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ON THE PRINCIPLE OF EXCHANGE OF STABILITIES IN RAYLEIGH–BÉNARD CONVECTION*

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This paper is dedicated to Professor Stephen H. Davis for his 60th birthday.

Abstract. The problem of Rayleigh–Bénard convection with internal heat sources and a variable gravity field is treated. For the case of stress-free boundary conditions, it is proved that the principle of exchange of stabilities holds as long as the product of gravity field and the integral of the heat sources is nonnegative throughout the layer. The proof is based on the idea of a positive operator, and uses the positivity properties of Green's function.

Key words. convection, stability

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1. Introduction.

Problems in fluid mechanics involving the onset of convection have been of great interest for some time. In the experimental area, in the area of numerical simulations, and in the area of geophysical applications, the importance of convection phenomena cannot be overestimated [1], [2], [9], [16]. Starting in the 1930s and 1940s the theoretical treatments usually invoked the so-called principle of exchange of stabilities (PES), which is demonstrated physically as convection occurring initially as a stationary convection. This has been stated as "all nondecaying disturbances are nonoscillatory in time" [3]. Alternatively, it can be stated as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero" [4], [12], [19]. For the Rayleigh–Bénard problem, the principle was first proved by Pellew and Southwell [14]. The case they considered was for a fluid in the Boussinesq approximation, with uniform heating from below, where it turns out that the governing instability equations have a particular symmetry which determine that all eigenvalues of the linearized problem are real. This result also plays an important role in the bifurcation theory of the instability [15]. In 1969, S. H. Davis proved an important theorem concerning this problem [3]. He proved that the eigenvalues of the linearized stability equations will continue to be real when considered as a suitably small perturbation of a selfadjoint problem, such as was considered by Pellew and Southwell [14]. This was one of the first instances in which operator theory was employed in hydrodynamic stability theory. As one of several applications of his theorem, he studied Rayleigh–Bénard convection with a constant internal heat source. One objective here is to consider a more general situation when the heat source is variable.

Here we follow the formulation of Straughan [18], who also made an important original contribution to the study of this problem [17], where a more general treatment is given, in which gravity is allowed to depend on the vertical coordinate z. To begin, the Boussinesq approximation is made. The nondimensional disturbance equations

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for the fluid are then

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + H(z) R \theta \mathbf{e}_z + \mathbf{\Delta u}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1.2}$$

$$Pr \left(\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta \right) = RN(z) w + \Delta \theta, \tag{1.3}$$

where $\mathbf{u} = (u, v, w)$ is the velocity; θ is the temperature; $\mathbf{e}_z = (0, 0, 1)$; R^2 is the Rayleigh number; Pr is the Prandtl number; $N(z) = 1 + \delta q(z)$, where $q(z)$ is proportional to the integral of the heat source, with δ a constant being a scale for $q(z)$, and $H(z) = 1 + \varepsilon h(z)$. The gravity $g(z)$ is defined by $g(z) = g[1 + \varepsilon h(z)]$, g constant, and ε being a scale for h. In his work, Straughan chose δ and ε as small quantities. That will not be necessary in the analysis to follow. The equations are assumed to hold in the layer

$$\Omega = \{(x, y, z) \mid -\infty < x, y < \infty,\ 0 < z < 1\}.$$

Next, linearize the perturbed system (1.1)–(1.3) and assume disturbances to be periodic in x (period $2\pi/\alpha$) and y (period $2\pi/\beta$) with growth rate σ, of the form

$$\mathbf{u} = e^{\sigma t + i\alpha x + i\beta y} \hat{\mathbf{u}}(z),$$

for the velocity components, with comparable representations for θ and p. Then take curl curl of the linearized momentum equations to obtain

$$\sigma (D^2 - k^2) w = (D^2 - k^2)^2 w - k^2 R H(z) \theta, \tag{1.4}$$

$$Pr \sigma \theta = (D^2 - k^2) \theta + RN(z) w, \tag{1.5}$$

where $D = d/dz$, $k^2 = \alpha^2 + \beta^2$, and the tildes have been dropped. The usual boundary conditions are either (i) fixed

$$w = Dw = \theta = 0, \quad z = 0, 1, \tag{1.6}$$

or (ii) free boundary conditions

$$w = D^2 w = \theta = 0, \quad z = 0, 1. \tag{1.7}$$

In case (ii), the possibility of a nonzero tangential velocity is allowed. It is case (ii), with stress-free boundary conditions which will be treated in this work. When $\varepsilon = 0$ and $\delta = 0$, $H(z) \equiv 1$ and $N(z) \equiv 1$, and it is possible to show without much difficulty that if $\sigma = \sigma_1 + i\sigma_2$ is complex, and $\sigma_2 \neq 0$, then $\sigma_1 < 0$. This was the original idea of Pellew and Southwell [14]. Straughan [18] was also able to handle the case where either $\varepsilon = 0$ or $\delta = 0$, with two free boundary conditions on w at $z = 0, 1$. He also points out that there is one interesting special case, when $H(z)$ is a multiple of $N(z)$, then the system can easily be made symmetric and σ is real. However, in general, when $H(z)$ and $N(z)$ are independent variables, no matter the boundary conditions, the only known results are those of Davis [3]. It is the purpose of this article to provide a new result on this problem. It is proved that if $H(z)N(z) \geq 0$ throughout the layer, the principle of exchange of stabilities holds. For example, for positive but possibly varying gravity, with a positive but varying source of heat, PES will undoubtedly hold. Nevertheless, the condition $H(z)N(z) \geq 0$ is a technical mathematical requirement of the method of proof. There are cases in which this condition is violated and PES still holds. For instance, Straughan [17] proved that when N is constant and $H(z)$ changes sign in the layer, the beginning of convection is stationary. He studied numerically the case $N \equiv 1$ and $H(z) = 1 - \varepsilon z$, $0 \leq \varepsilon \leq 1.5$, providing results to show how variable gravity affects the critical Rayleigh number. Applications to the phenomena of penetrative and non-Boussinesq convection are clearly in view with results of this type [3], [18, pp. 95–101].

It is desirable then to be able to search for stationary convection by setting $\sigma = 0$ in the linearized stability equations. If all of the eigenvalues of the problem are real, this is the natural place to begin. When all of the eigenvalues are not real, a criterion such as the one to be derived here is a valuable resource. In the next section the proposed technique is described. Then in the succeeding section the proof is carried out.

2. The method of positive operators. The idea of the method of solution is based on the notion of a *positive* operator [5], [10], a generalization of a positive matrix, that is, one with all of its entries positive. Such matrices have the property that they possess a single greatest positive eigenvalue, identical to the spectral radius. To apply the method the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of compositions of certain integral operators. When the Green's function kernels for these operators are all nonnegative, the resulting operator is termed positive. The infinite dimensional counterpart of this property is contained in the following.

THEOREM 1 (see [13]). *If a linear, compact operator A, leaving invariant a cone $\mathfrak{K}$, has a point of the spectrum different from zero, then it has a positive eigenvalue λ, not less in modulus than every other eigenvalue, and to this number corresponds at least one eigenvector $\phi \in \mathfrak{K}$ of the operator A ($A\phi = \lambda\phi$), and at least one eigenvector $\psi \in \mathfrak{K}^*$ of the operator A^*.*

For this problem the cone consists of the set of nonnegative functions.

In the formulation (1.4) and (1.5) with the boundary conditions (1.7), it is possible to rewrite the equations in terms of certain operators,

$$\left(\tilde{M}^2 + \sigma\tilde{M}\right) w - k^2 RH(z)\theta = 0, \tag{2.1}$$

$$-RN(z)w + \left(\tilde{M} + Pr\sigma\right)\theta = 0, \tag{2.2}$$

which are defined as follows. The formal differential operator $(-D^2 + k^2)w := mw$ is obtained from the reduced Laplacian; then define

$$\begin{aligned} \tilde{M}w &= mw, \quad w \in \operatorname{dom} \tilde{M}, \\ \tilde{M}^2 w &= m^2 w, \quad w \in \operatorname{dom}(\tilde{M}\tilde{M}), \\ \tilde{M}\theta &= m\theta, \qquad \theta \in \operatorname{dom} \tilde{M}. \end{aligned}$$

The domains are contained in $\mathfrak{H}$, where

$$\mathfrak{H} = L^2(0,1) = \left\{ \phi \mid \int_0^1 |\phi|^2 \, dz \right\} < \infty,$$

with scalar product

$$\langle \phi, \psi \rangle = \int_0^1 \phi(z)\bar{\psi}(z) dz, \quad \phi, \psi \in \mathfrak{H},$$

and norm

$$\| \phi \| = \langle \phi, \phi \rangle^{1/2}.$$

So, the domain of $\tilde{M}$ is

$$\operatorname{dom} \tilde{M} = \{ \phi \in \mathfrak{H} \mid D\phi, m\phi \in \mathfrak{H}, \, \phi(0) = \phi(1) = 0 \}.$$

With the above definitions, it is not difficult to verify the following properties of the operator just defined.

Remark 1. $\tilde{M}$ is self-adjoint and positive definite. Furthermore, $\Gamma(\sigma) = (\tilde{M}+\sigma)^{-1}$ exists for

$$\sigma \notin \Sigma_k = \left\{ \sigma \in \mathbf{C} \mid \operatorname{Re}(\sigma) \leq -k^2, \operatorname{Im}(\sigma) = 0 \right\},$$

and $\|\Gamma(\sigma)\|^{-1} > \left|\sigma + k^2\right|$ for $\operatorname{Re}(\sigma) > -k^2$ [11, p. 272]. Explicitly, $\Gamma(\sigma)$ is the integral operator such that for $f \in \mathfrak{H}$,

$$\Gamma(\sigma) f = (\tilde{M} + \sigma)^{-1} f = \int_0^1 g(z, \xi; \sigma) f(\xi) d\xi,$$

where

$$g(z, \xi; \sigma) = \frac{\cosh[r(1 - |z - \xi|)] - \cosh[r(-1 + z + \xi)]}{2r \sinh r} \tag{2.3}$$

is the appropriate Green's function and

$$r = \sqrt{k^2 + \sigma}$$

is the positive square root.

It is now possible to write the system as a single equation in w. Solving for θ from (2.2) it follows that

$$\theta = \left(\tilde{M} + Pr\sigma\right)^{-1} RN(z) w = \Gamma(Pr\sigma) RN(z) w.$$

Similarly, in (2.1),

$$\begin{aligned} w &= \left(\tilde{M}^2 + \sigma \tilde{M}\right)^{-1} k^2 RH(z)\theta \\ &= \tilde{M}^{-1}\Gamma(\sigma) k^2 RH(z)\theta. \end{aligned}$$

So substituting for θ,

$$w = k^2 R^2 \tilde{M}^{-1} \Gamma(\sigma) H(z) \Gamma(Pr\sigma) N(z) w.$$

In a more compact form this equation is written as

$$w = K(\sigma) w. \tag{2.4}$$

What is to be studied in what follows is the resolvent of K defined as $[I - K(\sigma)]^{-1}$. Suppose $K(\sigma)$ depends analytically on σ in a certain right half of the complex plane. Furthermore, let

$$[I - K(\sigma)]^{-1} = \left\{ I - [I - K(\sigma_0)]^{-1} [K(\sigma) - K(\sigma_0)] \right\}^{-1} [I - K(\sigma_0)]^{-1}. \tag{2.5}$$

So, if for all real σ_0 greater than some a, (P1) $[I - K(\sigma_0)]^{-1}$ is positive,

(P2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients; i.e., $(-d/d\sigma)^n K(\sigma_0)$ is positive for all n. Then the right side of (2.5) has an expansion in $(\sigma_0 - \sigma)$ with positive coefficients. Moreover the methods of [19] and [15] apply, showing that "there exists a real eigenvalue $\sigma_1 \leq a$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma \mid \mathrm{Re}\,(\sigma) \leq \sigma_1\}$." This is equivalent to PES, which was stated earlier as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero."

To verify conditions (P1) and (P2), the structure of the operator $\Gamma(\sigma)$ will be utilized. It is assumed that the product of the functions $H(z)N(z) \geq 0$ on $[0, 1]$.

3. The principle of exchange of stabilities. Condition (P1) is treated first. The operator $\tilde{M}^{-1} = \Gamma(0)$ is an integral operator whose Green's function $g(z, \xi; 0)$ is nonnegative so $\tilde{M}^{-1}$ is a positive operator. Next observe that by Remark 1, $\Gamma(Pr\sigma)$ is also an integral operator, but its Green's function kernel $g(z, \xi; \sigma)$ in (2.3) is the Laplace transform of the Green's function $G(z, \xi; t)$ for the initial-boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2} + k^2 + Pr\frac{\partial}{\partial t}\right) G = \delta(z - \xi, t), \tag{3.1}$$

$$G(0, \xi; t) = G(1, \xi; t) = G(z, \xi; 0) = 0. \tag{3.2}$$

Using the method of images, or by direct calculation of the inverse Laplace transform we find

$$G(z, \xi; t) = \frac{e^{-k^2\tau}}{\sqrt{4\pi\tau}} \sum_{j=-\infty}^{\infty} \left\{ e^{-(z-\xi+2j)^2/4\tau} - e^{-(z+\xi+2j)^2/4\tau} \right\}, \tag{3.3}$$

where $\tau = t/Pr$. It follows easily that

$$G(z, \xi; t) \geq 0, \quad 0 \leq z, \xi \leq 1, \quad t > 0. \tag{3.4}$$

Since

$$g(z, \xi; \sigma) = \int_0^\infty e^{-\sigma t} G(z, \xi; t) dt, \tag{3.5}$$

we see that

$$\left(-\frac{d}{d\sigma}\right)^n g(z, \xi; \sigma) = \int_0^\infty t^n e^{-\sigma t} G(z, \xi; t) dt \geq 0 \tag{3.6}$$

for all n and for all real $\sigma > -k^2$.

It was shown in (3.3)–(3.6) that $\Gamma(\sigma) = (\tilde{M} + \sigma)^{-1}$ is a positive operator for all real $\sigma_0 > -k^2$ and that $\Gamma(\sigma)$ has a power series expansion about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients; i.e., $\left(-\frac{d}{d\sigma}\right)^n \Gamma(\sigma_0)$ is positive for all n . Thus the expansion

$$\Gamma(\sigma) = \Gamma(\sigma_0)[I - (\sigma_0 - \sigma)\Gamma(\sigma_0)]^{-1}$$

$$= \Gamma(\sigma_0)[I + (\sigma_0 - \sigma)\Gamma(\sigma_0) + (\sigma_0 - \sigma)^2(\Gamma(\sigma_0))^2 + \cdots] \tag{3.7}$$

is valid for $|\sigma_0 - \sigma|\|\Gamma(\sigma_0)\| < 1$. The coefficients are positive operators when $\sigma_0 > -k^2$. The expansion (3.7) may be analytically continued to the whole half-plane $\mathrm{Re}\,(\sigma) > -k^2$.

THEOREM 2. *The PES holds for* (2.1)–(2.2) *when the product of integrated internal heat sources* $N(z)$ *and variable gravity ratio* $H(z)$ *is nonnegative throughout the layer.*

Proof. The system (2.1)–(2.2) may be written as the single equation suggested by (2.4),

$$u = K(\sigma)u, \tag{3.8}$$

where

$$K(\sigma) = k^2 R^2 \tilde{M}^{-1}\Gamma(\sigma)H(z)\Gamma(Pr\sigma)N(z).$$

The resolvent is examined as defined in (2.5). It has been demonstrated that the original system (2.1)–(2.2) and the transformed system (3.8) have spectra that agree except on the set Σ_k, when $0 < Pr \le 1$, or on the set $\Sigma_{k/\sqrt{Pr}}$, when $Pr > 1$, which in either case is a subset of the negative real half-line. We have shown in (3.1)–(3.7) that $\Gamma(\sigma)$ is a positive operator and that $\Gamma(Pr\sigma)$ and $\Gamma(\sigma)$ have power series expansions for real $\sigma_0 > -k^2/Pr$ and $\sigma_0 > -k^2$, respectively.

To verify condition (P2), again note that it is assumed that $H(z)N(z) \ge 0$, while k^2 and R^2 are clearly positive. Therefore, by the product rule for differentiation, one concludes that $K(\sigma)$ in (3.8) satisfies condition (P2).

It has been demonstrated that all of the terms in $K(\sigma)$ determine positive operators. Moreover, for σ real and sufficiently large, by Remark 1, the norms of the operators $\Gamma(\sigma)$ and $\Gamma(Pr\sigma)$ become arbitrarily small. Hence, $\|K(\sigma)\|$ will be less than 1. Then $[I - K(\sigma)]^{-1}$ has a convergent Neumann series and hence is positive. This is the content of condition (P1). □

4. Concluding comments. Despite the simplicity of the method employed here for stress-free boundary conditions, the difficulty occurring with no-slip boundary conditions has always impeded solution of that problem. It is more difficult because the operator occurring in the counterpart of (2.1) does not factor as does

$$\left(\tilde{M}\tilde{M} + \sigma\tilde{M}\right)^{-1} = \tilde{M}^{-1}\Gamma(\sigma) \tag{4.1}$$

in the current notation. It is such a factorization (4.1), which made possible the solution of analogous problems for Görtler flow [6] and for Langmuir circulations [7]. There is another problem, convection in a porous medium with an internal heat source and variable gravity which can be handled in much the same way as the problem treated here [8]. Davis [3] had also considered convection with radiative transfer. In this case, the temperature coefficient $N(z)$ is a variable, but does not change sign. A direct application is not relevant since an additional term involving an integral operator occurs in the disturbance equation of energy. Nevertheless, it is possible that the techniques of this paper will apply in that case also.

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