Trigonometric Series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

How do you compute \( a_n \) and \( b_n \)?

For the moment, cast aside convergence issues.

[Focus on:]

\[ \int_{-\pi}^{\pi} \sin nx \, dx = 0 \]

\[ \int_{-\pi}^{\pi} \cos nx \, dx = 0 \]

\[ \int_{-\pi}^{\pi} \sin nx \cos nx \, dx = \frac{\pi}{n} \quad \text{if } m = n \]

\[ \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad \text{if } m \neq n \]
\[
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \neq 0 \\ \pi & \text{if } m = 0, n = 0 \end{cases}
\]

(Orthogonality of trigonometric functions)

Proof of the last: \[
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{n} \left[ \sin (m+n)x + \sin (m-n)x \right] \, dx \bigg|_{-\pi}^{\pi} = 0
\]

\[\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx\]

\[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx\]

\[b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx\]
Def: \( f(x) \) is periodic with period \( T \) if \( f(x + T) = f(x) \) for all \( x \in \mathbb{R} \).

Thm: For any \( a, b, c, d \in \mathbb{R} \),
\[
\int_a^b f(x) dx = \int_c^d f(x) dx
\]
if \( f \) is periodic with period \( T \).

Let \( f(x) \) be periodic with period \( 2\pi \).
Integrable. Define
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt
\]
and
\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt
\]
If \( f(x) \) is piecewise continuous (continuous except at finitely many discrete points) within one period, where \( f \) has finite jumps) and \( f(x) \) is piecewise continuous.

**THEOREM** If \( f(x) \) is piecewise continuous

Then

\[
\lim_{n \to \infty} \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)
\]

converges pointwise to

\[
\frac{1}{2} \left( f(x^+) + f(x^-) \right).
\]

In particular, if converges
to \( f(x) \) where \( f(x) \) is continuous.
Proof

Let

\[ S_n(x) = \frac{a_0}{2} + \sum_{m=1}^{N} \left( a_m \cos mx + b_m \sin mx \right) \]

Then

\[ S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin \frac{1}{2} x}{\sin \frac{1}{2} t} \sum_{m=1}^{N} \left( \cos mx \sin mx \right) dt \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{1}{2} \cos \frac{1}{2} x \right) \sin \left( \frac{1}{2} t \right) \sum_{m=1}^{N} \left( \cos mx \sin mx \right) dt \]
**Lemma 1** \[ \frac{1}{2} + \sum_{m=1}^{N} \cos m(t-x) = \] \[ \frac{\sin \left( N \frac{1}{2} \right) \left( \frac{1}{2} \right)}{\sin \left( \frac{1}{2} \right)} \]

**Proof:** \[ \frac{1}{2} + \sum_{m=1}^{N} \cos m \alpha = \] \[ \frac{1}{2} \sum_{m=-N}^{N} e^{i m \alpha} = \] \[ \frac{1}{2} e^{i N \alpha} \left( 1 + e^{i \alpha} + \ldots + e^{i N \alpha} \right) = \] \[ \frac{1}{2} e^{i N \alpha} \frac{1 - e^{i (N+1) \alpha}}{1 - e^{i \alpha}} = \] \[ \frac{1}{2} e^{i \frac{1}{2} \alpha} \left( e^{i (N+\frac{1}{2}) \alpha} - e^{-i (N+\frac{1}{2}) \alpha} \right) = \] \[ \frac{1}{2} \sin \left( N + \frac{1}{2} \right) \alpha \]
\[ S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\tau)}{\sin \left( \frac{\tau}{2} \right)} \sin \left( \frac{\tau}{2} (t - x) \right) d\tau \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\tau)}{\sin \left( \frac{\tau}{2} \right)} \sin \left( \frac{\tau}{2} x \right) d\tau \]

We must show:

\[ S_n(x) \rightarrow f(x) \quad \text{as} \quad n \rightarrow \infty \]

Let \( M \geq 2 \) \( \forall x \) as piecewise continuous on \( a \leq x \leq b \).

Then \( \int_{-\pi}^{\pi} f(x) \sin \left( \frac{\tau}{2} x \right) d\tau = 0 \), \( \lambda \rightarrow \infty \).

**Proof:** Can assume \( f(x) \) is continuous. Let \( |f(x)| \leq M \).

\[ S_a^b \sin \left( \frac{\tau}{2} x \right) dx = \left[ S_a^\frac{b-\pi}{2} + S_{\frac{b-\pi}{2}}^b \right] \sin \left( \frac{\tau}{2} x \right) dx \]

Let \( \frac{b-\pi}{2} = x + \frac{\pi}{2} \)

\[ \int_{a}^{b} \frac{b-\pi}{2} \sin \left( \frac{\tau}{2} x \right) dx = -\int_{a+\frac{\pi}{2}}^{b+\frac{\pi}{2}} \sin \left( \frac{\tau}{2} x \right) dx \]
\[ \Rightarrow \int_a^b \int_a^b [S(x) - S(x - \frac{\pi}{x})] \sin x \, dx \, dx \]

If \( x \) is large enough, by continuity
\[
|S(x) - S(x - \frac{\pi}{x})| < \epsilon \Rightarrow |\frac{1}{x} \int \frac{1}{a} \int_a^b \sin x \, dx| < \epsilon (b-a) + \frac{2\pi M}{x} \xrightarrow{x \to \infty} 0.
\]

If \( S(x) \) is piecewise continuous
\[ a = x_0 < x_1 < x_2 < \ldots < x_n = b \]
apply their proof on every
\[ \left[x_0, x_{k+1}\right]. \]

**Intuition:** Why does this work?

Because \( x \to \infty \), oscillations become stronger and stronger and since \( S(x) \) doesn't change, these are cancellations.
Areas cancel out.

Reflected w.r.t. $b - \frac{\pi}{2}$

$$\int_{a + \frac{\pi}{2}}^{b} [f(x) - s(x - \frac{\pi}{2})] \sin \theta \, dx \to 0$$

**Proof of the theorem again:**

We must show

$$\lim_{n \to \infty} \frac{\sin \sum_{k=1}^{n} \sin f(x+k \frac{\pi}{n}) \sin \left( m + \frac{\pi}{2} \right) \, dt}{\sin \frac{\pi}{2}}$$

$$= \frac{1}{2} \left( \frac{f(x+1)}{2} + \frac{f(x)}{2} \right).$$

Look at

$$s(t) = \frac{f(x+1) - f(x)}{2 \sin \frac{\pi}{2}}.$$
Since \( f(x) \) is piecewise smooth

\[
\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = 0
\]

\[
= \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t}
\]

\[
\geq \lim_{t \to 0^+} \frac{\epsilon \sin \left( \frac{t}{2} \right)}{t}
\]

\[
\Rightarrow \frac{1}{\pi} \int_{0}^{\pi} \sin \left( \frac{2\theta}{\pi} \right) \cos \left( \frac{\theta}{2} \right) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} f(x+t) \cos \left( \frac{\theta}{2} \right) \, d\theta
\]

\[
\Rightarrow \frac{1}{\pi} \int_{0}^{\pi} \cos \left( \frac{2\theta}{\pi} + \frac{\theta}{2} \right) \, d\theta
\]

by Lemma 2, \( \Rightarrow m \to \infty \)

But

\[
\frac{1}{2\pi} \int_{0}^{\pi} \cos \left( \frac{2\theta}{\pi} + \frac{\theta}{2} \right) \, d\theta = \frac{1}{2}
\]

\[
= \frac{1}{2} \left( \frac{\pi}{2} + \sum_{k=1}^{\infty} \cos \left( \frac{\pi k}{2} + \frac{\theta}{2} \right) \right) \to \frac{1}{2}
\]
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{t=0}^{m-1} \frac{f(x + t)}{m} \cos \left( \frac{\pi}{m} + \frac{\pi}{2} \right) dt = \frac{1}{2} \left( f(x^+) + f(x^-) \right)
\]

Similarly

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{t=0}^{m-1} \frac{f(x + t)}{m} \cos \left( \frac{\pi}{m} + \frac{\pi}{2} \right) dt = \frac{1}{2} f(x^-)
\]

\[
\lim_{m \to \infty} m(x) = \lim_{m \to \infty} \sum_{t=0}^{m-1} f(x + t) \cos \left( \frac{\pi}{m} + \frac{\pi}{2} \right) dt
\]

\[
= \frac{1}{2} \left( f(x^+) + f(x^-) \right)
\]

**Examples:**

\[f(x) = x, \quad -\pi < x < \pi\]

\[f(x) = \frac{\sin x}{x}, \quad -\pi < x < \pi\]
\[ x = \frac{\pi}{2} \; \text{if} \; \left( \frac{\pi}{2} \right) = 2 \left( \frac{\min \frac{B}{2} - \min \frac{E}{2} \right) \]

\[ \Rightarrow \frac{\pi}{4} = 1 - \text{constant} \]

\[ f(x) = x^2 \]

\[ a_m = \frac{1}{\pi} \int_0^\pi x^2 \cos mx \, dx = \left( -1 \right)^m \frac{4}{m^2} \]

\[ a_0 = \frac{2\pi^2}{3} \]

\[ f(x) = \frac{\pi^2}{3} - 4 \left( \frac{\cos x}{2^2} - \frac{\cos 2x}{3^2} + \frac{\cos 3x}{4^2} \right) \]

\[ f'(x) \]

\[ x \cos x = -\frac{1}{2} \sin x + 2 \frac{(-1)^{\frac{1}{2}}}{2^2 - 1} \]
BESSEL'S INEQUALITY

Let \( f(x) \) be piecewise continuous (not necessarily differentiable). Then,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} - \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right) \right]^2 dx \geq 0
\]

by the orthogonality of trig functions and the definition of \( a_n, b_n \)

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx - \left[ \frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right) \right] \geq 0
\]

Since \( \frac{1}{\pi} \) is finite for all \( n, m \),

\[
\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx
\]

(The fact = sign holds for every integrable \( f(x) \), but we won't prove it.)
Uniform convergence of Fourier series for continuous $f(x)$ with piecewise continuous $f'(x)$:

Let the Fourier coefficients for $f(x)$ be $c_n, d_n$. Then $c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$.

Proof: $c_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} S f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} S f(x) \sin nx \, dx = n b_n$

By Besel's inequality for $f'(x)$:

$$\sum_{n=1}^{\infty} n^2 (c_n^2 + d_n^2) = \sum_{n=1}^{\infty} n^2 (c_n^2 + d_n^2) \leq \frac{4}{\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx$$
By the Cauchy–Schwarz inequality,

\[ |a_n \cos nx + b_n \sin nx| \leq \left( a_n^2 + b_n^2 \right)^{1/2} \left( \cos^2 nx + \sin^2 nx \right)^{1/2} = a_n + b_n \]

Now use \( p^2 \leq \frac{1}{2} (p^2 + q^2) \)

with \( p = \frac{1}{n} \), \( q = \sqrt{a_n^2 + b_n^2} \):

\[ |a_n \cos nx + b_n \sin nx| \leq \frac{1}{n} \left( \frac{1}{n} + n^2 (a_n^2 + b_n^2) \right)^{1/2} \]

The term inside the right-hand side is convergent.

Therefore \( \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) = f(x) \) converges uniformly.

\[ \text{THUS IF f(x) is continuous and piecewise continuously differentiable, then } \text{its Fourier series converges uniformly.} \]
**Functional Version of Abel's Test:** Let

(i) \(|S_n(x)| = |a_1(x) + \cdots + a_n(x)| < M\), independently of \(m\) and \(x\).

(ii) \(p_1 \geq p_2 \geq \cdots \geq p_m \geq \cdots > 0\), \(p_m \to 0\)

Then \(\frac{1}{n} \sum_{k=1}^{n} a_k(x)\) converges uniformly.

**Proof:** \[|p_{m+1} a_{m+1}(x) + \cdots + p_m a_m(x)| = |p_{m+1} (S_{m+1}(x) - S_m(x)) + \cdots + p_m (S_m(x) - S_0(x))| = |p_{m+1} S_m(x) - p_m S_m(x) + (p_{m+1} - p_{m+2}) S_{m+1}(x) + \cdots + (p_{m+2} - p_{m+3}) S_{m+2}(x) + \cdots + (p_{m-1} - p_m) S_{m-1}(x)| \leq p_{m+1} M + p_m M + (p_{m+1} - p_{m+2} + p_{m+2} - p_{m+3} + \cdots + p_{m-1} - p_m) M = 2 p_{m+1} M \to 0\] uniformly on \(x\).
Go back to $\phi(x) = \sum_{n=1}^{\infty} (-1)^n \min_{x} x = x$ \( \sum \), \(-\pi \leq x < \pi \)

\[ \sum_{n=1}^{N} (-1)^{n+1} \sin n x \]

\[ = \int_{-\pi}^{\pi} \sum_{n=1}^{N} (-1)^{n+1} e^{inx} \]

\[ = e^{i x} \frac{1 - (-1)^N e^{i N x}}{1 + e^{-i x}} \]

\[ = \frac{e^{i x} - (-1)^N e^{i (N+1) x}}{2 \cos \frac{x}{2}} \]

\[ \Rightarrow \sum_{n=1}^{N} (-1)^{n+1} \sin n x = \frac{\min x - (-1)^{N+1} \sin \left( N+\frac{1}{2} \right) x}{2 \cos \frac{x}{2}} \]

\[ \Rightarrow \sum_{n=1}^{\infty} (-1)^n \sin n x \leq \frac{1}{\cos \frac{x_0}{2}} \quad |x| \leq x_0 < \pi \]
By Abel's Test, \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) converges uniformly in \( -\frac{\pi}{2} + \epsilon < x < \frac{\pi}{2} - \epsilon \), for any \( \epsilon > 0 \).

By periodicity the series converges uniformly to the periodic extension \( f(x) \) of \( f(x) \) for all \( x \) except in the intervals \( [\pi n - \epsilon, \pi n + \epsilon] \)

\[ n = 0, \pm 1, \pm 2, \ldots \]

**Theorem:** If \( f(x) \) is piecewise smooth and in-periodic, then its Fourier series converges uniformly on all closed sub-intervals on which \( f(x) \) is continuous.

**Proof:**

\[ f(x) = \frac{2}{n} \sum_{b=1}^{M} [f(x_{b+}) - f(x_{b-})] \phi_{b}(x - x_{b}) \]

where \( x_{b}, b = 1, \ldots, M \) are the discontinuities of \( f(x) \) on \( -\pi < x < \pi \), and \( f(x) \) is continuous and piecewise smooth.