_2\omega_2 = 0$, $k_1, k_2 \in \mathbb{Z}$ implies $k_1 = k_2 = 0$.

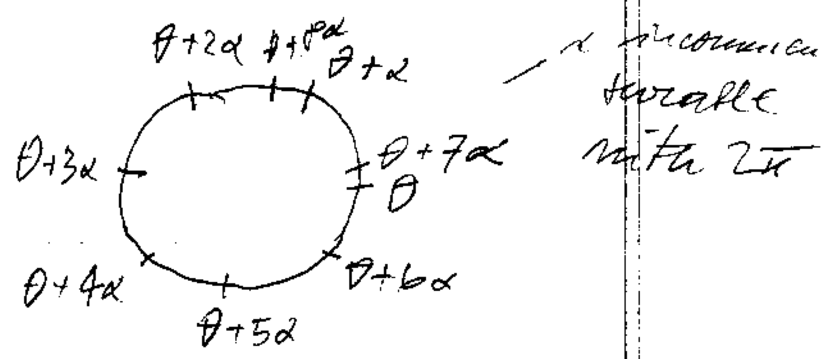
EXAMPLE $4\sqrt{2} + \sqrt{2} \neq 0$

$\sqrt{6}$ and $\sqrt{2}$ are <u>rationally indep.</u>

<u>THM</u> If ω_1 and ω_2 are rationally dependent, then every trajectory of equation $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ on T^2 is closed. If ω_1 and ω_2 are rationally independent, then every trajectory of this equation is everywhere dense on T_2.

($A \subset B$ is everywhere dense if there is at least one point of A in every nbhd. of every pt. of A.)





<u>Lemma</u> Let α be incommensurable with 2π. Then θ, $\theta+\alpha$, $\theta+2\alpha$, $\theta+3\alpha$, $\theta+4\alpha$, ... (mod 2π) are dense on the circle

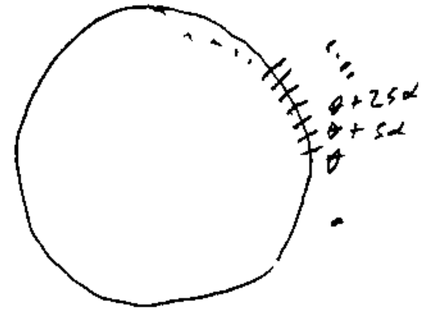
Proof of lemma:

Pigeonhole principle: If $k+1$ objects are placed in k cells, there should be two objects in at least one cell.

Divide S^1 into k half-open intervals of length $\frac{2\pi}{k}$. Among $\theta + j\alpha$, $j = 0, 1, \dots, k$, $\exists\, p > q$ s.t. $\theta + p\alpha$ and $\theta + q\alpha$ are in the same interval. Let $s = p - q$

$\Rightarrow$ $s\alpha \bmod 2\pi < \frac{2\pi}{k}$, and any two consecutive pts. in θ, $\theta + s\alpha$, $\theta + 2s\alpha$, $\theta + 3s\alpha$, ... are the same distance $d < \frac{2\pi}{k}$ apart

$\Rightarrow$ $\forall\, \varepsilon$ nbhd. of $\forall$ pt. of S^1 contains points of $\{\theta + ms\alpha\}$ if we make $k > \frac{2\pi}{\varepsilon}$.



PROOF OF THE THEOREM: Solution of $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ is

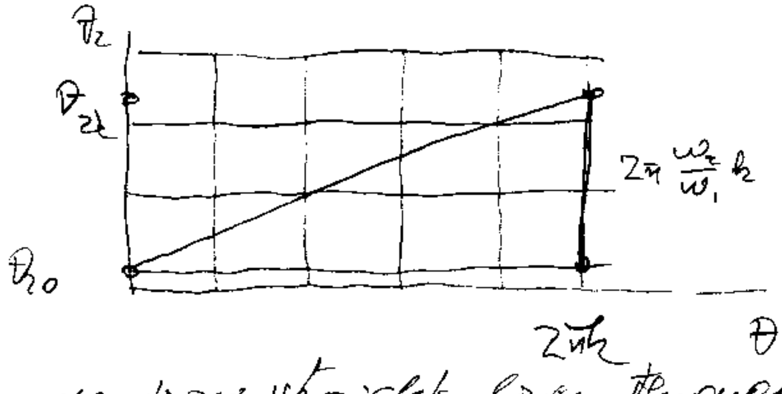
(*) $$\theta_1(t) = \theta_1(0) + \omega_1 t, \quad \theta_2(t) = \theta_2(0) + \omega_2 t$$

Let $k_1\omega_1 + k_2\omega_2 = 0$ for $k_1, k_2 \in \mathbb{Z}$, $k_1^2 + k_2^2 > 0$

Then equations $\omega_1 T = 2\pi k_2$, $\omega_2 T = -2\pi k_1$ are compatible & T is the period of a periodic orbit.

Let ω_1 & ω_2 be rationally indep. $\Rightarrow$ $\Rightarrow \frac{\omega_1}{\omega_2}$ is irrational. Consider consecutive intersection pts. of (*) with $\theta_1 = 0 \mod 2\pi$. Their latitudes are

$$\theta_{2k} = \theta_{20} + 2\pi \frac{\omega_2}{\omega_1} k \mod 2\pi.$$



By lemma, they are dense on $\theta_1 = 0 \mod 2\pi$.

If L is a straight line in $\mathbb{R}^2$ and if we draw straight lines through a set everywhere dense in L in a direction different from the direction of L $\Rightarrow$ these lines are dense in $\mathbb{R}^2$. $\Rightarrow$ Let $[x]$ = integer part of x. Then the image $\hat{\theta}_j(t) = \theta_j(t) - 2\pi\left[\frac{\theta_j(t)}{2\pi}\right]$ $j = 1, 2$ of (*) on the square $0 \le \hat{\theta}_j < 2\pi$, $j = 1, 2$ is everywhere dense $\Rightarrow$ Trajectories of $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$ and so of $\dot{q}_j = \omega_j p_j$, $\dot{p}_j = -\omega_j q_j$, $j = 1, 2$ are dense on T^2.

STABILITY AND LINEARIZATION

THM Let $\dot{x} = Ax + f(x)$ (*)

where $A \in \mathbb{R}^{n\times n}$ is constant, and all the eigenvalues of A have nonzero real parts.

Let $f \in C(|x|<R)$ for some R, f - real and

$$\frac{|f(x)|}{|x|} \to 0 \quad \text{as } |x| \to 0$$

Then the identically zero solution of (*) is stable.

LEMMA (The Grönwall Inequality)

Let $\varphi, \psi, \chi : [a,b] \to \mathbb{R}$ be continuous, $\chi(t) > 0$ on $[a,b]$, and let

(1) $\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\varphi(s)\,ds$, for $\forall t \in [a,b]$

Then on $[a,b]$

(2) $\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s)\, e^{\int_s^t \chi(u)\,du}\, ds$

PROOF: Let $R(t) = \int_a^t \chi(s)\varphi(s)\,ds$. (3)

Then

$$R' - \chi R = \chi(t)\varphi(t) - \chi(t)\int_a^t \chi(s)\varphi(s)\,ds$$

$$\leq \chi(t)\psi(t) \qquad \text{by (1)}$$

By variation of constants, multiply by $e^{-\int_a^t \chi(u)\,du}$ and integrate

$$\left[R(s)\, e^{-\int_a^s \chi(u)\,du}\right]\Big|_a^t \leq \int_a^t \chi(s)\psi(s)\, e^{-\int_a^s \chi(u)\,du}\,ds$$

Now, $R(a) = 0$ by (3), so

$$R(t)\, e^{-\int_a^t \chi(u)\,du} \leq \int_a^t \chi(s)\psi(s)\, e^{-\int_a^s \chi(u)\,du}\,ds$$

$$R(t) \leq \int_a^t \chi(s)\psi(s)\, e^{\int_s^t \chi(u)\,du}\,ds$$

By (1) and (3)

$$\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s)\, e^{\int_s^t \chi(u)\,du}\,ds,$$

which is (2).

PROOF of the THM: The solution φ of

$$\dot{x} = Ax + f(x) \qquad (*)$$

with $|\varphi(0)|$ small can be continued for increasing t as long as $|\varphi(t)|$ remains small.

Use variation of constants and integrate, to conclude that so long as $\varphi(t)$ exists,

$$(\bigstar) \quad \varphi(t) = e^{At}\varphi(0) + \int_0^t e^{(t-s)A} f(\varphi(s))\,ds.$$

Because the real parts of all the eigenvalues of A are negative, it follows that

$$(\square) \quad \|e^{At}\| \leq K e^{-\sigma t}$$

for some $K, \sigma > 0$, $\forall t \geq 0$.

By $(\square)$, $(\bigstar)$ gives

$$|\varphi(t)| \leq K|\varphi(0)|e^{-\sigma t} + K\int_0^t e^{-\sigma(t-s)} |f(\varphi(s))|\,ds$$

Now for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $|x| \leq \delta$ then $|f(x)| < \dfrac{\varepsilon|x|}{2K}$.

So, as long as $|\varphi(t)| \leq \delta$, it follows that

$$e^{\delta t}|\varphi(t)| \leq K|\varphi(0)| + \varepsilon \int_0^t e^{\delta s}|\varphi(s)|\, ds$$

<u>Replace</u> $\varphi \to e^{\delta t}|\varphi(t)|$

$\psi \to K|\varphi(0)|$

$\chi \to \varepsilon$, $a = 0$

in the Gronwall inequality to get

$$e^{\delta t}|\varphi(t)| \leq K|\varphi(0)|\left(1 + \int_0^t \varepsilon\, e^{\varepsilon\int_s^t du}\, ds\right)$$

$$= K|\varphi(0)|\left(1 + \varepsilon\int_0^t e^{\varepsilon(t-s)}\, ds\right) =$$

$$= K|\varphi(0)|\left(1 - e^{\varepsilon t}\left(e^{-\varepsilon s}\right)\Big|_0^t\right)$$

$$= K|\varphi(0)|\, e^{\varepsilon t}$$

$$\Rightarrow |\varphi(t)| \leq K|\varphi(0)|\, e^{-(\delta-\varepsilon)t} \qquad (\Delta)$$

If $\varepsilon < \delta$, then $|\varphi(t)| < K|\varphi(0)|$ so long as $|\varphi(t)| < \delta$. Thus, if $|\varphi(0)| < \frac{\delta}{K}$, (Δ) is valid for $\forall t \geq 0$, and 0 is asymptotically stable!