89.7

(a) $z_t = \alpha^2 z_{xx}$

$z(0,t) = 0, \quad z(\ell,t) = 60$

$z(x,0) = 25$

STEADY STATE: $v = \frac{60}{\ell} x$

$$z(x,t) = 60\frac{x}{\ell} + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2}{\ell^2}t} \sin\frac{n\pi x}{\ell}$$

$$z(x,0) = 60\frac{x}{\ell} + \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell} = 25$$

$$\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{\ell} = -60\frac{x}{\ell} + 25$$

$$c_n = \frac{2}{\ell}\int_0^{\ell}\left(25 - 60\frac{x}{\ell}\right)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{50}{\ell}\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Big|_0^{\ell} +$$

$$+ \frac{120}{\ell^2}\frac{x\ell}{n\pi}\cos\frac{n\pi x}{\ell}\Big|_0^{\ell} - \frac{120\ x\ell}{\ell^2 n\pi}\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\Big|_0^{\ell}$$

$$c_n = \frac{50}{n\pi}(1-\cos n\pi) + \frac{120}{n\pi}\cos n\pi = \frac{50}{n\pi} + \frac{70}{n\pi}\cos n\pi$$

$$z(x,t) = \frac{60x}{\ell} + \sum_{n=1}^{\infty} \frac{50+70\cos n\pi}{n\pi} e^{-\frac{n^2\pi^2\alpha^2}{\ell^2}t} \sin\frac{n\pi x}{\ell}$$

$$= \frac{60x}{\ell} + \sum_{n=1}^{\infty}\left(\frac{50+70\cos n\pi}{n\pi}\right) e^{-0.86 n^2\pi^2 t/\ell^2} \sin\frac{n\pi x}{\ell}$$

(b) $u(5,30) \approx 12.6°C$

$u(5,60) = 13.7°C$

(c) $u(5,30) \approx 14.1°C$; 12% BETWEEN ONE-TERM AND TWO-TERM APPROX.

THIRD TERM = $-0.005°C$

(d) $t \cong 160$ sec.

90) $u_t = \alpha^2 u_{xx}$ (5)

$u_x(0,t) = u_x(\ell, t)$

$u(x,0) = \sin \frac{\pi x}{\ell}$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2 t}{\ell}} \cos \frac{n\pi x}{\ell}$$

$$c_0 = \frac{2}{\ell}\int_0^\ell \sin\frac{\pi x}{\ell}\,dx = -\frac{2}{\pi}\cos\frac{\pi x}{\ell}\Big|_0^\ell =$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\ell}\int_0^\ell \sin\frac{\pi x}{\ell}\cos\frac{n\pi x}{\ell}\,dx =$$

$$= \frac{2}{\ell}\int_0^\ell \frac{1}{2}\left[\sin\frac{(n+1)\pi x}{\ell} - \sin\frac{(n-1)\pi x}{\ell}\right]dx$$

$$= \frac{1}{\ell}\Big[\frac{\ell}{(n+1)\pi}\cos\frac{(n+1)\pi}{\ell}x\Big|_0^\ell -$$

$$- \frac{\ell}{(n-1)\pi}\cos\frac{(n-1)\pi}{\ell}x\Big|_0^\ell\Big] =$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right]\left(1-(-1)^{n+1}\right)$$

$$= -\frac{1}{\pi}\frac{2}{n^2-1}\left(1-(-1)^{n+1}\right) =$$

$$= \begin{cases} -\dfrac{4}{\pi(n^2-1)} & n \text{ EVEN} \\ 0 & n \text{ ODD} \end{cases}$$

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\frac{4n^2\pi^2\alpha^2 t}{l^2}}}{(4n^2-1)} \cos\frac{2n\pi x}{l}$$

$$u(x, t\to\infty) = \frac{2}{\pi}$$

91) $u_t = \alpha^2 u_{xx}$

$u_x(0,t) = 0$, $u(l,t) = T$

STEADY STATE $u(x,t) = v(x)$

$u_t = 0$

$v_{xx} = 0 \implies v = Ax + B$

$v'(0) = A = 0$

$v(l) = B = T$

STEADY STATE $v(x) = T$

92.) $u_t = \alpha^2 u_{xx}$

$u(0,t) = 30 \qquad u(40,t) = -20$

Steady state: $u(x,t) = v(x)$

$u_t = 0$

$v_{xx} = 0 \Rightarrow v = Ax + B$

$v(0) = B = 30 \qquad v(40) = 40A + 30$

$= -20$

$A = -\frac{5}{4}$

STEADY STATE $\quad v(x) = -\frac{5}{4}x + 30$

93.) $u_t = \alpha^2 u_{xx}$

$u(0,t) = T \quad , \quad u_x(\ell,t) = 0$

STEADY STATE $\quad u(x,t) = v(x)$

$v_{xx} = 0 \Rightarrow v = Ax + B$

$v(0) = B = T \qquad v_x(\ell,t) = A = 0$

STEADY STATE: $\quad v(x) = T$

94.) $u_{xx} - u_x + u_t = 0$

$u(0,t) = 0 \;, \quad u_x(1,t) = e$

STEADY STATE: $u(x,t) = v(x)$

$u_t = 0$

$v_{xx} - v_x = 0$

$v_x - v = C$

1.) $\mu(x) = e^{-\int dx} = e^{-x}$

2.) $\left(e^{-x} v\right)_x = C e^{-x} \Rightarrow e^{-x} v = -C e^{-x} + D$

$v = De^{x} - C$

$v(0) = D - C = 0$

$v'(1) = De = e \Rightarrow D = C = 1$

STEADY STATE

$v(x) = e^{x} - 1$

95.) $u_t = \alpha^2 u_{xx} \quad 0<x<\ell, \quad t>0$ ⑦

$u(0,t)=0, \quad u_x(\ell,t)=0, \quad t>0$

$u(x,0)=f(x)$

$u(x,t) = X(x)\,T(t)$

$X\dot{T} = \alpha^2 X'' T$

$\frac{1}{\alpha^2}\frac{\dot{T}}{T} = \frac{X''}{X} = S$

$X'' - SX = 0 \qquad X(0)=0, \; X'(\ell)=0$

$S>0 \Rightarrow X=0$

$S=0 \qquad X''=0$

$X = Ax+B$

$X(0) = B = 0, \quad X'(\ell) = A = 0$

$X=0$

$S<0 \qquad S=-\lambda^2$

$X''+\lambda^2 X = 0$

$X = A\cos\lambda x + B\sin\lambda x$

$X(0) = A = 0$

$X'(\ell) = \lambda B\cos\lambda\ell = 0$

$$\lambda l = \frac{2n-1}{2}\pi \qquad n = 1, 2, \ldots$$

$$X_n(x) = \sin\frac{(2n-1)\pi x}{2l}$$

$$T_n(t) = e^{-\frac{(2n-1)^2\pi^2\alpha^2 t}{4l^2}}$$

$$u_n(x,t) = e^{-\frac{(2n-1)^2\pi^2\alpha^2 t}{4l^2}} \sin\frac{(2n-1)\pi x}{2l}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, e^{-\frac{(2n-1)^2\pi^2\alpha^2 t}{4l^2}} \sin\frac{(2n-1)\pi x}{2l}$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi x}{2l} = f(x)$$

$$c_n \int_0^l \sin^2\frac{(2n-1)\pi x}{2l}\,dx = \int_0^l f(x)\sin\frac{(2n-1)\pi x}{2l}\,dx$$

$$\int_0^l \sin^2\frac{(2n-1)\pi x}{2l}\,dx = \frac{1}{2}\int_0^l \left(1-\cos\left(\frac{(2n-1)\pi x}{l}\right)\right)dx$$

$$= \frac{l}{2} - \frac{l}{2(2n-1)\pi}\sin\frac{(2n-1)\pi x}{l}\Big|_0^l = \frac{l}{2}$$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\frac{(2n-1)\pi x}{2l}\,dx$$