FOURIER SERIES

Trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

How do you compute a_n, b_n?

For the moment cast aside

convergence issues.

Focus on:

$$\int_{-\pi}^{\pi} \sin nx \, dx = \int_{-\pi}^{\pi} \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}$$

①

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \pi & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$$

(Orthogonality of trigonometric functions)

(Proof of the last: $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx =$

$= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x] \, dx = 0$)

$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Def: $f(x)$ is periodic with period T if $f(x+T)=f(x)$ for $\forall x \in \mathbb{R}$.

THM For any $a, b \in \mathbb{R}$, $\int_a^{a+T} f(x)dx = \int_b^{b+T} f(x)dx$ if f is periodic with period T.

Let $f(x)$ be periodic with 2π, integrable. Define:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

$f(x)$ is piecewise smooth if it is piecewise continuous (continuous except at finitely many discrete points within one period, where it has finite jumps) and $f'(x)$ is piecewise continuous.

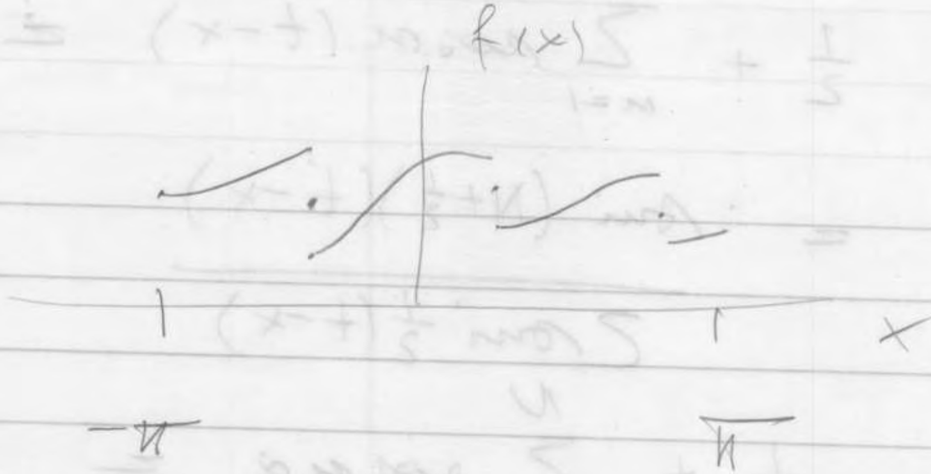
THEOREM If $f(x)$ is piecewise smooth, then

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

converges pointwise to

$$\frac{1}{2}\left(f(x+)+f(x-)\right).$$

In particular it converges to $f(x)$ where $f(x)$ is continuous



Proof Let

$$S_n(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right)$$

Then

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{n=1}^{N} (\cos nt \cos nx + \sin nt \sin nx) \right\} dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{n=1}^{N} \cos n(t-x) \right\} dt$$

LEMMA 1 $\frac{1}{2} + \sum_{n=1}^{N} \cos n(t-x) =$

$$= \frac{\sin (N+\frac{1}{2})(t-x)}{2\sin \frac{1}{2}(t-x)}$$

PROOF: $\frac{1}{2} + \sum_{n=1}^{N} \cos n\alpha =$

$$= \frac{1}{2}\sum_{n=-N}^{N} e^{in\alpha} =$$

$$= \frac{1}{2} e^{-iN\alpha}\left(1 + e^{i\alpha} + \cdots + e^{2iN\alpha}\right)$$

$$= \frac{1}{2} e^{-iN\alpha} \frac{1 - e^{i(2N+1)\alpha}}{1 - e^{i\alpha}} =$$

$$= \frac{1}{2} \frac{e^{-i(N+\frac{1}{2})\alpha} - e^{i(N+\frac{1}{2})\alpha}}{e^{-\frac{i}{2}\alpha} - e^{\frac{i}{2}\alpha}}$$

$$= \frac{\sin (N+\frac{1}{2})\alpha}{2\sin \frac{1}{2}\alpha}$$

$$\Rightarrow S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin (N+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tau)\,\frac{\sin (N+\frac{1}{2})\tau}{\sin\frac{1}{2}\tau}\,d\tau$$

We must show $S_n(x) \to$

$\to \frac{1}{2}\left(f(x+)+f(x-)\right)$

LEMMA 2 If $S(x)$ is piecewise continuous on $a \le x \le b$, then

$$\lim_{\lambda\to\infty}\int_a^b S(x)\sin\lambda x\,dx = 0.$$

PROOF: Can assume $f(x)$ is continuous. Let $|s(x)|<M$.

$$\int_a^b S(x)\sin\lambda x\,dx = \left(\int_a^{b-\frac{\pi}{\lambda}} + \int_{b-\frac{\pi}{4}}^{b}\right) S(x)\sin\lambda x\,dx$$

Let $t = x+\frac{\pi}{\lambda}$

$$\int_a^{b-\frac{\pi}{\lambda}} S(x)\sin\lambda x\,dx = -\int_{a+\frac{\pi}{\lambda}}^{b} S(t-\tfrac{\pi}{\lambda})\sin\lambda x\,dx$$

$$\Rightarrow 2\int_a^b f(x)\sin\lambda x\,dx = \int_{a+\frac{\pi}{\lambda}}^{b-\frac{\pi}{\lambda}} \left(f(x)-f(x-\tfrac{\pi}{\lambda})\right)\sin\lambda x\,dx$$

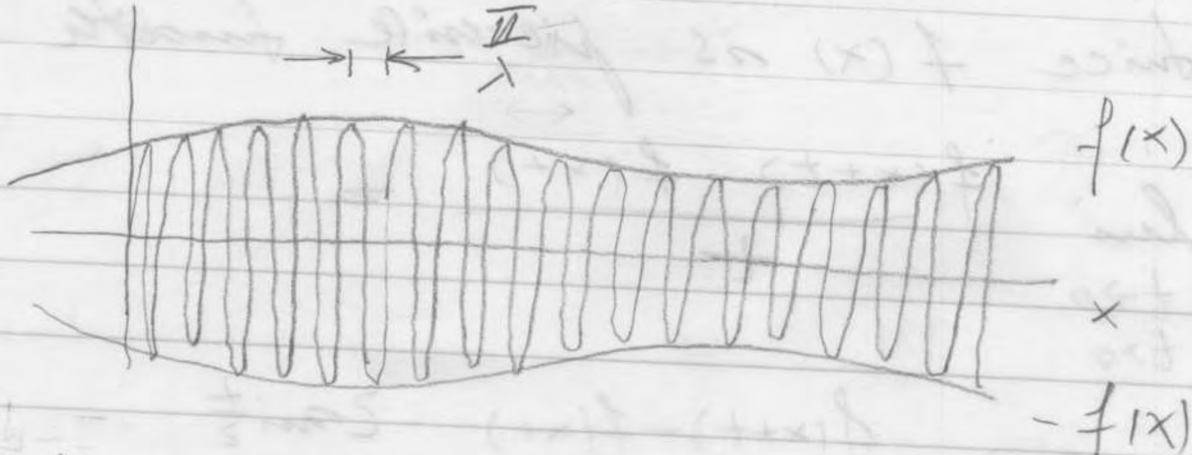
$$+\left(\int_a^{a+\frac{\pi}{\lambda}} + \int_{b-\frac{\pi}{\lambda}}^{b}\right) f(x)\sin\lambda x\,dx$$

Since $|f(x)| < M$, $\left|\int_a^{a+\frac{\pi}{\lambda}}\right|$, $\left|\int_{b-\frac{\pi}{\lambda}}^{b}\right| < \frac{\pi M}{\lambda}$

If λ is large enough, by continuity $|f(x)-f(x-\frac{\pi}{\lambda})| < \varepsilon \Rightarrow \left|2\int_a^b f(x)\sin\lambda x\,dx\right| <$
$< \varepsilon(b-a) + \frac{2\pi M}{\lambda} \to 0$.

If $f(x)$ is piecewise continuous on $a = x_0 < x_1 < x_2 < \dots x_n = b$, apply this proof on every $[x_k, x_{k+1}]$.

INTUITION: Why does this work?
Because as $\lambda \to \infty$, oscillations become stronger and stronger and since $f(x)$ doesn't change, there are cancellations



Areas cancel out.

Reflected in

$$\int_{a+\frac{\pi}{\lambda}}^{b-\frac{\pi}{\lambda}} \left[f(x)-f\left(x-\frac{\pi}{\lambda}\right)\right] \sin \lambda x \, dx \longrightarrow 0$$

PROOF OF THE THEOREM AGAIN:

WE MUST SHOW

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x+t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \, dt$$

$$= \frac{1}{2}\left(f(x+) + f(x-)\right).$$

Look at $s(t) = \dfrac{f(x+t) - f(x+)}{2\sin\frac{t}{2}}$

Since $f(x)$ is piecewise smooth

$$\lim_{\substack{t\to 0\\ t>0}} \frac{f(x+t)-f(x+)}{t} =$$

$$= \lim_{t\to 0+} \frac{f(x+t)-f(x+)}{2\sin\frac{t}{2}} \, \frac{2\sin\frac{t}{2}}{t}$$

$$= \lim_{t\to 0+} \frac{f(x+t)-f(x+)}{2\sin\frac{t}{2}} \text{ exists.}$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi} S(t)\sin(n+\tfrac{1}{2})t\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(x+t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt$$

$$- \frac{1}{2\pi}\int_0^{\pi} f(x+)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt \to 0$$

by Lemma 2. as $n\to\infty$

But

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt =$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \sum_{k=1}^{n}\int_0^{\pi}\cos kt\,dt\right) = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{\pi} f(x+t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt = \frac{1}{2} f(x+)$$

Similarly

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{0} f(x+t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt$$

$$= \frac{1}{2} f(x-)$$

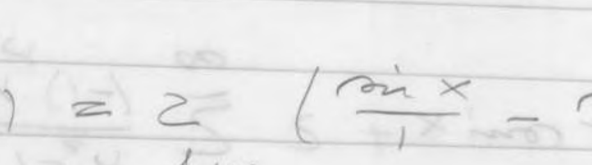
$$\Rightarrow \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(x+t)\,\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt$$

$$= \frac{1}{2}\left(f(x+) + f(x-)\right)$$

<u>EXAMPLES</u> :

$f(x) = x$ on $-\pi < x < \pi$

$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right.$



$x = \frac{\pi}{2}$: $\frac{\pi}{2} = 2\left(\frac{\sin\frac{\pi}{2}}{1} - \frac{\sin 2\frac{\pi}{2}}{2} + \frac{\sin 3\frac{\pi}{2}}{3} + \cdots\right)$

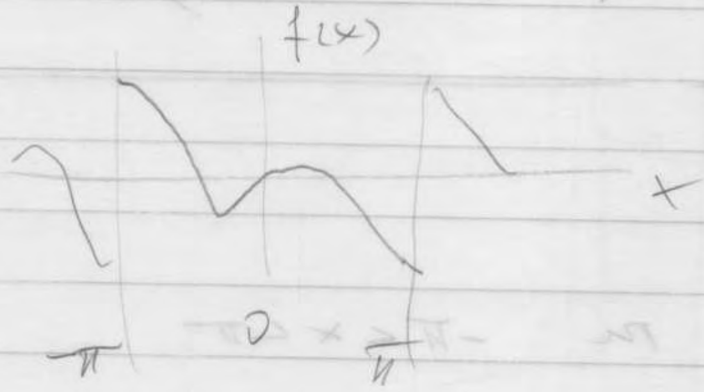
$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$f(x) = x^2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = (-1)^n \frac{4}{n^2}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right)$$



$$x \cos x = -\frac{1}{2}\sin x + 2\sum_{\nu=2}^{\infty} \frac{(-1)^\nu \nu}{\nu^2 - 1} \sin \nu x$$

BESSEL'S INEQUALITY

Let $f(x)$ be piecewise continuous (not necessarily differentiable). Then,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)-\frac{a_0}{2}-\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right)\right]^2dx\ge 0$$

By the orthogonality of trig functions and the definition of a_n, b_n

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx-\left[\frac{a_0^2}{2}+\sum_{k=1}^{n}\left(a_k^2+b_k^2\right)\right]\ge 0$$

Since this is true for all n,

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)\le\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx$$

(In fact, $=$ sign holds for every integrable $f(x)$, but we won't prove it.)

Uniform convergence of Fourier series for continuous $f(x)$ with piecewise continuous $f'(x)$:

Let the Fourier coefficients for $f'(x)$

be c_n, d_n. Then $c_n = nb_n$, $d_n = -na_n$

(Proof $c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx =$

$= \frac{1}{\pi} f(x)\cos nx \Big|_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = nb_n$)

By Bessel's inequality for $f'(x)$

$$\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) = \sum_{n=1}^{\infty}(c_n^2+d_n^2) \le \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}[f'(x)]^2\,dx}_{M^2}$$

By the Cauchy-Schwartz inequality,

$|a_n \cos nx + b_n \sin nx| \le$

$\le (a_n^2 + b_n^2)(\cos^2 nx + \sin^2 nx) = a_n^2 + b_n^2$

Now use $pq \le \frac{1}{2}(p^2 + q^2)$ with $p = \frac{1}{n}$, $q = \sqrt{a_n^2 + b_n^2}$:

$|a_n \cos nx + b_n \sin nx| \le \frac{1}{n}\, n (a_n^2 + b_n^2)$

$\le \frac{1}{2}\left[\frac{1}{n^2} + n^2(a_n^2 + b_n^2)\right]$

The sum over n on the right-hand side is convergent.

Therefore $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = f(x)$ converges uniformly.

⇒ <u>THM</u> If $f(x)$ is continuous and piecewise continuously differentiable, then its Fourier series converges uniformly.

FUNCTIONAL VERSION OF ABEL'S TEST: Let

(i) $|S_n(x)| = |a_1(x) + \cdots + a_n(x)| < M$ independently of n and x

(ii) $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \cdots \geq 0$, $p_n \to 0$

Then $\sum_{n=1}^{\infty} p_n a_n(x)$ converges uniformly.

Proof $|p_{n+1} a_{n+1}(x) + \cdots\ p_m a_m(x)|$

$= |p_{n+1}(S_{n+1}(x) - S_n(x)) + p_{n+2}(S_{n+2}(x) - S_{n+1}(x)) + \cdots + p_m(S_m(x) - S_{m-1}(x))|$

$= |-p_{n+1} S_n(x) - p_m S_m(x) + (p_{n+1} - p_{n+2}) S_{n+1}(x) +$

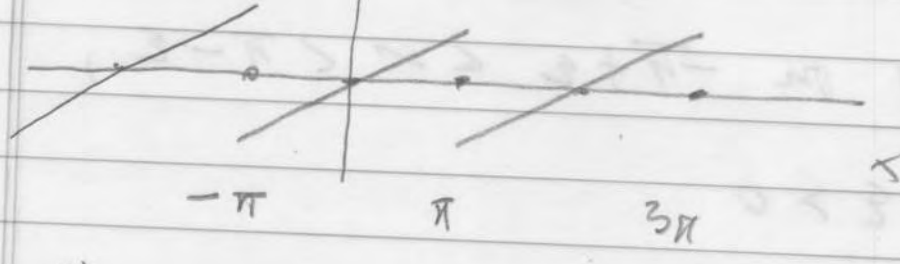
$+ (p_{n+2} - p_{n+3}) S_{n+2}(x) + \cdots + (p_{m-1} - p_m) S_{m-1}(x)|$

$\leq p_{n+1} M + p_m M + (p_{n+1} - p_{n+2} + p_{n+2} - p_{n+3} + \cdots$

$+ \cdots + p_{m-1} - p_m) M = 2 p_{n+1} M \to 0$

uniformly in x.

GO BACK TO $\phi(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} = \frac{x}{2}, \quad -\pi < x < \pi$



$$\sum_{n=1}^{N} (-1)^{n+1} \sin nx = \underset{\text{Imaginary part}}{\mathrm{Im}} \left[\sum_{n=1}^{N} (-1)^{n+1} e^{inx} \right]$$

NOW $\sum_{n=1}^{N} (-1)^{n+1} e^{inx} = e^{ix} \sum_{n=1}^{N-1} \left(-e^{inx}\right)$

$$= e^{ix} \frac{1 - (-1)^N e^{iNx}}{1 + e^{ix}} =$$

$$= \frac{e^{i\frac{x}{2}} - (-1)^N e^{i(N+\frac{1}{2})x}}{2 \cos \frac{x}{2}}$$

$$\Rightarrow \sum_{n=1}^{N} (-1)^{n+1} \sin nx = \frac{\sin \frac{x}{2} - (-1)^N \sin (N+\frac{1}{2})x}{2 \cos \frac{x}{2}}$$

$$\Rightarrow \left| \sum_{n=1}^{N} (-1)^{n+1} \sin nx \right| \leq \frac{1}{\cos \frac{x_0}{2}}, \quad |x| \leq x_0 < \pi$$

$\Rightarrow$ By Abel's Test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n} \to \frac{x}{2}$

uniformly on $-\pi + \varepsilon < x < \pi - \varepsilon$,

for any $\varepsilon > 0$

By periodicity the series converges uniformly to the periodic extension $\phi(x)$ of $\frac{x}{2}$ for all x except in the intervals $[(2n+1)\pi - \varepsilon, (2n+1)\pi + \varepsilon]$

$n = 0, \pm 1, \pm 2, \ldots$

<u>THM</u> If $f(x)$ is piecewise smooth and 2π-periodic, then its Fourier series converges uniformly on all closed subintervals on which $f(x)$ is continuous.

<u>PROOF</u>: $f(x) - \frac{2}{\pi} \sum_{k=1}^{M} [f(x_k+) - f(x_k-)] \phi(x - x_k)$, where x_k, $k = 1..M$ are the discontinuities of $f(x)$ on $-\pi < x \leq \pi$, is continuous and piecewise smooth.