2×2 MATRICES

COMPONENT

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ - MATRIX

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$ - VECTOR

$A = [a_{ij}]$ $\qquad$ $B = [b_{ij}]$

$A + B = [a_{ij} + b_{ij}]$

ADD COMPONENTS IN THE SAME POSITION

$\lambda A = [\lambda a_{ij}]$, λ — NUMBER

$$A\underline{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}$$

$$\lambda \underline{v} = (\lambda v_1, \lambda v_2)$$

IDENTITY: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I$

E.g. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A + B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$3A = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$

$v = (1, 0)$

$Av = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$2v = (2, 0)$

LINEAR EQUATIONS

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$Ax = b$$

HOMOGENEOUS EQUATION:

$Ax = 0$

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

UNIQUE SOLUTION IF
THE TWO EQUATIONS NOT PROPORTIONAL;
THE SOLUTION IS $x_1 = x_2 = 0$

WHEN ARE THE EQUATIONS
PROPORTIONAL?
(i.e. $\alpha a_{11}x_1 + \alpha a_{12}x_2 =$
$= a_{21}x_1 + a_{22}x_2$)

IF $\alpha a_{11} = a_{21}$
$\alpha a_{12} = a_{22}$
FOR THE SAME α

$$\alpha = \boxed{\frac{a_{21}}{a_{11}} = \frac{a_{22}}{a_{12}}}$$

$$\downarrow$$

$$\boxed{a_{11}a_{22} - a_{21}a_{12} = 0}$$

DETERMINANT MEASURES IF THE
EQUATIONS ARE PROPORTIONAL.

BY DEFINITION

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

IF $\det A = 0 \Rightarrow$ MORE THAN ONE SOLUTION (x_1, x_2)

IN FACT IF $Ax = 0$, THEN $A\lambda x = 0$ FOR ANY # λ

$\Rightarrow$ If x is a solution, so is λx.

EXAMPLE: $Ax = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\det A = 1 \neq 0 \Rightarrow x = 0$ IS THE ONLY SOLUTION

$Ax = 0 \qquad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \qquad \begin{matrix} x_1 + x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{matrix}$

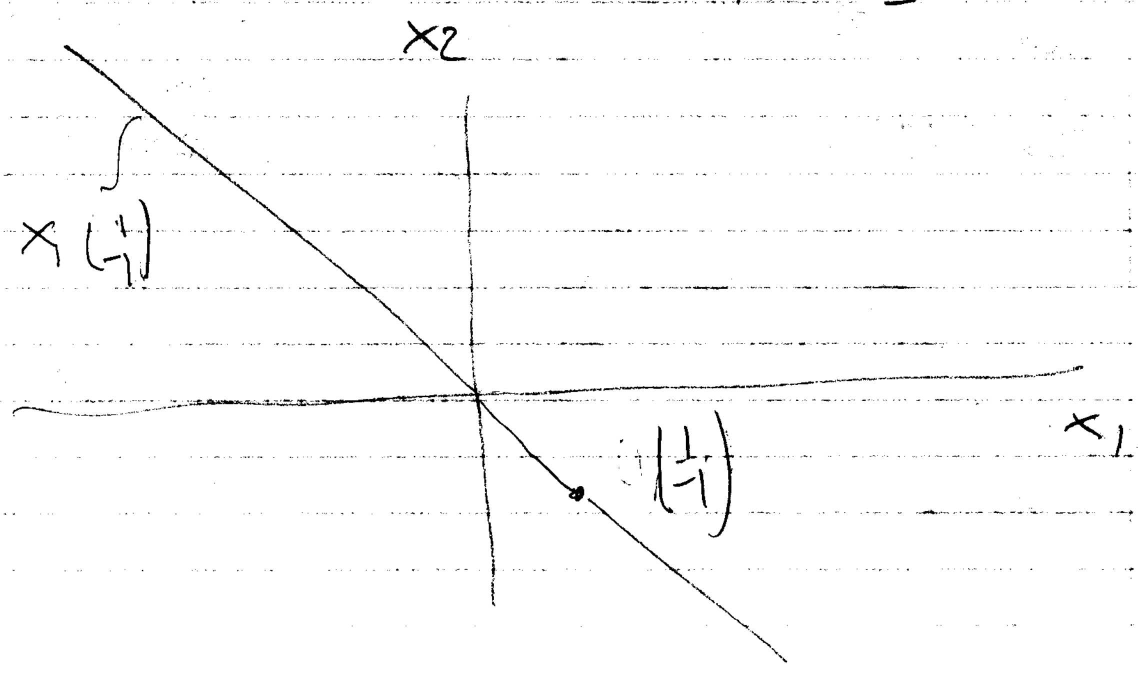
$\det A = 3 - 3 = 0$

EQUATIONS ARE PROPORTIONAL:
ONLY NEED TO LOOK AT
ONE OF THEM:

$$x_1 + x_2 = 0$$

$$x_2 = -x_1$$

$$x = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \underbrace{x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\text{A LINE OF SOLUTIONS}}$$



$$\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\det(A) = -1 - (-1) = 0$

EQUATIONS ARE PROPORTIONAL!

IN FACT, THE SECOND IS i TIMES THE FIRST, DROP FIRST:

$$i x_1 - x_2 = 0$$

$$x_2 = i x_1$$

$$x = \begin{pmatrix} x_1 \\ i x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

EIGENVALUES AND EIGENVECTORS:

LOOK FOR NONZERO SOLUTIONS OF THE PROBLEM

UNKNOWN NUMBER

$$A x = r x$$

UNKNOWN VECTOR

KNOWN 2x2 MATRIX

$$Ax = rIx$$

$$(A - rI)x = 0$$

$$\begin{pmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

HAS NONZERO SOLUTIONS IF

$$\det(A - rI) = 0$$

$\Rightarrow$ A QUADRATIC EQUATION FOR r

$\Rightarrow$ TWO ROOTS CALLED <u>EIGENVALUES</u>

r_1, r_2

$$\left.\begin{array}{ll} (A - r_1 I)\, x^{(1)} = 0; & x^{(1)} \\ (A - r_2 I)\, x^{(2)} = 0; & x^{(2)} \end{array}\right\} \text{EIGENVECTORS}$$

<u>EXAMPLE:</u> $(A - rI)x = 0$

$$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$$

$$\det(A - rI) = \begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = (5-r)(1-r) + 3 = r^2 - 6r + 8$$

$$r^2 - 6r + 8 = 0$$

$$(r-4)(r-2) = 0$$

$$r_1 = 2, \quad r_2 = 4$$

$\underline{r_1 = 2}$: $\quad A - 2I = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} =$

$$= \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow 3x_1 - x_2 = 0$$

$$x_2 = 3x_1$$

$$x^{(1)} = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

EIGENVECTOR

$\underline{r_2 = 4}$: $\quad (A - 2I) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 - x_2 = 0$$

$$x_2 = x_1$$

$$x^{(2)} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

EXAMPLE $(A - rI)x = 0$

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

$$\det(A - rI) = \begin{vmatrix} 3-r & -2 \\ 4 & -1-r \end{vmatrix} =$$

$$= (3-r)(-1-r) + 8 = r^2 - 2r + 5$$

$$r_{1,2} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{\sqrt{-16}}{2} =$$

$$= 1 \pm 2i$$

RULE: IF $r_1 = \lambda + i\mu$,

THEN $r_2 = \lambda - i\mu$

AND IF $x^{(1)} = a + ib$,

THAT IS $\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

THEN

$$x^{(2)} = a - ib$$

$\Rightarrow$ JUST CALCULATE $x^{(1)}$ CORRESPONDING TO $r_1 = 1+2i$

$$A - (1+2i)I$$

$$= \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix} =$$

$$= \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}$$

$$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

EQUATIONS <u>MUST</u> BE PROPORTIONAL (IN FACT, THE SECOND ONE IS $1+i$ TIMES THE FIRST ONE)

$$(2-2i)x_1 - 2x_2 = 0$$

$$(1-i)x_1 - x_2 = 0$$

$$x_2 = (1-i)x_1$$

$$x^{(1)} = \begin{pmatrix} x_1 \\ (1-i)x \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$x^{(2)} = x_1 \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

(LECTURE 6)

SYSTEMS / 2nd ORDER EQUATIONS

$$ay'' + by' + cy = 0$$

$$a \neq 0$$

$$\Rightarrow \quad y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

$$y'' + Ay' + By = 0$$

WRITE AS A SYSTEM

INTRODUCE NEW VARIABLE $y' = z$

$$y' = z$$

$$y'' = z' = -Ay' - By = -Az - By$$

$$\left.\begin{aligned} y' &= z \\ z' &= -Az - By \end{aligned}\right\}$$ system, we know how to solve it, plot trajectories etc.

LECTURE 6

HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

$$\underline{\dot{x}} = A\,\underline{x}$$

A ← CONSTANT MATRIX

Try $\underline{x} = \underline{z}\, e^{rt}$

$\underline{z}$ ← CONSTANT VECTOR

$$\underline{\dot{x}} = \underline{z}\, r e^{rt} = r\underline{z}\, e^{rt}$$

$$\Rightarrow \quad r\underline{z}\, e^{rt} = A\underline{z}\, \cancel{e^{rt}}$$

$$A\underline{z} = r\underline{z} \quad \Rightarrow \quad (A - rI)\underline{z} = 0$$

EIGENVALUE → r

EIGENVECTOR → $\underline{z}$

EXAMPLE: $\underline{\dot{x}} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underline{x}$

$$\underline{x} = \underline{z}\, e^{rt} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} e^{rt}$$

$$\Rightarrow \quad \left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - rI \right] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = 0$$

$(r-3)(r+1) = 0$

$r_1 = 3, \quad r_2 = -1$ } EIGENVALUES

EIGENVECTORS

$\underline{r_1:}$ $z^{(1)} = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}$

$$\left[\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - r_1 I\right] \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix} = 0$$

$$\Rightarrow \quad -2 z_1^{(1)} + z_2^{(1)} = 0$$

SECOND EQUATION IS $-2 \times$ FIRST.

$$\Rightarrow \quad z_2^{(1)} = 2 z_1^{(1)}$$

WE CAN CHOOSE $z_1^{(1)} = 1$

$$\Rightarrow \quad z_2^{(1)} = 2$$

$$z^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

r_2 : $z^{(2)} = \begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix}$

$$\left[\begin{pmatrix}1 & 1\\4 & 1\end{pmatrix} - r_1 I\right]\begin{pmatrix} z_1^{(2)}\\ z_2^{(2)}\end{pmatrix} = \begin{pmatrix}2 & 1\\4 & 2\end{pmatrix}\begin{pmatrix}z_1^{(2)}\\ z_2^{(2)}\end{pmatrix} = 0$$

$$\Rightarrow \quad 2z_1^{(1)} + z_2^{(1)} = 0$$

SECOND EQUATION IS 2× FIRST

$$\Rightarrow \quad z_2^{(2)} = -2z_1^{(2)}$$

CHOOSE $z_1^{(2)} = 1 \Rightarrow z_2^{(2)} = -2$

$$z^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}$$

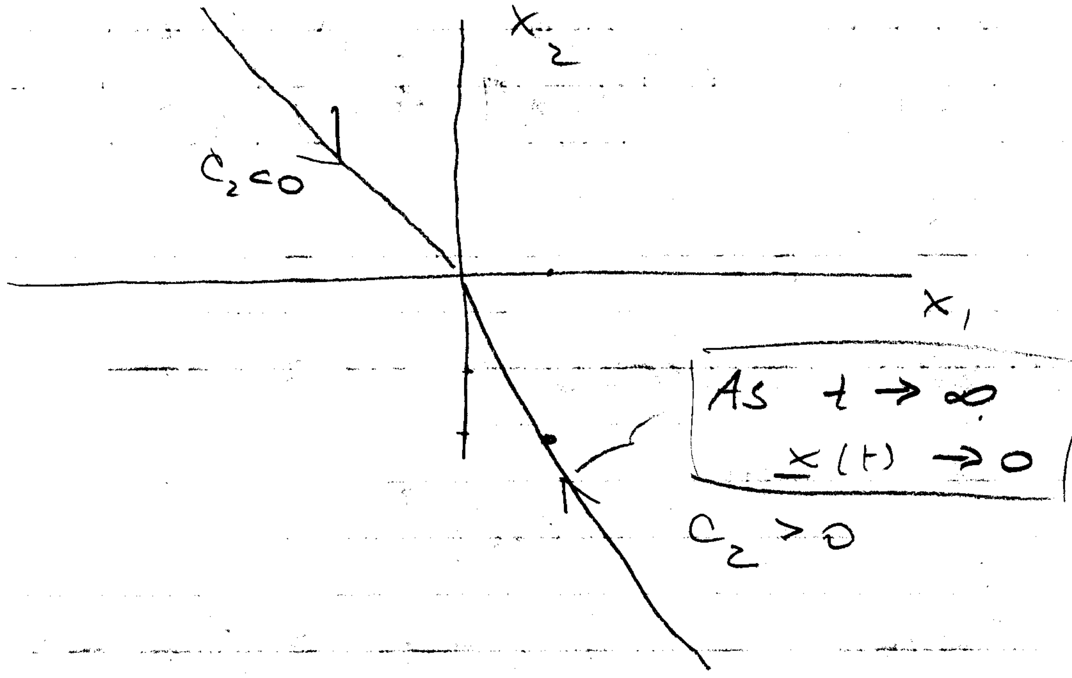
$$x^{(1)} = \begin{pmatrix}1\\2\end{pmatrix} e^{3t}, \quad x^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix} e^{-t}$$

$$x(t) = c_1 \begin{pmatrix}1\\2\end{pmatrix} e^{3t} + c_2 \begin{pmatrix}1\\-2\end{pmatrix} e^{-t}$$

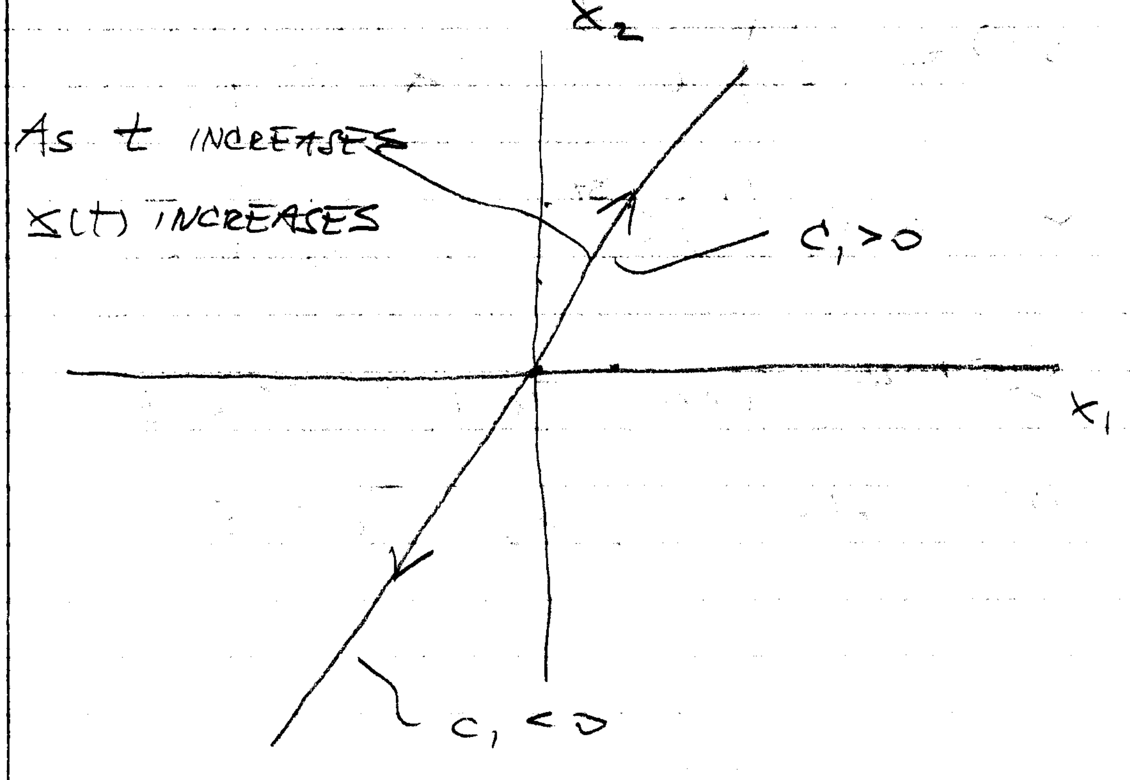
SKETCH SOME TRAJECTORIES

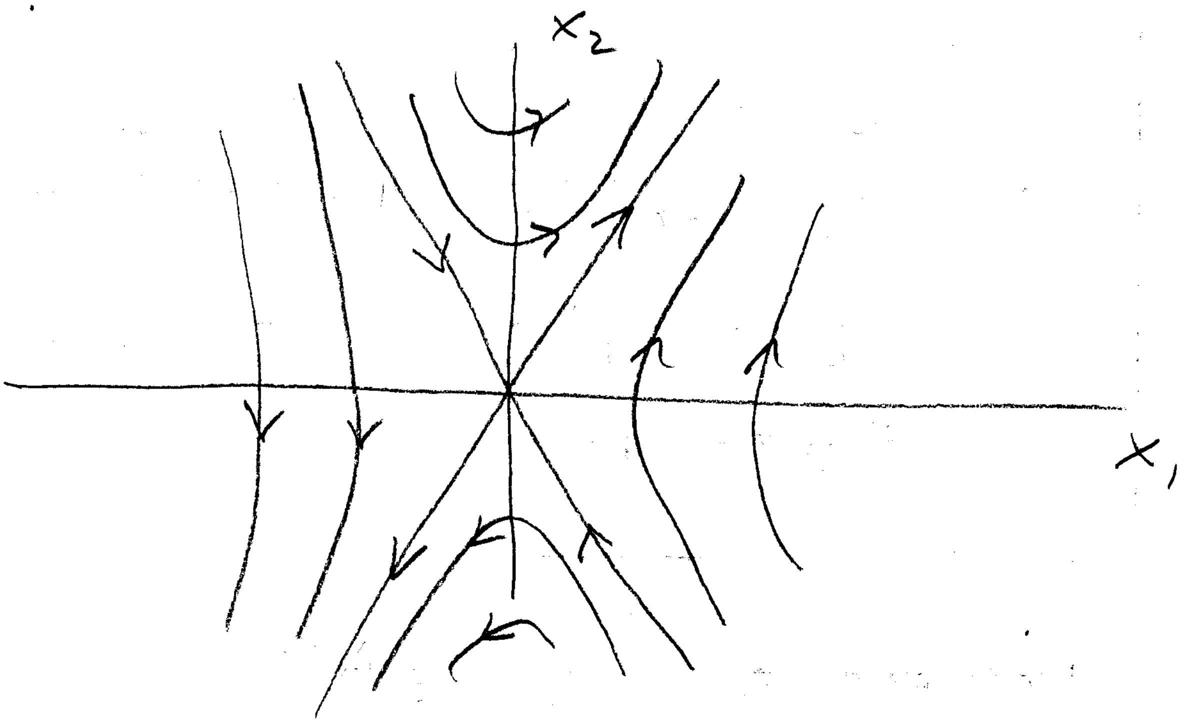
IN THE $x_1 - x_2$ PLANE:

$\underline{c_1 = 0} \Rightarrow \underline{x}(t) = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$



$\underline{c_2 = 0} \Rightarrow \underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$





IN GENERAL: A - $n \times n$ matrix

$r_1, r_2, \ldots r_n$ - DIFFERENT REAL EIGENVALUES

$\Rightarrow \underline{z}^{(1)}, \underline{z}^{(2)}, \ldots \underline{z}^{(n)}$ — (LINEARLY INDEPENDENT) EIGENVECTORS

GENERAL SOLUTION:

$$\underline{x}(t) = \underline{z}^{(1)} e^{r_1 t} + \underline{z}^{(2)} e^{r_2 t} + \ldots + \underline{z}^{(n)} e^{r_n t}$$

EXAMPLE: $\dot{\underline{x}} = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \underline{x}$

$$\underline{x} = \underline{z} e^{rt} \Rightarrow \begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} = (-3-r)(-2-r) - 2 =$$

$$= 6 + 5r + r^2 - 2 = r^2 + 5r + 4 =$$

$$= (r+4)(r+1)$$

$r_1 = -1, \quad r_2 = -4$

EIGENVECTORS: $r_1 \Rightarrow z^{(1)}$

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix} = 0$$

$$\sqrt{2}\, z_1^{(1)} - z_2^{(1)} = 0$$

1^{st} EQUATION IS $\sqrt{2} \times$ SECOND

$\Rightarrow \quad z_2^{(1)} = \sqrt{2}\, z_1^{(1)}$

CHOOSE $z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = \sqrt{2}$

$$z^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$r_2 \Rightarrow z^{(2)}$ $(r_2 = -4)$

$$\begin{pmatrix} -3-r_2 & \sqrt{2} \\ \sqrt{2} & -2-r_2 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix} = 0$$

$$z_1^{(2)} + \sqrt{2}\, z_2^{(2)} = 0$$

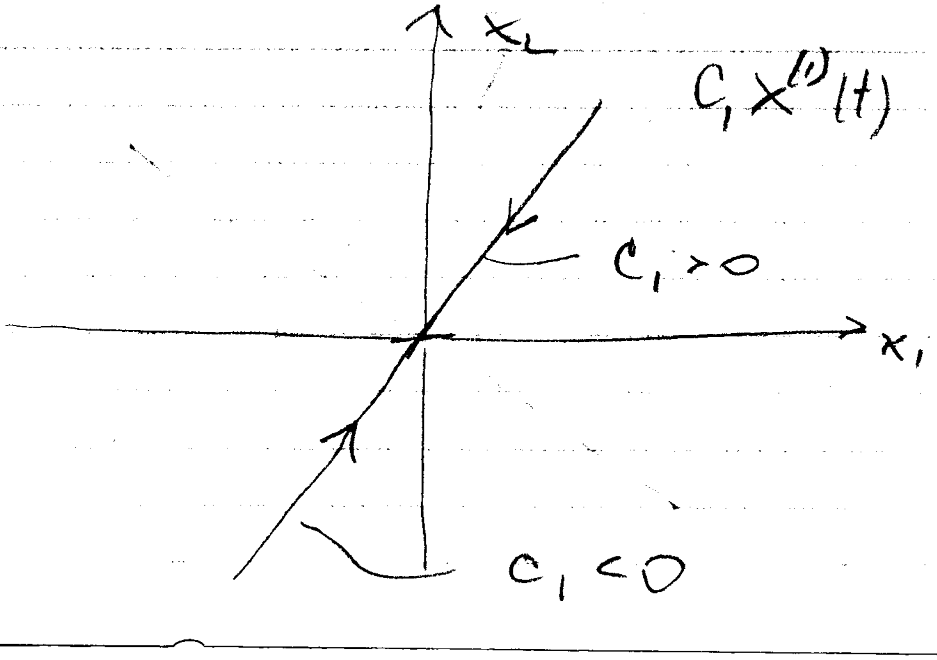
SECOND EQUATION IS $\sqrt{2} \times$ FIRST

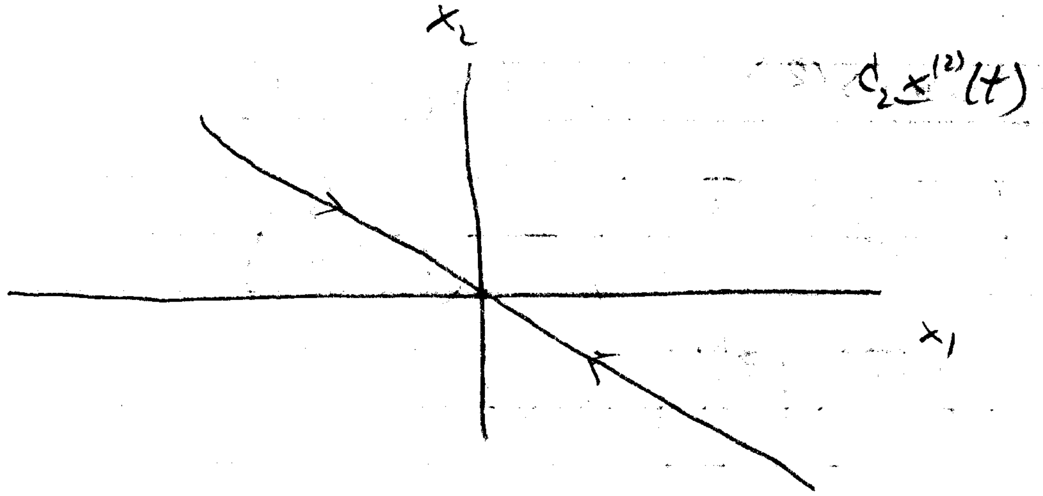
CHOOSE $z_2^{(2)} = 1 \Rightarrow z_1^{(2)} = -\sqrt{2}$

$$z^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\Rightarrow x^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \quad x^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

$$x(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

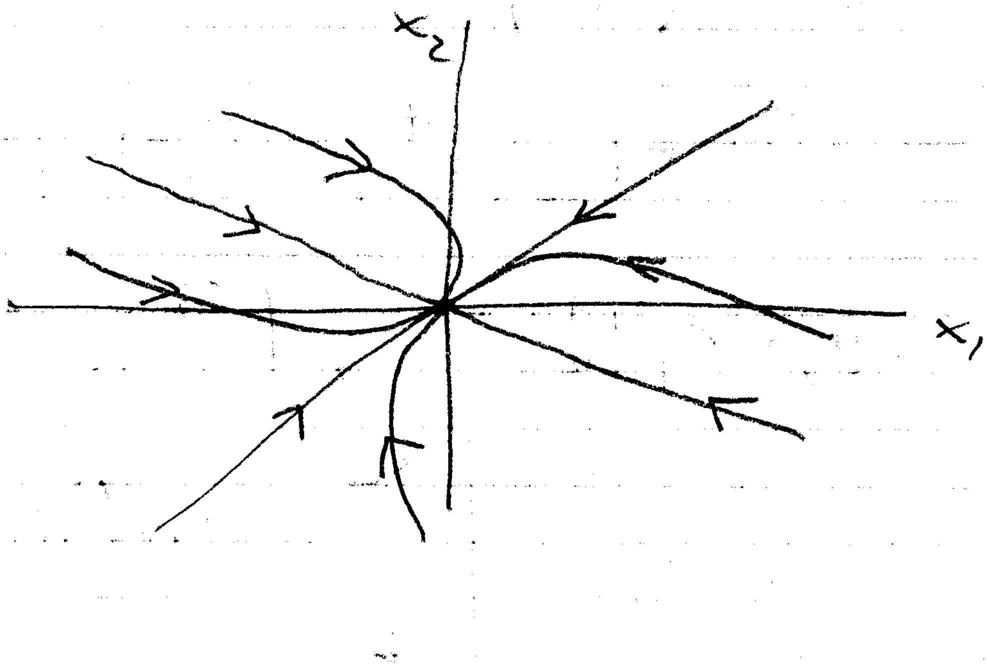




$t \to 0 \Rightarrow x^{(2)}(t)$ MUCH SMALLER THAN $x^{(1)}(t)$

$\Rightarrow$ SOLUTIONS ALMOST $x^{(1)}(t)$

$\Rightarrow$ SOLUTIONS TANGENT TO $x^{(1)}(t)$



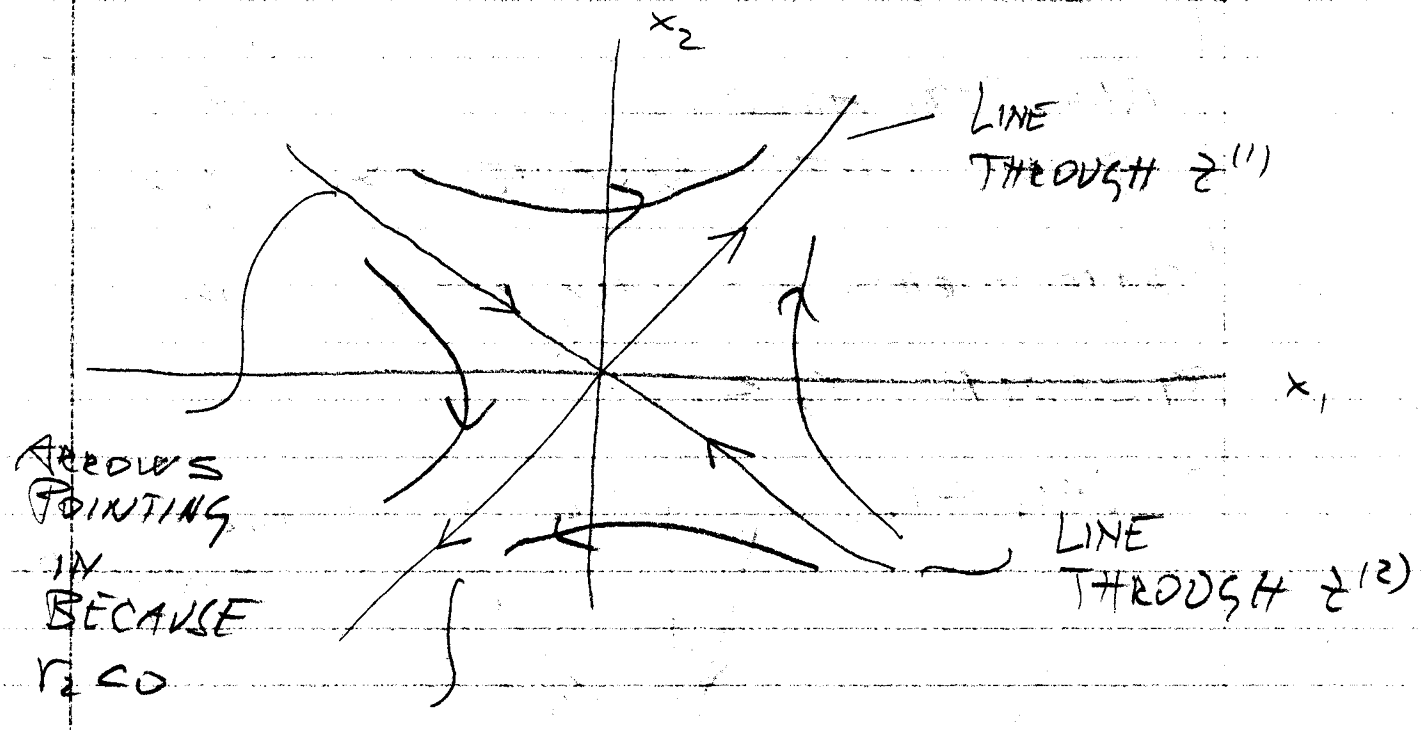
DRAWING TRAJECTORIES OF 2×2 SYSTEMS

$$\dot{\underline{x}} = A\underline{x} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

r_1, r_2 – EIGENVALUES OF A; REAL, DIFFERENT

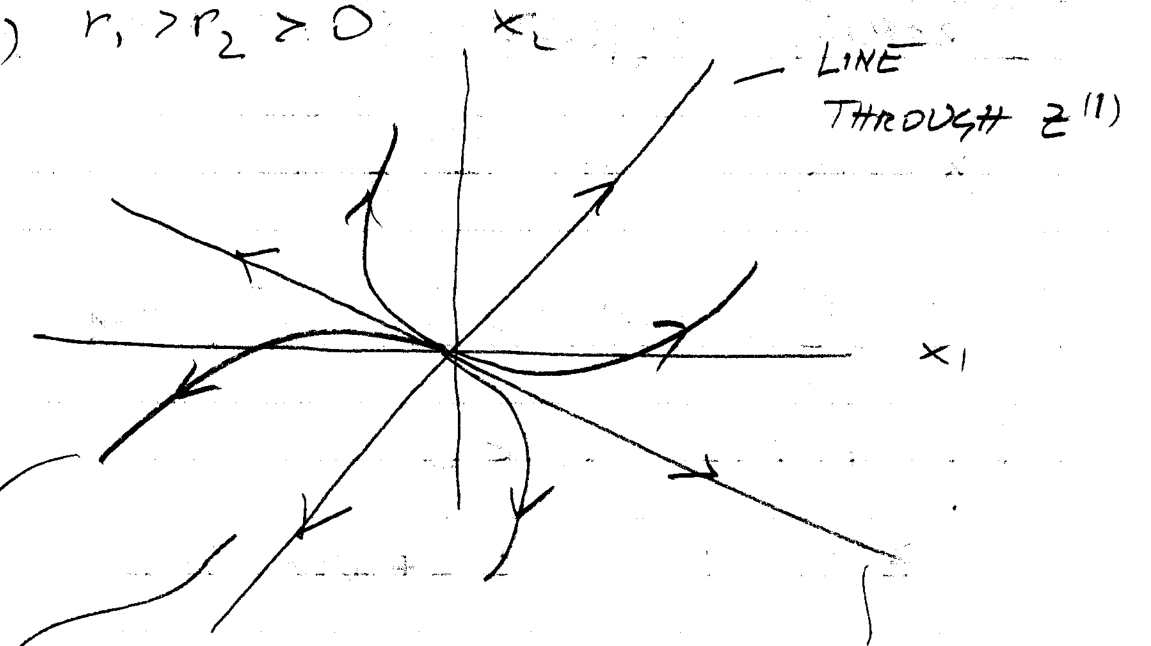
1.) $r_1 > 0$, $r_2 < 0$

$z^{(1)}, z^{(2)}$ – EIGENVECTORS



ARROWS POINTING OUT BECAUSE $r_1 > 0$

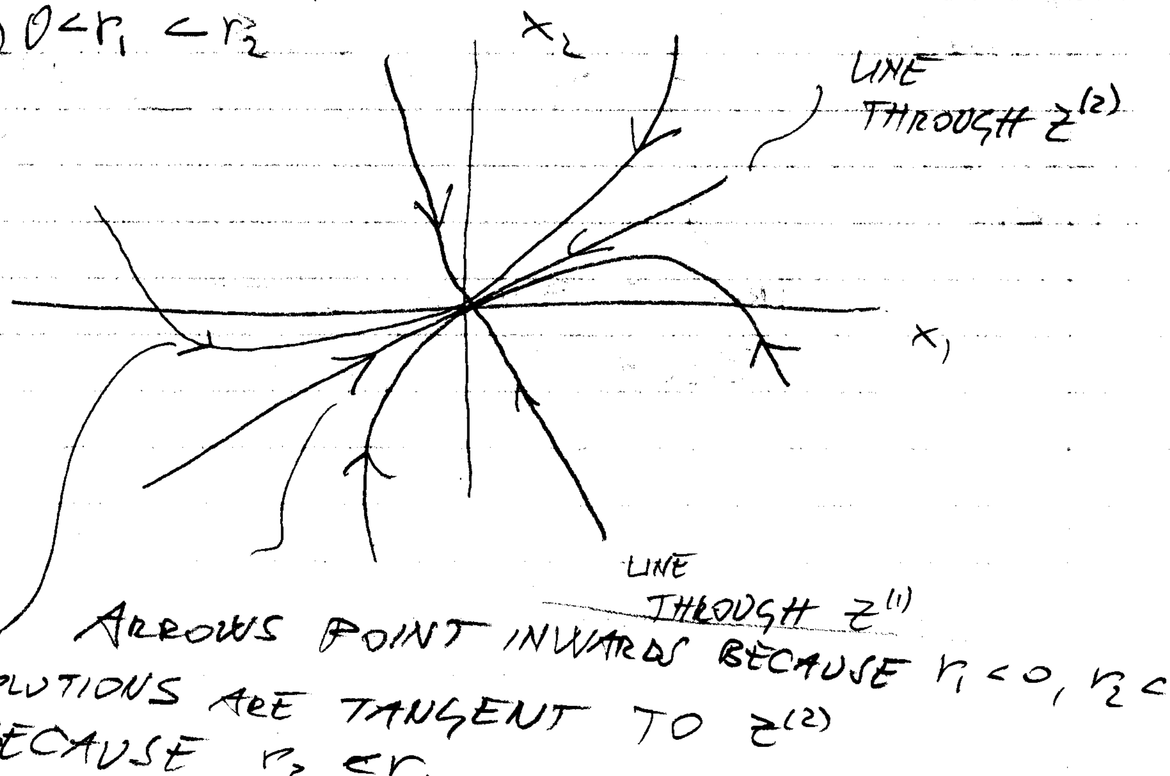
2.) $r_1 > r_2 > 0$



ALL THE ARROWS POINT OUT BECAUSE $r_1 > 0$, $r_2 > 0$

TRAJECTORIES ARE TANGENT TO $z^{(2)}$ BECAUSE $r_2 < r_1$

3.) $0 < r_1 < r_2$



ARROWS POINT INWARDS BECAUSE $r_1 < 0$, $r_2 < 0$

SOLUTIONS ARE TANGENT TO $z^{(2)}$ BECAUSE $r_2 < r_1$

EXAMPLE: $\dot{\underline{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \underline{x}$

$$\det \begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = (3-r)(-2-r)+4 =$$

$$= -6 - r + r^2 + 4 =$$

$$= r^2 - r - 2 = (r-2)(r+1)$$

$r_1 = -1, \quad r_2 = 2$

$r_1 = -1$: $\quad \underline{z}^{(1)} = \begin{pmatrix} z_1^{(1)} \\ z_1^{(2)} \end{pmatrix}$

$$4 z_1^{(1)} - 2 z_2^{(1)} = 0$$

$$z_2^{(1)} = 2 z_1^{(1)}$$

$$z_1^{(1)} = 1 \Rightarrow z_2^{(1)} = 2$$

$$\underline{z}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

$r_2 = 2$:

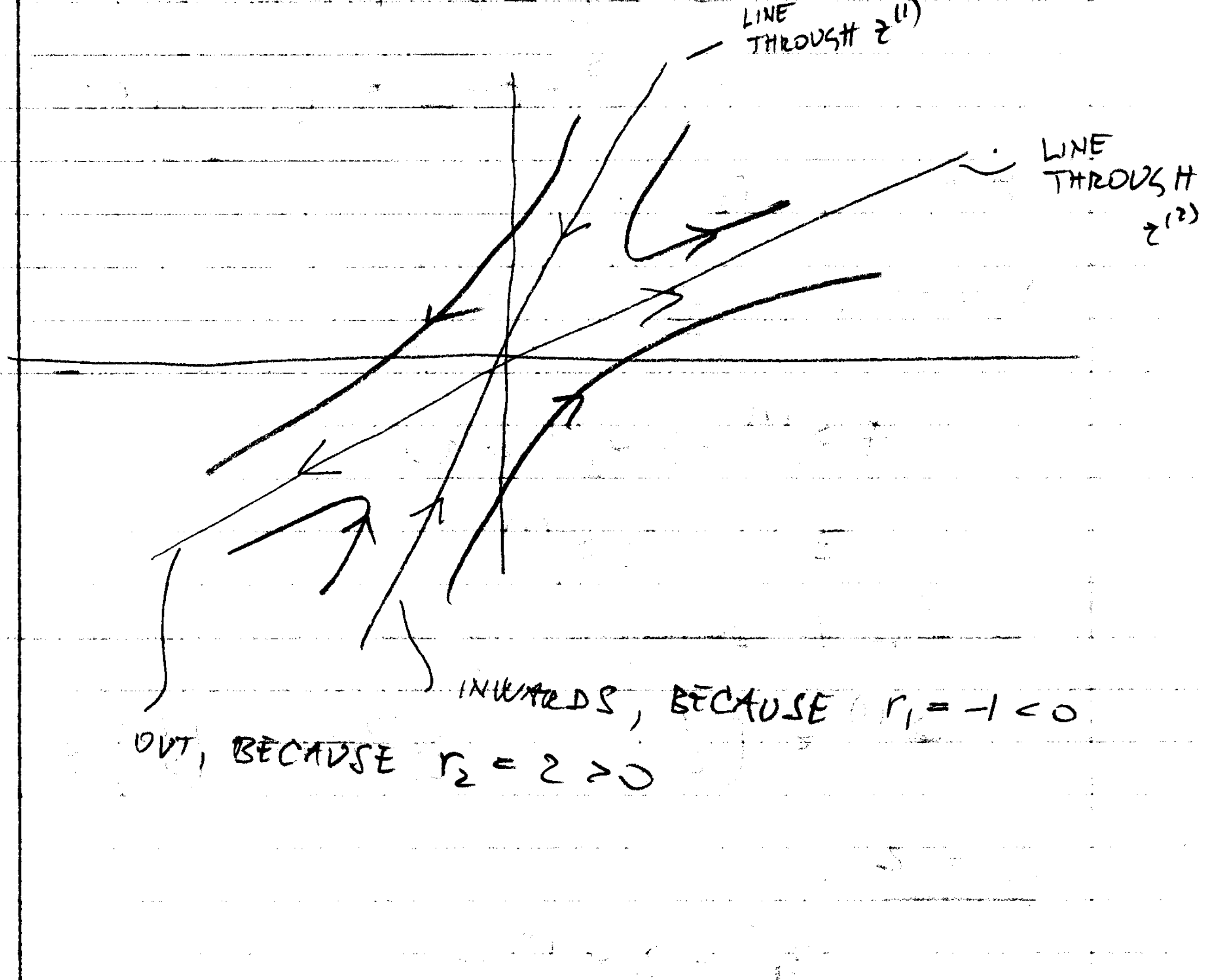
$$z_1^{(2)} - 2 z_2^{(2)} = 0$$

$$z_1^{(2)} = 2 z_2^{(2)}$$

$$z_2^{(2)} = 1 \Rightarrow z_1^{(2)} = 2$$

$$\underline{z}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \underline{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$



INWARDS, BECAUSE $r_1 = -1 < 0$

OUT, BECAUSE $r_2 = 2 > 0$

COMPLEX EIGENVALUES

$$\dot{\underline{x}} = A\underline{x}$$

$$\underline{x}(t) = \underline{z}\, e^{rt} \Rightarrow (A - rI)\underline{z} = 0$$

Let A be real, $r = \lambda + i\mu$

$$\Rightarrow 0 = \left[(A - rI)\underline{z} \right]^* = (A^* - r^* I^*)\,\underline{z}^* =$$

$$= (A - r^* I)\,\underline{z}^*$$

$\Rightarrow$ IF $r = \lambda + i\mu$ IS AN EIGENVALUE OF A AND $\underline{z}$ THE CORRESPONDING EIGENVECTOR, THEN $r^* = \lambda - i\mu$ IS ALSO AN EIGENVALUE OF A AND $\underline{z}^*$ IS THE CORRESPONDING EIGENVECTOR

IF $\underline{z} = \underline{a} + i\underline{b}$ ($\underline{a}, \underline{b}$ - real)

$$\Rightarrow \underline{z}^* = \underline{a} - i\underline{b}$$

SOLUTIONS:

$\lambda+i\mu$: $\underline{x}^{(1)}(t) = (\underline{a}+i\underline{b})\, e^{(\lambda+i\mu)t} =$

$= e^{\lambda t}(\underline{a}+i\underline{b})(\cos\mu t + i\sin\mu t)$

$= e^{\lambda t}(\underline{a}\cos\mu t - \underline{b}\sin\mu t)$

$+ i\, e^{\lambda t}(\underline{b}\cos\mu t + \underline{a}\sin\mu t)$

$\lambda-i\mu$: $[\underline{x}^{(1)}(t)]^* =$ (change i to $-i$ in $\underline{x}(t)$)

REAL SOLUTIONS

$\underline{u}(t) = e^{\lambda t}(\underline{a}\cos\mu t - \underline{b}\sin\mu t)$

$\underline{v}(t) = e^{\lambda t}(\underline{a}\sin\mu t + \underline{b}\cos\mu t)$

RULE: $\dot{\underline{x}} = A\underline{x}$

$r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, $\underbrace{r_3, r_4, \ldots, r_n}_{\text{REAL, DISTINCT}}$

$(A - r_1 I)\underline{z}^{(1)} = 0$: FIND $\underline{z}^{(1)} = \underline{a} + i\underline{b}$

Let $\underline{u}(t)$ AND $\underline{v}(t)$ AS ABOVE

FOR $i = 3, 4, \ldots, n$, FIND

$\underline{z}^{(3)}, \ldots, \underline{z}^{(n)} \Rightarrow$

$\Rightarrow \underline{x}^{(3)}(t) = \underline{z}^{(3)} e^{r_3 t}, \ldots, \underline{x}^{(n)}(t) = \underline{z}^{(n)} e^{r_n t}$

GENERAL SOLUTION:

$$\underline{x}(t) = c_1 \underline{u}(t) + c_2 \underline{v}(t) + $$
$$+ c_3 \underline{z}^{(3)} e^{r_3 t} + \ldots + c_n \underline{z}^{(n)} e^{r_n t}.$$

EXAMPLE: $\underline{\dot{x}} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \underline{x}$

$\underline{x} = \underline{z} e^{rt}$

$\Rightarrow \begin{bmatrix} -\frac{1}{2}-r & 1 \\ -1 & -\frac{1}{2}-r \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$

$\det \begin{pmatrix} -\frac{1}{2}-r & 1 \\ -1 & -\frac{1}{2}-r \end{pmatrix} =$

$= \left(-\frac{1}{2}-r\right)^2 + 1 =$

$= \frac{1}{4} + r + r^2 + 1 = r^2 + r + \frac{5}{4}$

$r_1, r_2 = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i$

$\underline{r_1 = -\frac{1}{2} + i}$

$\begin{bmatrix} -\frac{1}{2} - (-\frac{1}{2}+i) & 1 \\ -1 & -\frac{1}{2} - (-\frac{1}{2}+i) \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0$

$$-i z_1 + z_2 = 0$$

(2^{nd} EQ = $(-i)$ TIMES 1^{st})

$$z_2 = i z_1$$

$$z_1 = 1 \Rightarrow z_2 = i$$

$$\underline{z} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\underline{b}}$$

$$\underline{u}(t) = e^{-\frac{1}{2}t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right]$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

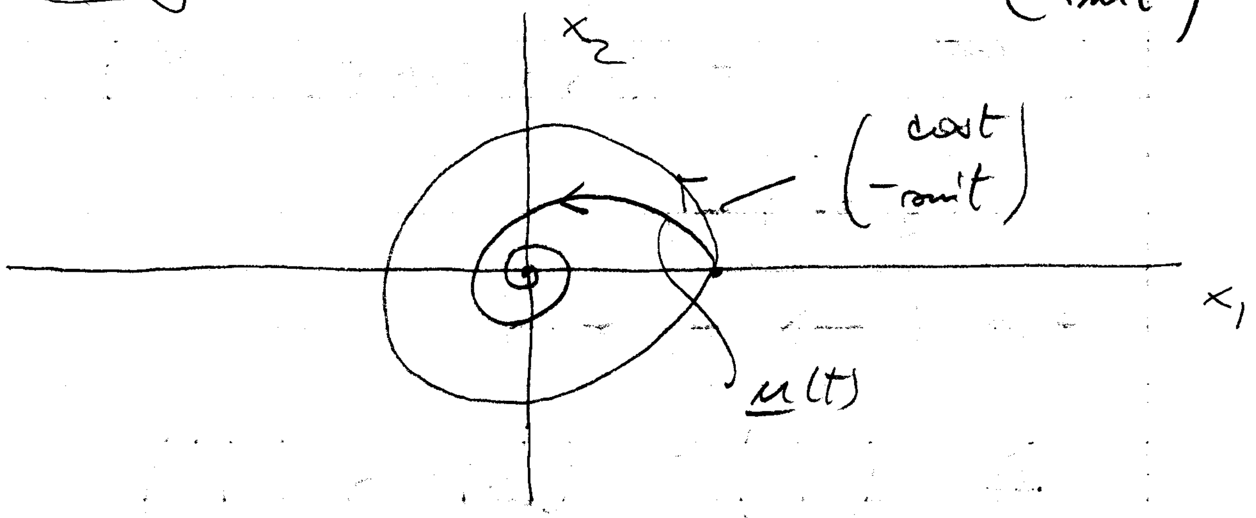
$$\underline{v}(t) = e^{-\frac{1}{2}t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right]$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

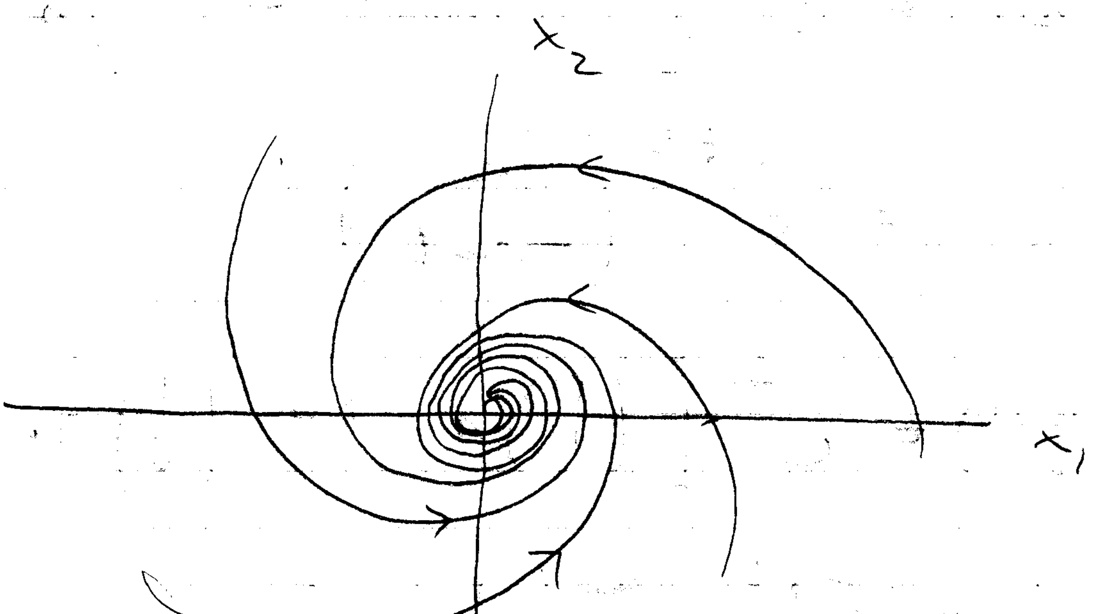
$$\underline{x}(t) = e^{-\frac{1}{2}t}\left[c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$

$$= e^{-\frac{1}{2}t}\begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

TRAJECTORIES: $\underline{u}(t) = e^{-\frac{t}{2}}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$



ALL OTHER TRAJECTORIES ARE SIMILAR



$$\dot{\underline{x}} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \underline{x}$$

$$\underline{x} = \underline{z} e^{rt} \Rightarrow \begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix} = (-1-r)^2 + 2 =$$

$$= 1 + 2r + r^2 + 2 = r^2 + 2r + 3 = 0$$

$$r_1, r_2 = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm i\sqrt{2}$$

$$\begin{pmatrix} -1-(-1+i\sqrt{2}) & -1 \\ 2 & -1-(-1+i\sqrt{2}) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -i\sqrt{2} & -1 \\ 2 & -i\sqrt{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

$$-i\sqrt{2}\, z_1 - z_2 = 0$$

$$z_2 = -i\sqrt{2}\, z_1$$

$$z_1 = 1 \Rightarrow z_2 = -i\sqrt{2}$$

$$\underline{z} = \begin{pmatrix} 1 \\ -i\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}$$

$$\underline{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}$$

$$\underline{u}(t) = e^{-t}\left[\begin{pmatrix}1\\0\end{pmatrix}\cos\sqrt{2}\,t - \begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\sin\sqrt{2}\,t\right]$$

$$= e^{-t}\begin{pmatrix}\cos\sqrt{2}\,t\\ \sqrt{2}\sin\sqrt{2}\,t\end{pmatrix}$$

$$\underline{v}(t) = e^{-t}\left[\begin{pmatrix}1\\0\end{pmatrix}\sin\sqrt{2}\,t + \begin{pmatrix}1\\-\sqrt{2}\end{pmatrix}\cos\sqrt{2}\,t\right]$$

$$= e^{-t}\begin{pmatrix}\sin\sqrt{2}\,t\\ -\sqrt{2}\cos\sqrt{2}\,t\end{pmatrix}$$

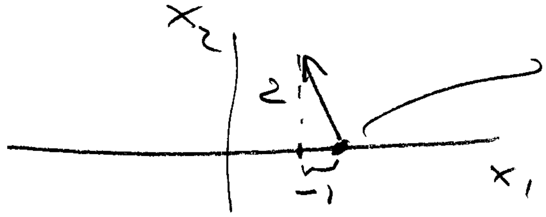
$$\underline{x}(t) = c_1 e^{-t}\begin{pmatrix}\cos\sqrt{2}\,t\\ \sqrt{2}\sin\sqrt{2}\,t\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}\sin\sqrt{2}\,t\\ -\sqrt{2}\cos\sqrt{2}\,t\end{pmatrix}$$

$$= e^{-t}\begin{pmatrix}c_1\cos\sqrt{2}\,t + c_2\sin\sqrt{2}\,t\\ \sqrt{2}\,c_1\sin\sqrt{2}\,t + \sqrt{2}\,c_2\cos\sqrt{2}\,t\end{pmatrix}$$

SKETCH TRAJECTORIES

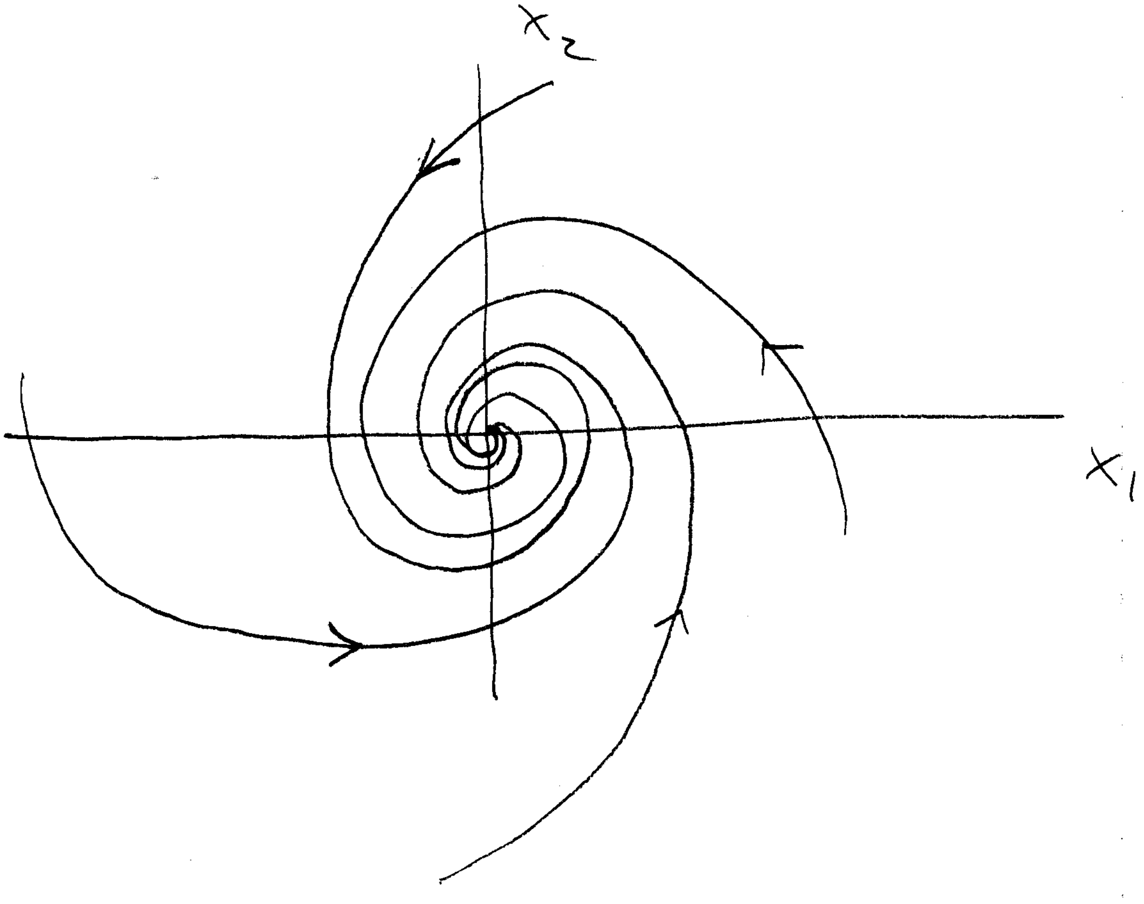
COMPLEX EIGENVALUES ALWAYS PRODUCE SPIRALS

$e^{-t} \Rightarrow$ CONTRACTING SPIRAL



look AT $\dot{\underline{x}}$ AT $\underline{x} = (1,0)$

$$\dot{\underline{x}} = \begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}-1\\ 2\end{pmatrix}$$



PLOTTING TRAJECTORIES OF 2x2 SYSTEMS

COMPLEX EIGENVALUES

$\dot{\underline{x}} = A\underline{x}$

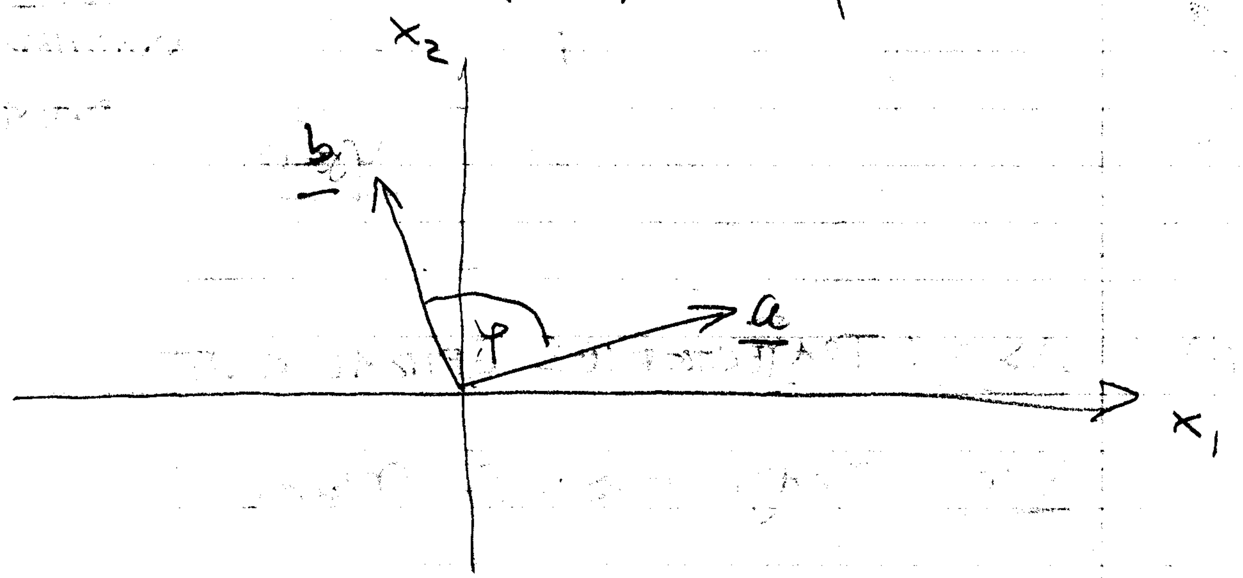
$\underline{x} = \underline{z}e^{rt} \Rightarrow (A - rI)\underline{z} = 0$

$r = \lambda \pm i\mu \Rightarrow \underline{z} = \underline{a} \pm i\underline{b} \qquad (\mu > 0)$

SOLUTION:

$\underline{x}(t) = c_1 e^{\lambda t}(\underline{a}\cos\mu t - \underline{b}\sin\mu t) +$
$\qquad + c_2 e^{\lambda t}(\underline{a}\sin\mu t + \underline{b}\cos\mu t)$

LOOK AT $\underline{a}\cos\mu t - \underline{b}\sin\mu t$



φ = ANGLE $< 180°$

LECTURE 16

PHASE PLANE

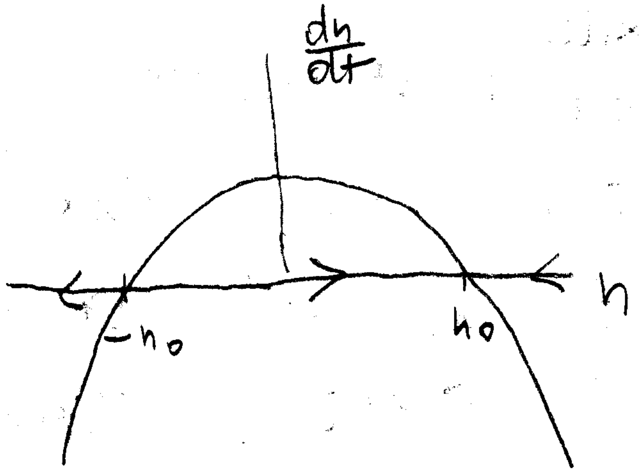
$$\frac{dx}{dt} = Ax \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

SOLUTION $x = z\, e^{rt}$

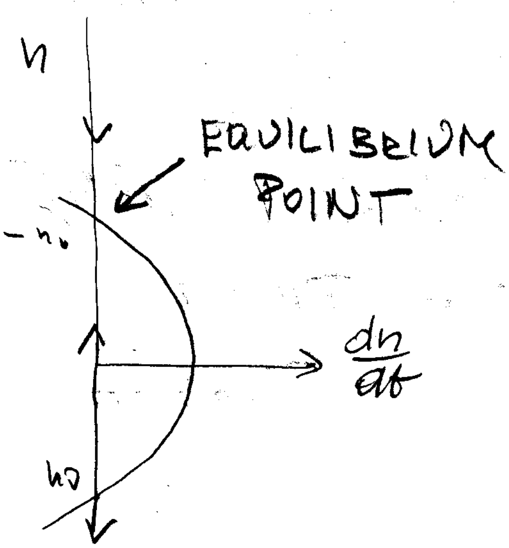
$$\Rightarrow (A - rI) z = 0$$

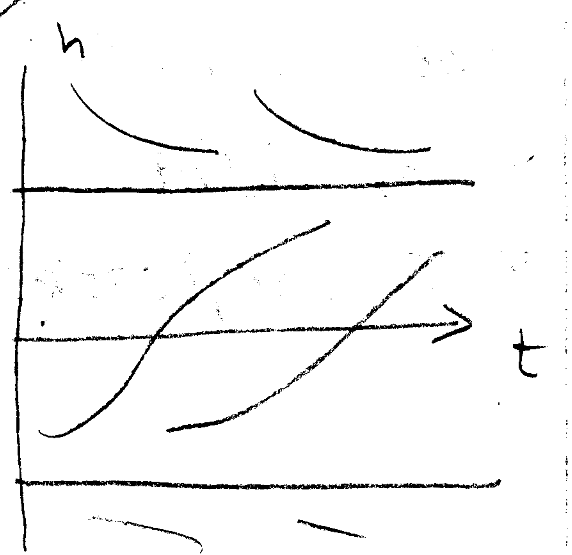
r - EIGENVALUE, z - EIGENVECTOR

$$\frac{dn}{dt} = \lambda (n_0^2 - n^2)$$



$$\boxed{\frac{dn}{dt} = \lambda (n_0^2 - n^2) = 0}$$





$$\frac{dx}{dt} = Ax \qquad (*)$$

EQUILIBRIUM POINTS = CRITICAL POINTS :

$$\frac{dx}{dt} = 0 = Ax$$

IF $\det A \neq 0 \Rightarrow x = 0$

$x(t) = 0$ – EQUILIBRIUM SOLUTION

$$\big((x_1(t), x_2(t)) = (0,0) \big)$$

LET $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ BE A SOLUTION

OF $(*)$. THEN $x(t)$ CAN

BE THOUGHT OF AS A PATH

IN THE x_1–x_2 PLANE, TRAVERSED

BY A MOVING PARTICLE WITH

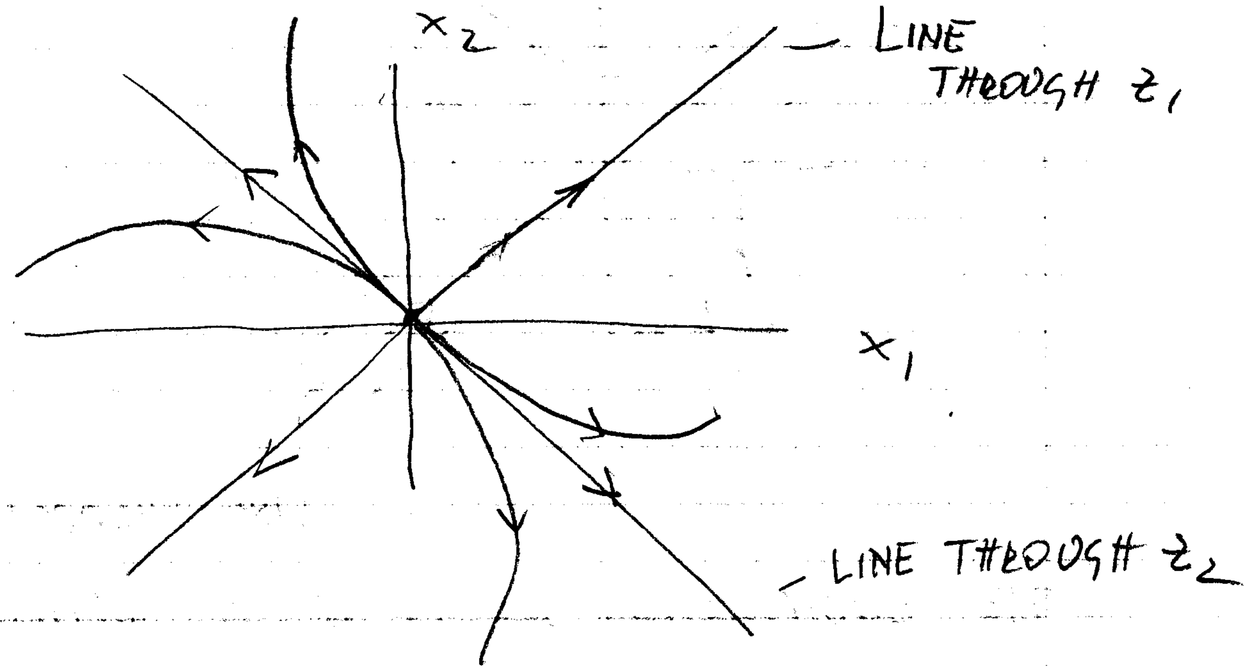
THE VELOCITY $\frac{dx}{dt} = Ax$.

$x_1 - x_2$ PLANE = PHASE PLANE

SET OF TRAJECTORIES = PHASE PORTRAIT

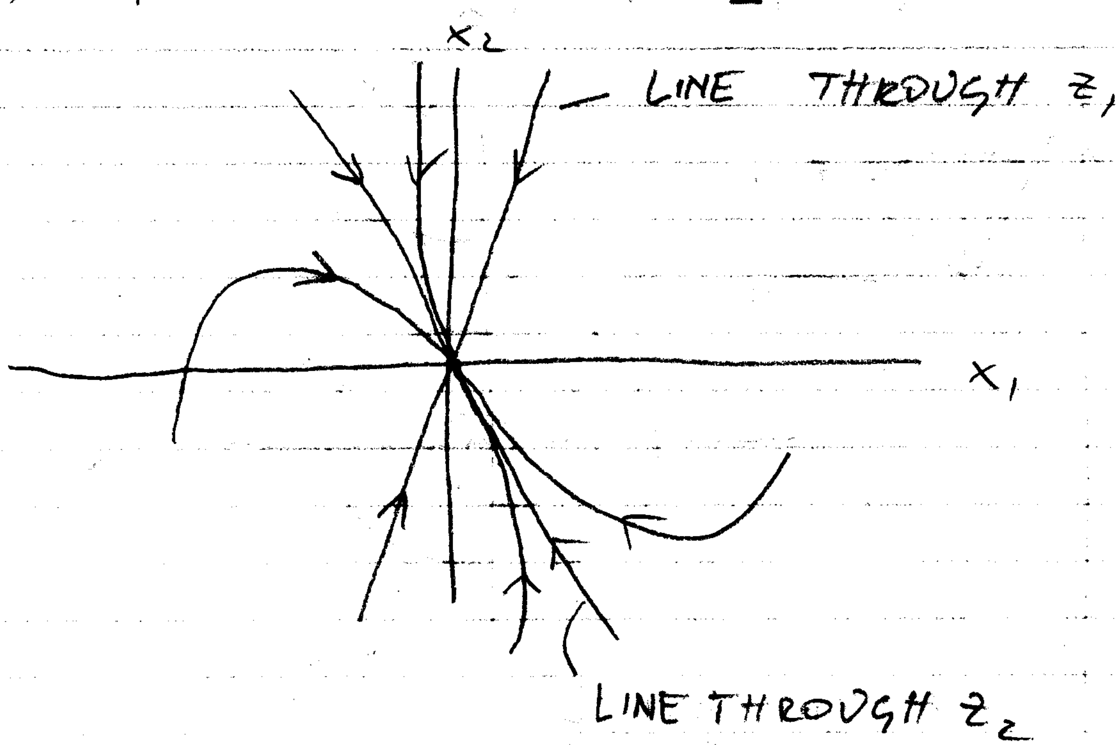
CASES:

1.) EIGENVALUES: $r_1 > r_2 > 0$



THE EQUILIBRIUM AT 0 IS A SOURCE.

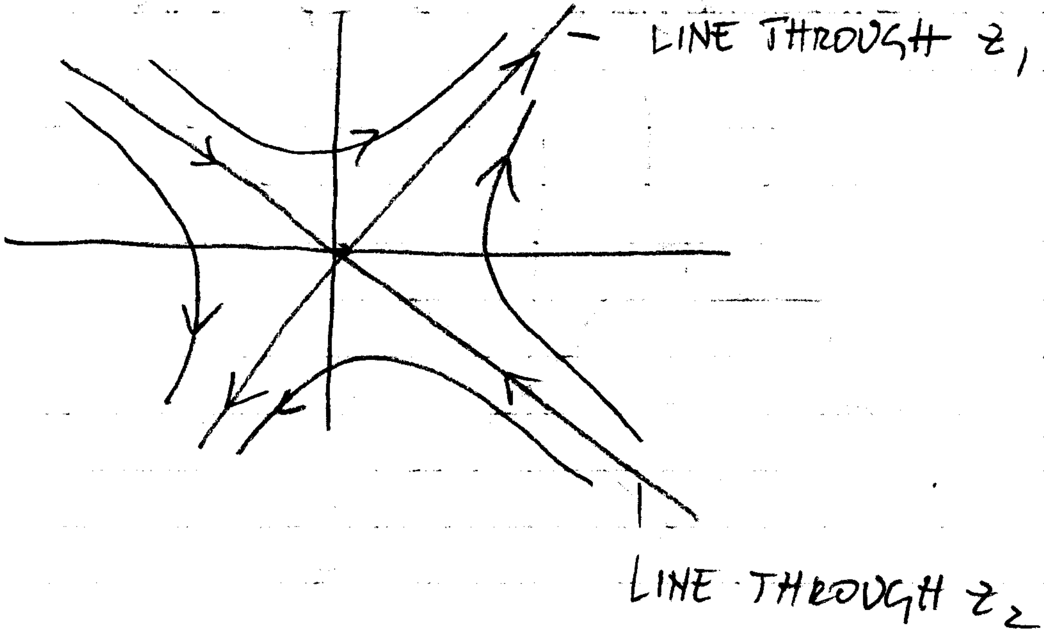
2.) EIGENVALUES $r_1 < r_2 < 0$



THE EQUILIBRIUM AT 0 IS A SINK.

SINK OR SOURCE = NODE

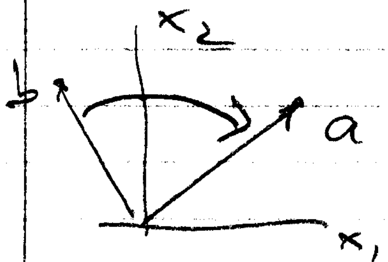
3.) EIGENVALUES $r_1 > 0 > r_2$

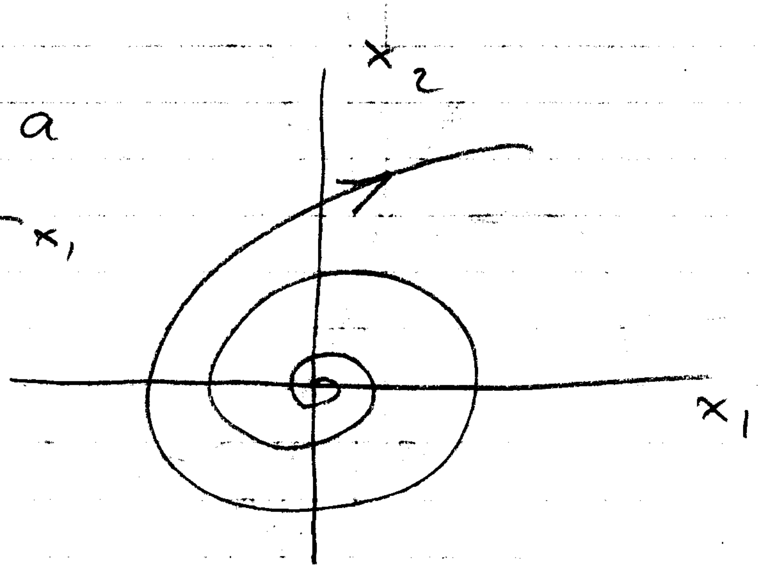


THE EQUILIBRIUM AT 0 IS A SADDLE.

4.) EIGENVALUES $\lambda \pm i\mu$, $\lambda > 0$

EIGENVECTORS $a \pm ib$

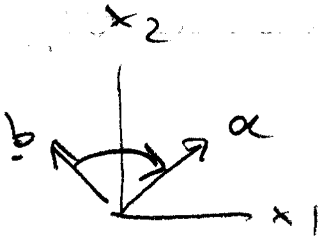


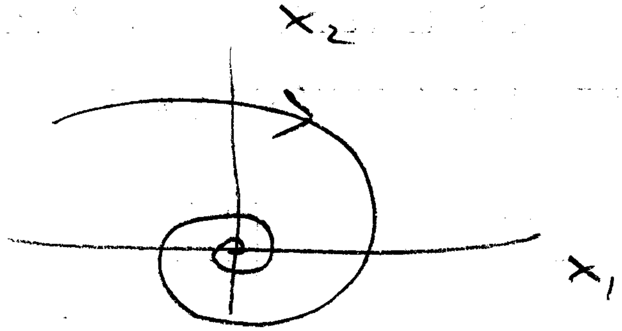


SPIRAL SOURCE

5.) EIGENVALUES $\lambda \pm i\mu$, $\lambda < 0$

EIGENVECTORS $a \pm ib$



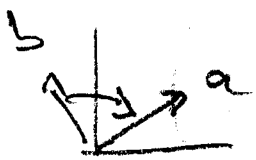


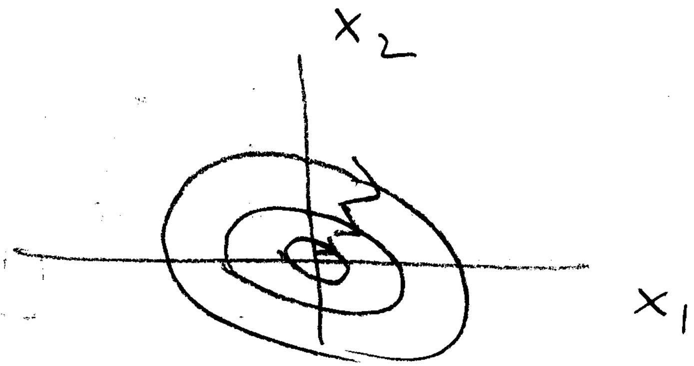
<u>SPIRAL SINK</u>

SPIRAL SOURCE OR SPIRAL SINK = SPIRAL POINT

6.) EIGENVALUES $\pm i\mu$

EIGENVECTORS $a \pm ib$



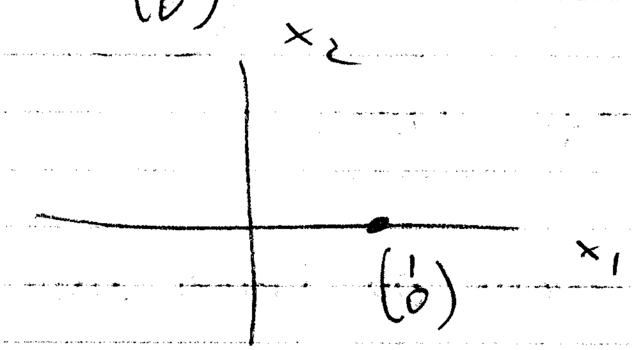


TRAJECTORIES ARE CLOSED CURVES

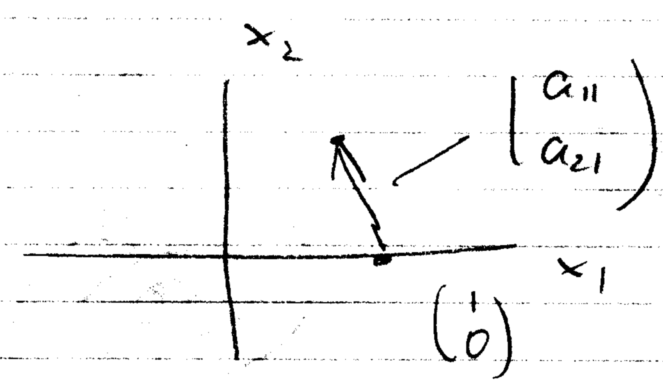
<u>CENTER</u>

EASY WAY OF FINDING THE DIRECTION (CLOCKWISE OR COUNTER CLOCKWISE) OF SPIRALING WITHOUT KNOWING THE EIGENVECTORS

LET $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$$\frac{dx}{dt} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

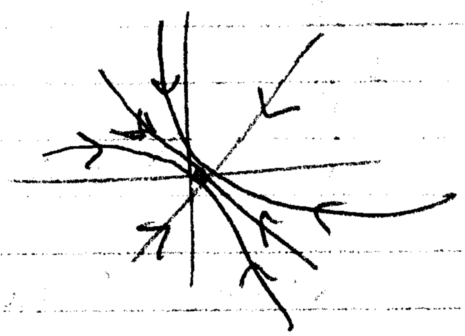


PUT THE TAIL OF THE VECTOR $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ AT THE POINT $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. THE DIRECTION OF THE SPIRALING IS THE SAME AS THE DIRECTION OF THIS VECTOR.

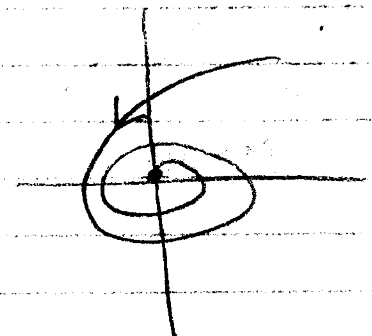
STABILITY:

- AN EQUILIBRIUM IS <u>STABLE</u> IF SOLUTIONS THAT START NEAR IT STAY NEAR IT.

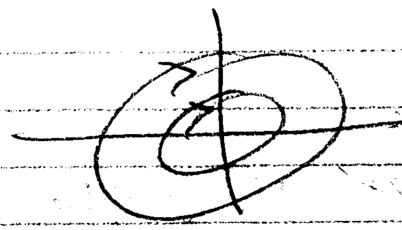
EXAMPLES:



SINK



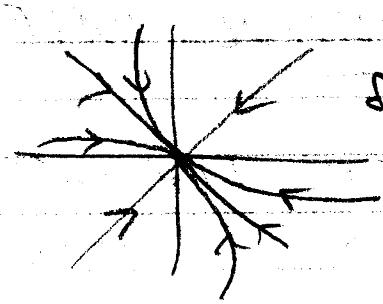
SPIRAL SINK



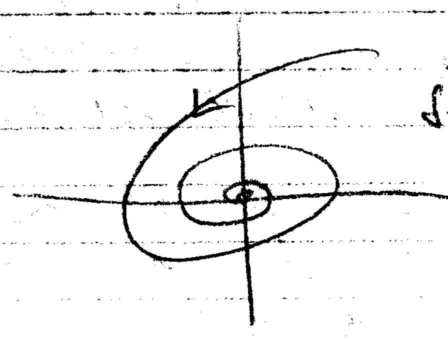
CENTER

- AN EQUILIBRIUM IS ASYMPTOTICALLY STABLE IF SOLUTIONS THAT START NEAR IT APPROACH IT

EXAMPLES:



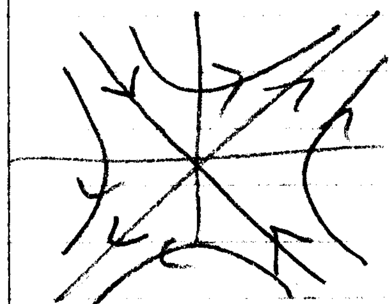
SINK



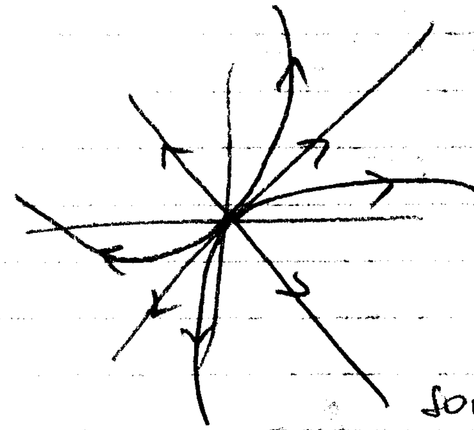
SPIRAL SINK

- AN EQUILIBRIUM IS UNSTABLE IF SOLUTIONS THAT START NEAR IT GO AWAY FROM IT

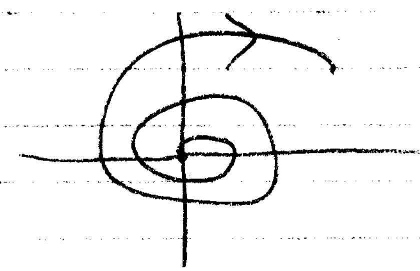
EXAMPLES:



SADDLE



SOURCE



SPIRAL SOURCE

$\dot{x} = Ax$; $x=0$ IS ASYMPTOTICALLY STABLE IF BOTH EIGENVALUES OF A ARE NEGATIVE ~~OR ARE COMPLEX WITH NEGATIVE REAL PART~~

$x=0$ IS UNSTABLE IF A HAS AT LEAST ONE EIGENVALUE WHICH IS POSITIVE OR ITS COMPLEX EIGENVALUES HAVE A POSITIVE REAL PART

$x=0$ IS STABLE IF THE EIGENVALUES OF A HAVE NEGATIVE OR 0 REAL PARTS.

AUTONOMOUS SYSTEMS AND STABILITY:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$$ – RIGHT-HAND SIDE DOES NOT INCLUDE t.

E.g.

$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

NOTE: WE CAN SKETCH TRAJECTORIES IN THE PHASE PLANE ONLY FOR AUTONOMOUS SYSTEMS.

EQUILIBRIA: $\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0$

$$f_1(x_1, x_2) = 0$$

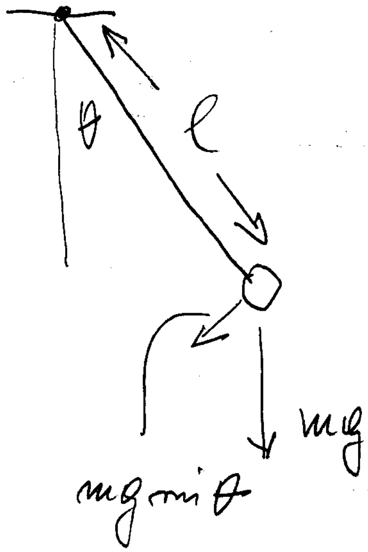
$$f_2(x_1, x_2) = 0$$

EQUILIBRIUM IS:

- STABLE, IF SOLUTIONS THAT START NEAR IT STAY NEAR IT (SINKS, CENTERS)
- ASYMPTOTICALLY STABLE, IF SOLUTIONS THAT START NEAR IT APPROACH IT
- UNSTABLE, IF SOLUTIONS TEND AWAY FROM IT.

(p. 444)

OSCILLATING PENDULUM



NEWTON'S LAW:

$$m \ell \ddot{\theta} = - mg \sin\theta$$

$$\ddot{\theta} = - \frac{g}{\ell} \sin\theta$$

ADD DAMPING: $\ddot{\theta} = - \alpha \dot{\theta} - \frac{g}{\ell} \sin\theta$

(Book: $\alpha = \frac{c}{m\ell}$)

$$\ddot{\theta} + \alpha \dot{\theta} + \frac{g}{\ell} \sin\theta = 0$$

LET $x = \theta$, $y = \frac{d\theta}{dt}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = - \alpha y + \frac{g}{\ell} \sin x \end{cases}$$

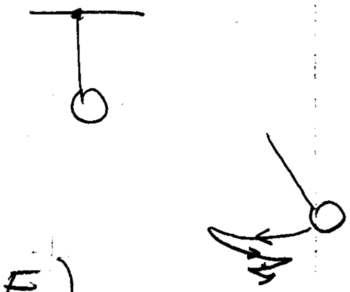
EQUILIBRIA: $y = 0$

$$- \alpha y + \frac{g}{\ell} \sin x = 0$$

$\Rightarrow y = 0, \quad x = n\pi$

$y = 0, \quad x = 0, \pm 2\pi, \pm 4\pi, \ldots$

CORRESPOND TO



THIS EQUILIBRIUM IS STABLE

(EVEN ASYMPTOTICALLY STABLE)

$y = 0, \quad x = \pm\pi, \pm 3\pi, \ldots$

CORRESPOND TO

THIS EQUILIBRIUM IS UNSTABLE:

SHOW DIR FIELD

NONLINEAR SYSTEMS NEAR EQUILIBRIA:

$$\dot{x} = F(x,y) \qquad \dot{y} = G(x,y) \qquad (*)$$

LET $x = x_0, \; y = y_0$ BE AN EQUILIBRIUM FOR $(*)$.

TAYLOR-EXPAND $(*)$ ABOUT (x_0, y_0)

$$\frac{dx}{dt} = \frac{d(x-x_0)}{dt} = F(x_0,y_0) +$$

$$+ F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

$+$ TERMS QUADRATIC IN $(x-x_0)$ AND $(y-y_0)$

$$= F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \text{HIGHER ORDER TERMS}$$

$$\frac{dy}{dt} = G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0)$$

$+$ HIGHER ORDER TERMS

$$\begin{pmatrix} \frac{d(x-x_0)}{dt} \\ \frac{d(y-y_0)}{dt} \end{pmatrix} \approx \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

LINEARIZATION OF (*) AROUND (x_0,y_0)

A NONLINEAR SYSTEM NEAR AN EQUILIBRIUM LOOKS APPROXIMATELY LIKE A LINEAR SYSTEM. ARE THE SOLUTIONS OF THE TWO SYSTEMS SIMILAR, TOO?

LET $u = x - x_0$, $v = y - y_0$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \qquad (\square)$$

$$A = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}$$

IF THE EIGENVALUES OF A ARE BOTH REAL AND NONZERO, OR BOTH COMPLEX AND HAVE NONZERO REAL PARTS, THEN THE SOLUTIONS OF $(*)$ NEAR (x_0, y_0) LOOK SIMILAR TO THE SOLUTIONS OF $(\square)$. IN PARTICULAR (x_0, y_0) IS A STABLE (UNSTABLE) EQUILIBRIUM OF $(*)$ IF $(0,0)$ IS A STABLE (UNSTABLE) EQUILIBRIUM OF $(\square)$.

IF AN EIGENVALUE OF A IS ZERO OR THE EIGENVALUES OF A ARE PURELY IMAGINARY, THEN $(\square)$ CAN NOT TELL US ANYTHING ABOUT $(*)$

Both: $\alpha = \frac{c}{ml}$

PENDULUM

$$\dot{x} = y$$
$$\dot{y} = -\alpha y - \frac{g}{l}\sin x$$

CALCULATE A:

$$A = \begin{bmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial(-\alpha y - \frac{g}{l}\sin x)}{\partial x} & \frac{\partial(-\alpha y - \frac{g}{l}\sin x)}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x & -\alpha \end{bmatrix}$$

EQUILIBRIA: $x = \pi \quad y = 0$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\alpha \end{bmatrix}$$

FIND EIGENVALUES:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{l} & -\alpha-\lambda \end{vmatrix} =$$

$$= \lambda(\alpha+\lambda) - \frac{g}{l} = \lambda^2 + \alpha\lambda - \frac{g}{l} = 0$$

$$\lambda = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \frac{g}{l}}$$

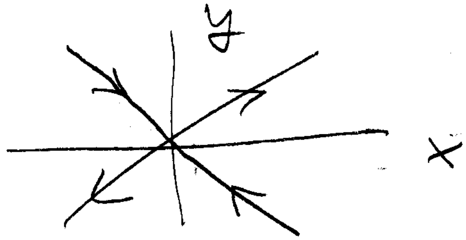
Now $\frac{\alpha^2}{4} + \frac{g}{\ell} > \frac{\alpha^2}{4}$

$$\Rightarrow \sqrt{\frac{\alpha^2}{4} + \frac{g}{\ell}} > \frac{\alpha}{2}$$

$$\Rightarrow \lambda_1 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \frac{g}{\ell}} > 0$$

$$\lambda_2 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \frac{g}{\ell}} < 0$$

$$\Rightarrow \lambda_1 > 0, \quad \lambda_2 < 0 \quad \Rightarrow \quad \text{SADDLE}$$



$x = 0, \ y = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\alpha \end{bmatrix}$$

FIND EIGENVALUES:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{\ell} & -\alpha - \lambda \end{vmatrix} =$$

$$= \lambda(\lambda + \alpha) + \frac{g}{\ell} = \lambda^2 + \alpha\lambda + \frac{g}{\ell} = 0$$

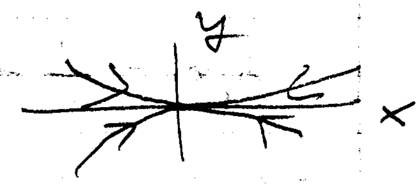
$$\lambda = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - \frac{g}{\ell}}$$

IF $\frac{\alpha^2}{4} - \frac{g}{\ell} > 0$,

THEN

$$\left.\begin{aligned} \lambda_1 &= -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \frac{g}{\ell}} \\ \lambda_2 &= -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \frac{g}{\ell}} \end{aligned}\right\} \text{REAL}, < 0$$

$\Rightarrow$ SINK

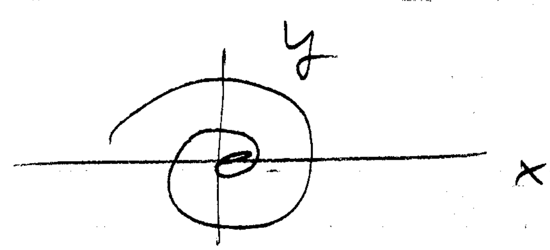


IF $\frac{\alpha^2}{4} - \frac{g}{\ell} < 0$,

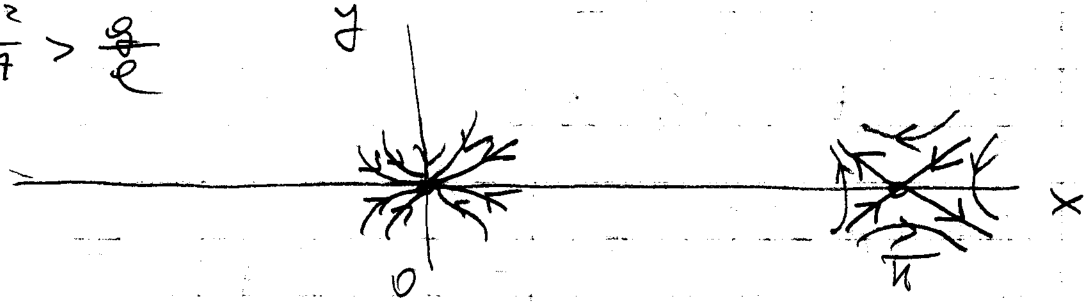
THEN

$$\lambda_{1,2} = -\frac{\alpha}{2} \pm i\sqrt{\frac{g}{\ell} - \frac{\alpha^2}{4}}$$

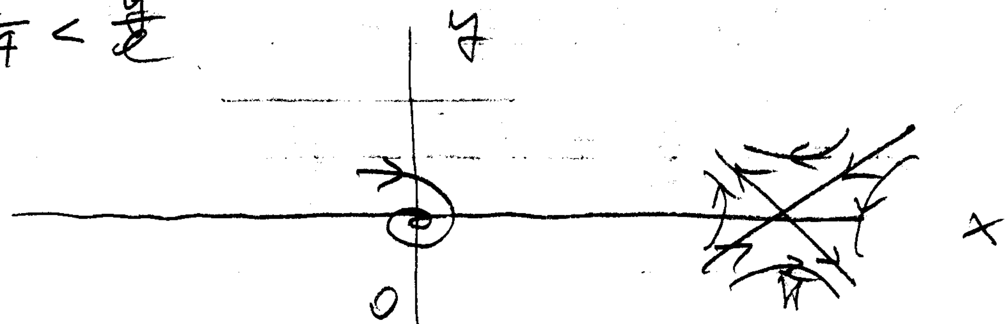
$\Rightarrow$ SPIRAL SINK



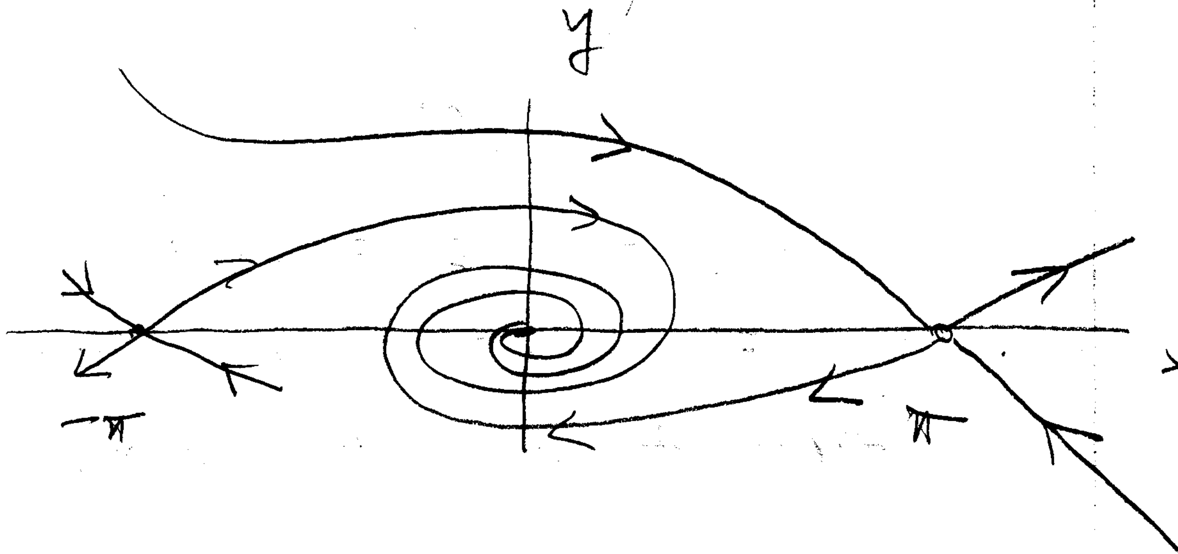
$\frac{\alpha^2}{4} > \frac{g}{\ell}$



$\frac{\alpha^2}{4} < \frac{g}{\ell}$



USE MAPLE'S PHASEPLOT TO GET THE FULL PHASE PORTRAIT



GENERAL RULE:

(*) $$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y)$$

1) FIND THE EQUILIBRIA:

ALL THE POINTS (x_0, y_0) SUCH THAT

$$F(x_0, y_0) = 0, \quad G(x_0, y_0) = 0$$

2.) COMPUTE LINEARIZED SYSTEMS AROUND EQUILIBRIA:

$$(1) \quad \frac{d}{dt}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}}_{A} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

4.) COMPUTE THE EIGENVALUES AND EIGENVECTORS OF THE MATRIX A AND THE SOLUTIONS OF THE SYSTEM (1).

NOW YOU CAN SKETCH TRAJECTORIES OF (*) NEAR THE EQUILIBRIA.

COMPETING SPECIES

TWO SPECIES THAT DO NOT INTERFERE WITH EACH OTHER

$$\dot{x} = x(\varepsilon_1 - \delta_1 x)$$

$$\dot{y} = y(\varepsilon_2 - \delta_2 y)$$

IF THE SPECIES COMPETE FOR FOOD:

$$\dot{x} = x(\varepsilon_1 - \delta_1 x - \alpha_1 y)$$

$$\dot{y} = y(\varepsilon_2 - \delta_2 y - \alpha_2 x)$$

EXAMPLE: $\dot{x} = x(1 - x - y)$

$$\dot{y} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$$

EQUILIBRIA: $x(1 - x - y) = 0$

$$y\left(\frac{3}{4} - y - \frac{1}{2}x\right) = 0$$

$\Rightarrow (0,0), \left(0, \frac{3}{4}\right), (1,0), \left(\frac{1}{2}, \frac{1}{2}\right)$

LINEARIZATION:

$$\frac{d}{dt}\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$= A(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$F(x,y) = x(1 - x - y)$$

$$G(x,y) = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$$

$$F_x(x,y) = 1 - 2x - y, \quad F_y(x,y) = -x$$

$$G_x(x,y) = -\frac{1}{2}y \qquad G_y(x,y) = \frac{3}{4} - 2y - \frac{1}{2}x$$

$$A(x,y) = \begin{bmatrix} 1 - 2x - y & -x \\ -\frac{1}{2}y & \frac{3}{4} - 2y - \frac{1}{2}x \end{bmatrix}$$

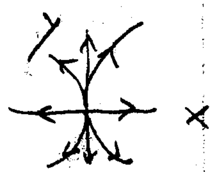
$\underline{x = y = 0}$ $\quad u = x - 0 = x, \quad v = y - 0 = y$

$$A(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

LINEARIZATION: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

EIGENVALUES: $1, \frac{3}{4} \Rightarrow$ SOURCE

EIGENVECTORS: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



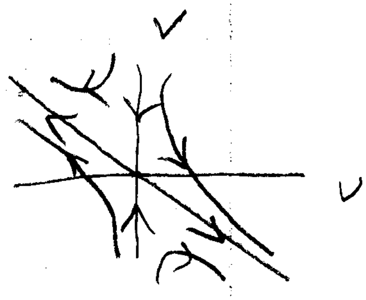
$\underline{x = 0, \ y = \frac{3}{4}}$: $\quad u = x - 0 = x, \quad v = y - \frac{3}{4}$

$$A(0, \tfrac{3}{4}) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix}$$

LINEARIZATION: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

EIGENVALUES: $\frac{1}{4}, -\frac{3}{4} \Rightarrow$ SADDLE

EIGENVECTORS: $\begin{pmatrix} 1 \\ -\frac{3}{8} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



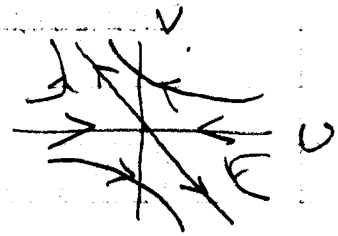
$\underline{x=1,\ y=0}$: $\quad u = x-1, \quad v = y-0 = y$

$$A(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix}$$

LINEARIZATION: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

EIGENVALUES: $-1, \ \frac{1}{4} \Rightarrow$ SADDLE

EIGENVECTORS: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{5}{4} \end{pmatrix}$



$\underline{x=\frac{1}{2},\ y=\frac{1}{2}}$: $\quad u = x-\frac{1}{2}, \quad v = y-\frac{1}{2}$

$$A\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

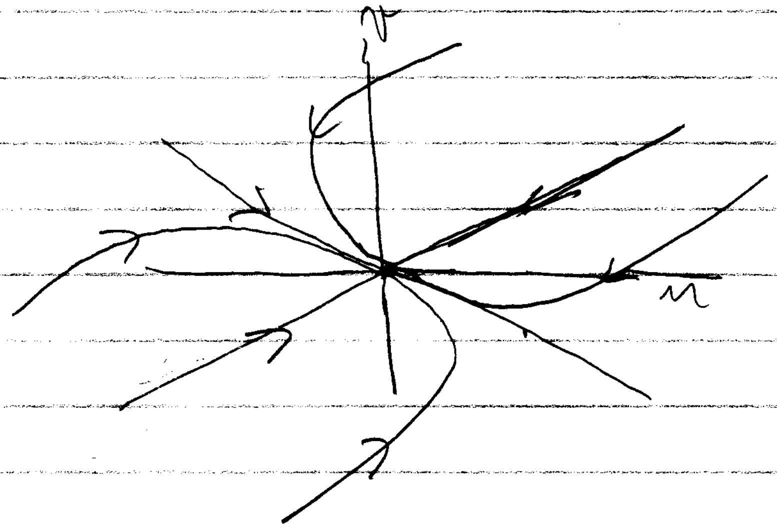
LINEARIZATION: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

EIGENVALUES:

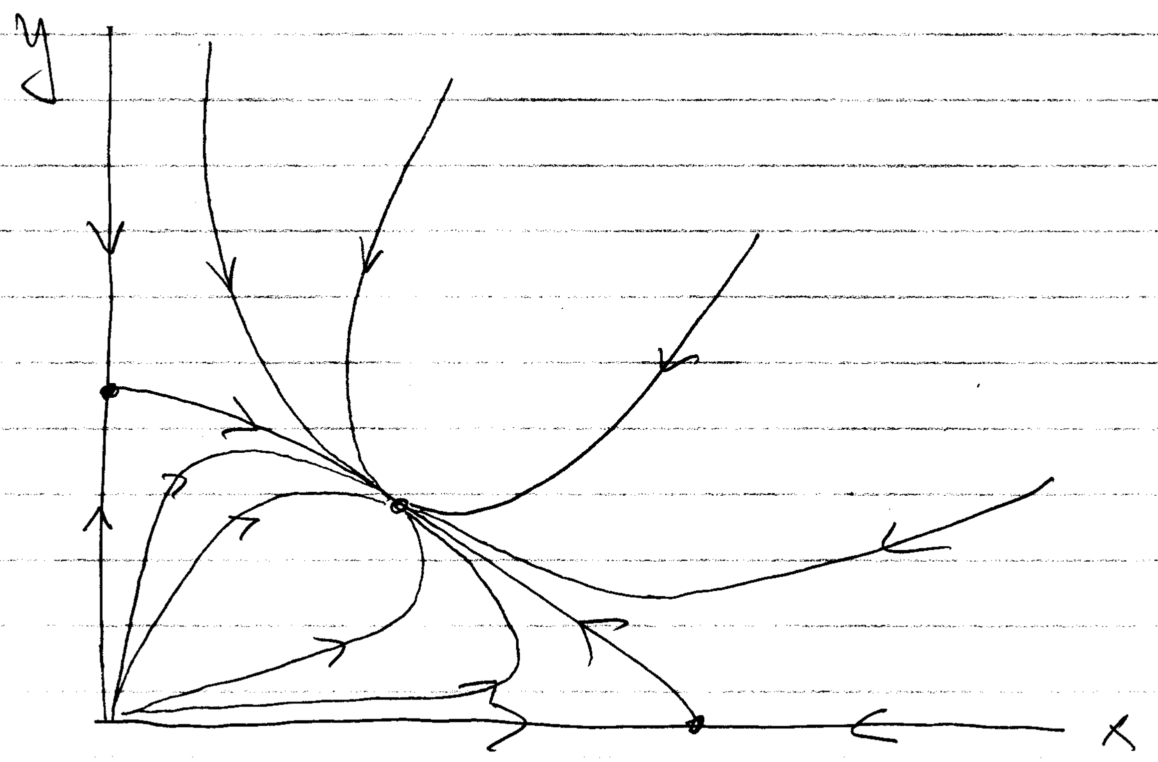
$$\det \begin{bmatrix} -\frac{1}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2}-\lambda \end{bmatrix} = \left(\lambda+\tfrac{1}{2}\right)^2 - \tfrac{1}{8} = 0$$

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2\sqrt{2}} < 0 \Rightarrow \text{SINK}$$

EIGENVECTORS: $\begin{pmatrix}\sqrt{2}\\-1\end{pmatrix}$, $\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$



PUT IT ALL TOGETHER



PREDATOR–PREY MODELS

x – PREY $\quad$ y – PREDATOR

1.) NO PREDATOR : $\dot{x} = ax \qquad a > 0$

2.) NO PREY : $\dot{y} = -cy \qquad c < 0$

3.) INTERACTION PROPORTIONAL TO xy

$$\dot{x} = ax - \alpha xy \qquad \dot{y} = -cy + \gamma xy$$

EXAMPLE :
$$\dot{x} = x\left(1 - \tfrac{1}{2}y\right)$$
$$\dot{y} = y\left(-\tfrac{3}{4} + \tfrac{1}{4}x\right)$$

EQUILIBRIA : $(0,0), (3,2)$

$$\dot{x} = F(x,y) = x\left(1 - \tfrac{1}{2}y\right)$$

$$\dot{y} = G(x,y) = y\left(-\tfrac{3}{4} + \tfrac{1}{4}x\right)$$

$$F_x(x,y) = 1 - \tfrac{1}{2}y \qquad F_y(x,y) = -\tfrac{1}{2}x$$

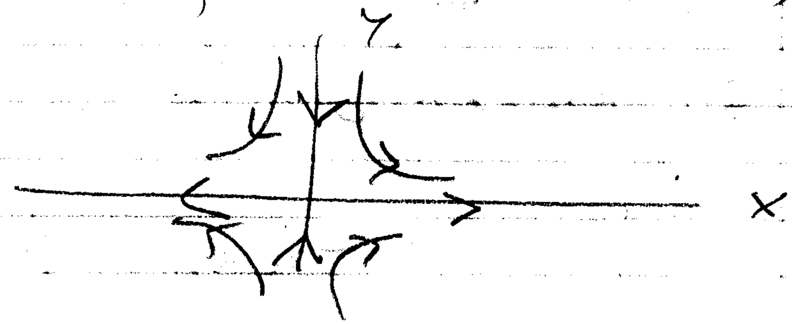
$$G_x(x,y) = \tfrac{1}{4}y \qquad G_y(x,y) = -\tfrac{3}{4} + \tfrac{1}{4}x$$

$\underline{x = y = 0}$

LINEARIZATION: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda_1 = 1, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = -\frac{3}{4}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

SADDLE;



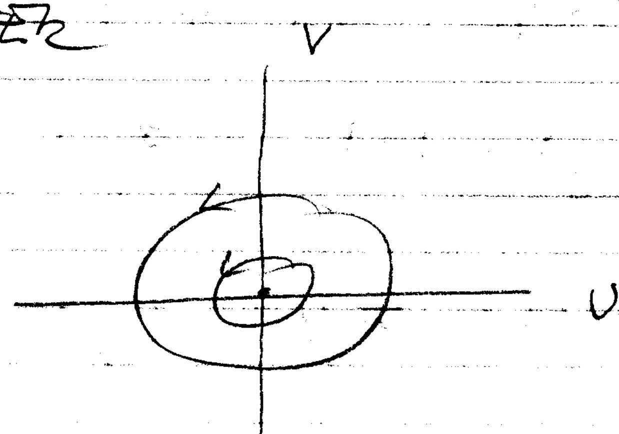
$\underline{x = 3, \quad y = 2}$: $\quad u = x - 3, \quad v = y - 2$

LINEARIZATION: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\underline{\text{EIGENVALUES}}$: $\lambda^2 + \frac{3}{4} = 0, \quad \lambda_{1,2} = \pm \frac{i\sqrt{3}}{2}$

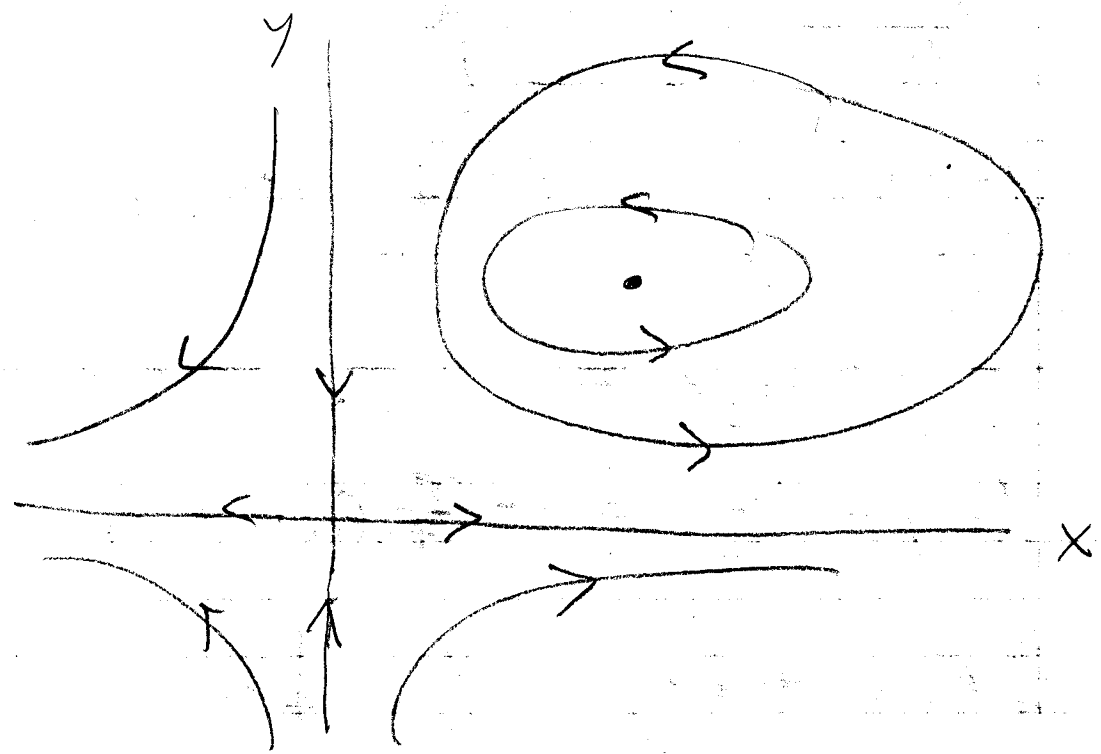
$$e_{1,2} = \begin{pmatrix} 1 \\ \mp \frac{i}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mp i \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

CENTER



(3,2) MAY NOT BE A CENTER FOR THE FULLY NONLINEAR SYSTEM.

HOWEVER, MAPLE INDICATES THE PICTURE:



IT SEEMS THAT (3,2) IS REALLY A CENTER, AND TRAJECTORIES IN THE FIRST QUADRANT ARE CLOSED CURVES.

IF YOU SUSPECT THIS KIND OF A SITUATION, DIVIDE THE $\dot{y}$ EQUATION BY THE x' EQUATION AND TRY TO INTEGRATE THE EQUATION FOR $\frac{dy}{dx}$.

$$\frac{dy}{dx} = y\frac{(-\frac{3}{4}+\frac{1}{4}x)}{x(1-\frac{1}{2}y)}$$

$$\frac{(1-\frac{1}{2}y)}{y}dy = \frac{(-\frac{3}{4}+\frac{1}{4}x)}{x}dx$$

$$\left(\frac{1}{y}-\frac{1}{2}\right)dy = \left(-\frac{3}{4x}+\frac{1}{4}\right)dx$$

$$\ln y - \frac{y}{2} = -\frac{3}{4}\ln x + \frac{x}{4} + C$$

$$\underbrace{\frac{3}{4}\ln x + \ln y - \frac{y}{2} - \frac{1}{4}x}_{\text{CONSTANT OF MOTION}} = C$$

TRAJECTORIES ARE LEVEL-LINES OF THIS FUNCTION;

WHEN YOU HAVE A CONSTANT OF MOTION, CENTERS OF THE LINEARIZATION ARE CENTERS OF THE ORIGINAL NONLINEAR SYSTEM.