LAPLACE TRANSFORM

$$(\mathcal{L}f)(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

E.g. $f(t) = e^{\alpha t}$, $\alpha > 0$

$$F(s) = \int_0^\infty e^{(\alpha - s)t}\, dt =$$

$$= \frac{1}{s-\alpha}, \quad s > \alpha$$

$f(t) = 1 \Rightarrow F(s) = \frac{1}{s}$ $(\alpha = 0)$

$f(t) = \sin at, \ \cos at$

CAN INTEGRATE BY PARTS AND SHOW

$$F(s) = \frac{a}{s^2+a^2}, \quad \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L} f_1 + c_2 \mathcal{L} f_2$$

What is $\mathcal{L} f'$?

$$(\mathcal{L} f')(s) = \int_0^\infty f'(t)\, e^{-st}\, dt =$$

$$= f(t)\, e^{-st} \Big|_0^\infty + s \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= s\, \mathcal{L} f(s) - f(0)$$

$$(\mathcal{L} f''(s)) = \int_0^\infty f''(t)\, e^{-st}\, dt =$$

$$= f'(t)\, e^{-st} \Big|_0^\infty + s \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= -f'(0) + s \left[f(t)\, e^{-st} \Big|_0^\infty + s \int_0^\infty f(t)\, e^{-st}\, dt \right]$$

$$= s^2 \int_0^\infty f(t)\, e^{-st}\, dt - s f(0) - f'(0)$$

$$= s^2 (\mathcal{L} f)(s) - s f(0) - f'(0)$$

INITIAL-VALUE PROBLEMS

$$y'' + ay' + by = f(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

$$\Rightarrow s^2 \mathcal{L}y - sy_0 - y_1 +$$

$$+ a(s\mathcal{L}y - y_0) + b\mathcal{L}y = \mathcal{L}f$$

Let $Y = \mathcal{L}y$, $F = \mathcal{L}F$

TRANSFER FUNCTION $\rightarrow$

$$\underbrace{(s^2 + as + b)}\, Y(s) = \underbrace{F(s) + (s+a)y_0 + y_1}$$

(RECALL: CHARACTERISTIC EQUATION)

INPUT (USUALLY $y_0 = y_1 = 0$)

ASSUME: $y_0 = y_1 = 0$

INPUT

$$Y(s) = \frac{F(s)}{(s^2 + as + b)}$$

TRANSFER FUNCTION

E.g. $y'' - y' - 2y = e^{-3t}$

$$y(0) = y'(0) = 0$$

$$\Rightarrow \quad (s^2 - s - 2)\,Y = \frac{1}{s+3}$$

$$Y = \frac{1}{(s^2 - s - 2)(s+3)} = \frac{1}{(s+1)(s+3)(s-2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s-2} =$$

$$= -\frac{1}{6}\,\frac{1}{s+1} + \frac{1}{10}\,\frac{1}{s+3} + \frac{1}{15}\,\frac{1}{s-2}$$

RECALL $\quad \mathcal{L}(e^{\alpha t})(s) = \dfrac{1}{s-\alpha}$

$$\Rightarrow \quad \mathcal{L}^{-1}\left(\frac{1}{s-\alpha}\right) = e^{\alpha t}$$

$$\Rightarrow \quad y(t) = -\frac{1}{6}\,e^{-t} + \frac{1}{10}\,e^{-3t} + \frac{1}{15}\,e^{2t}$$

$$y'' + 4y' + 3y = \sin t$$

$$y(0) = y'(0) = 0$$

$$Y(s^2 + 4s + 3) = \frac{1}{1+s^2}$$

$$Y = \frac{1}{(1+s^2)(s+1)(s+3)} =$$

$$= \frac{1}{4}\,\frac{1}{s+1} - \frac{1}{20}\,\frac{1}{(s+3)} -$$

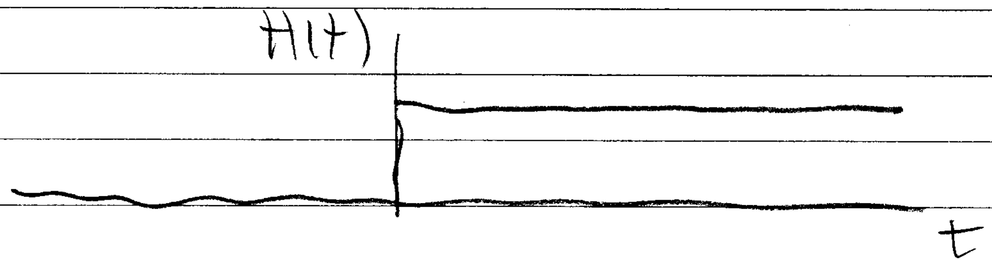
$$- \frac{1}{10}\,\frac{-1+2s}{s^2+1}$$

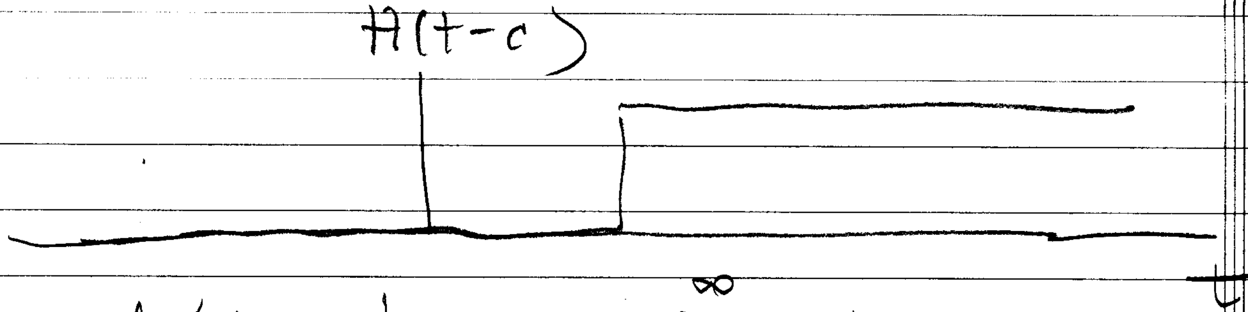
$$y = \frac{1}{4} e^{-t} - \frac{1}{20} e^{-3t} + \frac{1}{10} \sin t -$$

$$- \frac{1}{5} \cos t$$

HEAVYSIDE FUNCTION

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$





$$\mathcal{L}\{H(t-c)\}(s) = \int_c^\infty e^{-st}\,dt =$$

$$= -\frac{1}{s} e^{-st}\Big|_c^\infty = \frac{e^{-sc}}{s}$$

$$y'' + y = H(t) \qquad y(0) = y'(0) = 0$$

$$s^2 Y + Y = \frac{1}{s} \Rightarrow Y = \frac{1}{s(1+s^2)} =$$

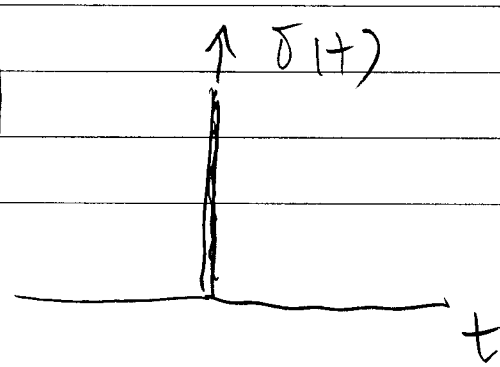
$$= \frac{1}{s} - \frac{s}{1+s^2}$$

$$y(t) = H(t) - \cos t$$

<u>DELTA</u> (IMPULSE) FUNCTION

$$\delta(t) = H'(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{-a}^{t>0} \delta(t)\,dt = 1$$



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$$\mathcal{L}(\delta(t-c))(s) = \int_0^\infty \delta(t-c)\, e^{-st}\, dt =$$

$$= e^{-sc}$$

$$\mathcal{L}(\delta(t))(s) = 1 \qquad (c=0)$$

$$y'' + y = \delta(t) \qquad y(0) = y'(0) = 0$$

$$(1+s^2)\, Y = 1$$

$$Y = \frac{1}{1+s^2}$$

$$y(t) = \sin t \qquad t > 0$$

$$y(t) = 0 \qquad t < 0$$