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On the Principle of Exchange of Stabilities in Rayleigh-Bénard Convection, II - No-slip Boundary Conditions.

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SUNTO - La convezione di Rayleigh-Benard con sorgenti interne di calore ed un campo di gravità variabile è trattato con condizioni di aderenza al bordo. È provato che il principio di scambio di stabilità vale per tutti i numeri di Prandtl, fintanto che il campo gravitazionale e l'integrale delle sorgenti di calore abbiano lo stesso segno. La dimostrazione è basata sull'idea di un operatore positvo, ed usa le proprietà di positività della funzione di Green. La funzione generalizzata di Green è anche usata.

ABSTRACT - Rayleigh-Bénard convection with internal heat sources and a variable gravity field is treated with no-slip boundary conditions. It is proved that the principle of exchange of stabilities holds at all Prandtl numbers, as long as the gravity field and the integral of the heat sources both have the same sign. The proof is based on the idea of a positive operator, and uses the positivity properties of Green's function. The generalized Green's function is also employed.

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1. – Introduction.

Problems in fluid mechanics involving the onset of convection have been of great interest for some time. Theoretical treatments usually invoke the so-called principle of exchange of stabilities (PES), that is demonstrated physically as convection occurring initially as a stationary convection. This has been stated as «all non-decaying disturbances are non-oscillatory in time» [1], [2], [3]. Alternatively, it can be stated as, «the first unstable eigenvalue of the linearized system has imaginary part equal to zero» [4], [18]. For the Rayleigh-

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Bénard problem, the principle was first proved by Pellew and Southwell [15]. The case they considered was for a fluid in the Boussinesq approximation, with uniform heating from below, where it turns out that the governing instability equations have a particular symmetry which determine that all eigenvalues of the linearized problem are real. This result also plays an important role in the bifurcation theory of the instability [16]. In 1969, Davis proved that the eigenvalues of the linearized stability equations will continue to be real when considered as a suitably small perturbation of a selfadjoint problem, such as was considered by Pellew and Southwell. As one of several applications of his theorem, he studied Rayleigh-Bénard convection with a constant internal heat source. In part I [6], a more general situation was considered with a variable heat source and a variable gravity. However, the boundary conditions were assumed to be the stress free conditions. The more complicated situation of no-slip boundary conditions is treated now in part II.

We follow the formulation of part I [6], [17]. The Boussinesq approximation is made

(1) $$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + H(z)\, R\theta \boldsymbol{e}_z + \boldsymbol{\Delta u}\,,$$

(2) $$\nabla\cdot\boldsymbol{u} = 0$$

$$Pr\left(\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta\right) = RN(z)\, w + \Delta\theta\,,$$

where $\boldsymbol{u} = (u, v, w)$, is the velocity, θ is the temperature, $\boldsymbol{e}_z = (0, 0, 1)$, R^2 is the Rayleigh number, Pr is the Prandtl number, $N(z) = 1 + \delta q(z)$, where $q(z)$ is proportional to the integral of the heat source, with δ, a constant being a scale for $q(z)$, and $H(z) = 1 + \varepsilon h(z)$; the gravity $g(z)$ being defined by $g(z) = g[1 + \varepsilon h(z)]$, g constant, and ε being a scale for h. The equations are assumed to hold in the layer

$$\Omega = \{(x, y, z)\,|\, -\infty < x, y < \infty\,, 0 < z < 1\}\,.$$

Next, linearize the perturbed system (1)-(3) and assume disturbances to be periodic in x (period $2\pi/\alpha$) and y (period $2\pi/\beta$) with growth rate σ, of the form

$$\boldsymbol{u} = e^{\sigma t + i\alpha x + i\beta y}\,\widehat{\boldsymbol{u}}(z)\,,$$

for the velocity components, with comparable representations for θ and p. Then take *curlcurl* of the linearized momentum equations to obtain

(4) $$\sigma(D^2 - k^2)\, w = (D^2 - k^2)^2 w - k^2 RH(z)\,\theta\,,$$

$$Pr\sigma\theta = (D^2 - k^2)\,\theta + RN(z)\, w,$$

where $D = d/dz$, $k^2 = \alpha^2 + \beta^2$, and the tildes have been dropped. The usual boundary conditions are either (i) fixed

$$w = Dw = \theta = 0\,, \qquad z = 0, 1\,, \tag{6}$$

or (ii) free boundary conditions

$$w = D^2 w = \theta = 0\,, \qquad z = 0, 1\,. \tag{7}$$

Case (ii), which is simpler, was solved in part I [6]. It is case (i), which has previously eluded successful resolution, that will studied in this work. However, it should be mentioned that when $\varepsilon = 0$ and $\delta = 0$, $H(z) \equiv 1$ and $N(z) \equiv 1$, and it is possible to show without much difficulty that if $\sigma = \sigma_1 + i\sigma_2$ is complex, and $\sigma_2 \neq 0$, then $\sigma_1 < 0$. This was the original idea of Pellew and Southwell [15]. However, in general, when $H(z)$ and $N(z)$ are independent variables, with no-slip boundary conditions, the only results known are those of Davis [3]. It is the purpose of this article to provide resolution to this problem. The method employed here is also successful in the stability problem of Couette flow [7]. In the next section, the now familiar technique is outlined. Then, in the succeeding section the proof is carried out in several steps. The underlying operators are described. Then a lemma for no-slip boundary conditions is introduced, which is the main advance from part I. Finally, the PES is proved.

2. – Abstract formulation.

2.1. *The method of positive operators.*

The idea of the method of solution is based on the notion of a *positive* operator [9], [5], a generalization of a positive matrix, that is, one with all of its entries positive. Such matrices have the property that they possess a single greatest positive eigenvalue, identical to the spectral radius. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of compositions of certain integral operators. When the Green's function kernels for these operators are all nonnegative, the resulting operator is termed positive. The infinite dimensional counterpart of this property is contained in the following theorem of Krein and Rutman.

THEOREM [13]. *If a linear, compact operator A, leaving invariant a cone $\mathfrak{K}$, has a point of the spectrum different from zero, then it has a positive eigenvalue λ, not less in modulus than every other eigenvalue, and to this number corresponds at least one eigenvector $\phi \in \mathfrak{K}$ of the operator A ($A\phi = \lambda\phi$), and at least one eigenvector $\psi \in \mathfrak{K}^*$ of the operator A^*.*

For this problem the cone consists of the set of nonnegative functions.

In the formulation (4) and (5) with the boundary conditions (6), it is possible to rewrite the equations in terms of certain operators

$$(M^*M + \sigma M)\, w - k^2 RH(z)\,\theta = 0\,, \tag{8}$$

$$-RN(z)\, w + (\widetilde{M} + Pr\sigma)\,\theta = 0\,, \tag{9}$$

where

$$Mw = (-D^2 + k^2)\, w := mw, \quad w \in \operatorname{dom} M\,,$$

$$M^*Mw = m^2 w, \quad w \in \operatorname{dom}(M^*M)\,,$$

$$\widetilde{M}\theta = m\theta, \quad \theta \in \operatorname{dom} \widetilde{M}\,.$$

The domains are contained in $\mathfrak{H}$, where

$$\mathfrak{H} = L^2(0, 1) = \left\{ \phi \,\middle|\, \int_0^1 |\phi|^2\, dz \right\} < \infty\,,$$

with scalar product

$$\langle \phi, \psi \rangle = \int_0^1 \phi(z)\, \overline{\psi}(z)\, dz, \quad \phi, \psi \in \mathfrak{H}\,,$$

and norm

$$\|\phi\| = \langle \phi, \phi \rangle^{1/2}.$$

The domains are given as follows

$$\operatorname{dom} M^* = \{\phi \in \mathfrak{H} \,|\, \phi',\ m\phi \in \mathfrak{H}\}\,,$$
$$\operatorname{dom} M = \{\phi \in \operatorname{dom} M^* \,|\, \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0\}\,,$$
$$\operatorname{dom} \widetilde{M} = \{\phi \in \operatorname{dom} M^* \,|\, \phi(0) = \phi(1) = 0\}\,.$$

Then with the above definitions, it is not difficult to verify the following properties of the operators just defined. The estimates are not the sharpest possible, but they will do for our purposes.

REMARK 1. M is closed, symmetric, but not maximal and hence not invertible. Moreover, M is positive definite, that is $\langle M\phi, \phi \rangle \geq k^2 \|\phi\|^2$, $\phi \in \operatorname{dom} M$, $k \neq 0$. The case $k = 0$ is excluded because stability is known to hold in that case.

REMARK 2. M^* is the adjoint of M and has no boundary conditions. The two-dimensional null space of M^*, $\ker M^*$, has the basis

$$\boldsymbol{q}(z) = \begin{pmatrix} e^{kz} \\ e^{-kz} \end{pmatrix}.$$

REMARK 3. $\widetilde{M}$ is a selfadjoint, positive definite extension of M. Furthermore, $\Gamma(\sigma) = (\widetilde{M} + \sigma)^{-1}$ exists for $\sigma \notin \Sigma_k = \{\sigma \in \boldsymbol{C} \mid \mathbf{Re}(\sigma) \leqslant -k^2,\ Im(\sigma) = 0\}$, and $\|\Gamma(\sigma)\|^{-1} > |\sigma + k^2|$, for $\mathrm{Re}(\sigma) > -k^2$ [12, p. 272]. Explicitly, $\Gamma(\sigma)$ is the integral operator such that for $f \in \mathfrak{H}$,

$$\Gamma(\sigma)\, f = (\widetilde{M} + \sigma)^{-1} f = \int_0^1 \tilde{g}(z,\ \xi;\ \sigma)\, f(\xi)\, d\xi\,,$$

where

$$\tilde{g}(z,\ \xi;\ \sigma) = \frac{\cosh[r(1 - |z - \xi|)] - \cosh[r(-1 + z + \xi)]}{2r \sinh r} \tag{10}$$

is the appropriate Green's function and

$$r = \sqrt{k^2 + \sigma}$$

is the branch of the square root which is positive for positive real numbers.

It is now possible to write the system as single equation in w. Eliminating θ from (9) it follows that

$$\theta = (\widetilde{M} + Pr\sigma)^{-1} RN(z)\, w = \Gamma(Pr\sigma)\, RN(z)\, w\,.$$

Similarly, in (8),

$$w = (M^* M + \sigma M)^{-1} k^2 RH(z)\, \theta = F(\sigma)\, k^2 RH(z)\, \theta\,,$$

where

$$F(\sigma) = (M^* M + \sigma M)^{-1}.$$

So substituting for θ,

$$w = k^2 R^2 F(\sigma)\, H(z)\, \Gamma(Pr\sigma)\, N(z)\, w\,.$$

In a more compact form this equation is written as

$$w = K(\sigma)\, w\,. \tag{11}$$

This formal derivation of (11) will be justified in the next section.

What is to be studied in what follows is $[I - K(\sigma)]^{-1}$. Suppose $K(\sigma)$ depends analytically on σ in a certain right half of the complex plane. Furthermore, for all σ_0 for which $K(\sigma)$ is defined,

(12) $[I - K(\sigma)]^{-1} = \left\{I - [I - K(\sigma_0)]^{-1}[K(\sigma) - K(\sigma_0)]\right\}^{-1}[I - K(\sigma_0)]^{-1}$.

So, if for all real σ_0 greater than some fixed constant a

(**P1**) $\left[I - K(\sigma_0)\right]^{-1}$ is positive,

(**P2**) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0 - \sigma)$ with positive coefficients, i.e. $(-d/d\sigma)^n K(\sigma_0)$ is positive for all n, then the right side of (12) has an expansion in $(\sigma_0 - \sigma)$ with positive coefficients. Then, the methods of [18] and [16] apply, showing that «there exists a real eigenvalue $\sigma_1 \leq a$ such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma | \mathrm{Re}(\sigma) \leq \sigma_1\}$". This is equivalent to PES.

To verify conditions (**P1**) and (**P2**), the structure of the operators $\Gamma(Pr\sigma)$ and $F(\sigma)$ will be examined. It is assumed that the product of the functions $H(z)\, N(z) \geq 0$ on $[0, 1]$.

3. – The principle of exchange of stabilities.

3.1. *The underlying operators.*

Condition (**P1**) is treated first. Each factor in $K(\sigma)$ is examined. Regarding $\Gamma(Pr\sigma)$ the following was proved in part I [6]:

LEMMA 1. The operator $\Gamma(\sigma) = (\widetilde{M} + \sigma)^{-1}$ is a positive operator for all real $\sigma > -k^2$. and $\Gamma(s)$ has a power series expansion about σ_0 in $(\sigma_0 - \sigma)$ with positive expansion coefficients, i.e. $\left(-\dfrac{d}{d\sigma}\right)^n \Gamma(\sigma_0)$ is positive for σ_0 *real*, for all n.

The operator $F(\sigma)$ is not as easily analyzed. The approach to be taken involves examining $F(\sigma)$ in terms of its factors. Use is made of the *generalized inverse* [14] of M, denoted by $M^\dagger$. First, the projection operator Q onto the $\ker M^*$ is defined by

$$(Q\psi)(z) = \int_0^1 g_Q(z, \xi)\, \psi(\xi)\, d\xi\,, \qquad \psi \in H\,,$$

where

$$g_Q(z, \xi) = \mathbf{q}^T(z)\left[\int_0^1 \boldsymbol{q}(s)\, \boldsymbol{q}^T(s)\, ds\right]^{-1} \boldsymbol{q}(\xi)\,.$$

Then $g^\dagger$, the generalized Green's function satisfies

$$\left(-\frac{\partial^2}{\partial z^2} + k^2\right) g^\dagger(z, \xi) = \delta(z - \xi) - g_Q(z, \xi)\,,$$

$$g^\dagger(0, \xi) = \frac{\partial}{\partial z} g^\dagger(0, \xi) = g^\dagger(1, \xi) = \frac{\partial}{\partial z} g^\dagger(1, \xi) = 0\,,$$

so that

$$(M^\dagger \phi)(z) = \int_0^1 g^\dagger(z, \xi)\, \phi(\xi)\, d\xi\,.$$

Some properties of $M^\dagger$ are

$$MM^\dagger = I - Q\,, \tag{13}$$

$$M^\dagger M = I\,, \tag{14}$$

since $\ker M$ is trivial.

LEMMA 2. $M^\dagger = \widetilde{M}^{-1}(I-Q)$.

PROOF. Since $\widetilde{M}$ is an invertible extension of M, $\widetilde{M}^{-1}M = I$ on $\operatorname{dom} M$, therefore operating on (13) with $\widetilde{M}^{-1}$

$$\widetilde{M}^{-1}MM^\dagger = \widetilde{M}^{-1}(I-Q).$$

and

$$\widetilde{M}^{-1}MM^\dagger = IM^\dagger = M^\dagger,$$

so

$$M^\dagger = \widetilde{M}^{-1}(I-Q)\,. \quad \blacksquare \tag{15}$$

LEMMA 3. The only solution of (8)-(9) is the trivial solution when $R=0$, and $\sigma \notin \Sigma_k$ for $0 < Pr \leqslant 1$, or $\sigma \notin \Sigma_{k/\sqrt{Pr}}$ for $Pr > 1$.

PROOF. Suppose that $R=0$, then (8)-(9) reduces to

$$(M^*M + \sigma M)\, w = 0\,, \tag{16}$$

$$(\widetilde{M} + Pr\sigma)\, \theta = 0\,. \tag{17}$$

Taking the scalar product of (16) with w gives $\langle M^*Mw, w\rangle = -\sigma\langle Mw, w\rangle$, or

$$-\sigma\langle Mw, w\rangle = \langle M^*Mw, w\rangle = \langle Mw, Mw\rangle =$$
$$= \|Mw\|^2 \geqslant \langle Mw, w\rangle^2/\|w\|^2 \geqslant k^2\langle Mw, w\rangle,$$

by Remarks 1 and 2. Thus, σ is real and $\sigma \leqslant -k^2$. Hence, when $\sigma \notin \Sigma_k$, $w=0$ and this implies $\theta = 0$, by (17) and Remark 3, when $Pr \leqslant 1$. If $Pr > 1$, then by Remark 3, $\theta = 0$, when $\sigma \notin \Sigma_{k/\sqrt{Pr}}$. $\blacksquare$

3.2. *A lemma for no-slip boundary conditions.*

LEMMA 4. The operator $L = (M^*M + \sigma M)$ has an inverse, positive for all real $\sigma > -k^2$. The inverse has a power series expansion about σ_0 in powers of $\sigma_0 - \sigma$ with positive expansion coefficients.

PROOF. On the basis of the preceding lemma, conclude that L is invertible for $\sigma \notin \Sigma_k$. Define

$$L^{-1} = F(\sigma) = (M^*M + \sigma M)^{-1} = (M^*M)^{-1}(I + \sigma B)^{-1}, \tag{18}$$

where $B = M(M^*M)^{-1}$ is a bounded operator. By Lemma 1, the expansion

$$\Gamma(\sigma) = \Gamma(\sigma_0)[I - (\sigma_0 - \sigma)\Gamma(\sigma_0)]^{-1} =$$

$$= \Gamma(\sigma_0)[I + (\sigma_0 - \sigma)\,\Gamma(\sigma_0) + (\sigma_0 - \sigma)^2(\Gamma(\sigma_0))^2 + \ldots]$$

is valid for $|\sigma_0 - \sigma|\|\Gamma(\sigma_0)\| < 1$. The coefficients are positive operators when $\sigma_0 > -k^2$. Analogously, by (18),

$$(M^*M + \sigma M)^{-1} = (M^*M + \sigma_0 M)^{-1}[I - (\sigma_0 - \sigma)\,M(M^*M + \sigma_0 M)^{-1}]^{-1}$$

so that

$$F(\sigma) = F(\sigma_0)[I - (\sigma_0 - \sigma)\,MF(\sigma_0)]^{-1} = \tag{19}$$

$$= F(\sigma_0)[I + (\sigma_0 - \sigma)\,MF(\sigma_0) + (\sigma_0 - \sigma)^2(MF(\sigma_0))^2 + \ldots],$$

for $|\sigma_0 - \sigma|\|MF(\sigma_0)\| < 1$, $\sigma \notin \Sigma_k$, $\sigma_0 > -k^2$. It is still necessary to establish the positivity of the coefficients in the expansion for $F(\sigma)$. Observe, also from (18), that

$$F(\sigma_0) = (M^*M)^{-1}[I + (-\sigma_0)\,B + (-\sigma_0)^2 B^2 + \ldots] \tag{20}$$

is defined for $|\sigma_0|\|B\| < 1$. If it can be shown that the coefficients in the above expansion (20) of $F(\sigma_0)$ in powers of $(-\sigma_0)$ are positive, then it follows that $F(\sigma_0)$ is a positive operator for real σ_0 in the interval $-k^2 < \sigma_0 < \|B\|^{-1}$. Then by a process of analytic continuation, which will be described, the positivity of $F(\sigma_0)$ for all real $\sigma_0 > -k^2$ will be proved.

The process thus begins to establish that $F(\sigma)$ in (19) has positive expansion coefficients. This is done by re-examining the expansion of $F(\sigma_0)$ (20). Define

$$S = I - Q\,,$$

the projection on the range of M, $\operatorname{ran} M$. We have, from (15), that

$$M^{\dagger *} = (\widetilde{M}^{-1}S)^* = S\widetilde{M}^{-1}, \tag{21}$$

since $\widetilde{M}^{-1}$, Q, and hence S are selfadjoint. Thus we have the equivalent factorizations

$$M^{\dagger}M^{\dagger *} = \widetilde{M}^{-1}SS\widetilde{M}^{-1} = M^{\dagger}\widetilde{M}^{-1} = \widetilde{M}^{-1}M^{\dagger *} = \widetilde{M}^{-1}S\widetilde{M}^{-1},$$

since S is a projection. But, by definition of the generalized inverse,

$$M^{\dagger}M^{\dagger *} = (M^*M)^{\dagger} = (M^*M)^{-1}, \tag{22}$$

since M^*M is invertible, being positive (definite) selfadjoint. Thus

$$B = M(M^*M)^{-1} = M^{\dagger *}, \tag{23}$$

and

$$\|B\| = \|S\widetilde{M}^{-1}\| \leqslant \|S\|\|\widetilde{M}^{-1}\| = \|\widetilde{M}^{-1}\|, \tag{24}$$

since S is projection, $\|S\| = 1$. Then the expansion of $F(\sigma_0)$, (20) may be re-written, using (23), as

$$\begin{aligned} F(\sigma_0) &= (M^*M)^{-1} + (-\sigma_0)(M^*M)^{-1}M(M^*M)^{-1} + \\ &\quad + (-\sigma_0)^2(M^*M)^{-1}(M(M^*M)^{-1})^2 + \dots \\ &= (M^*M)^{-1} + (-\sigma_0)M^{\dagger}\widetilde{M}^{-1}M^{\dagger *} + \sigma_0^2(M^{\dagger}M^{\dagger *})(M^{\dagger *})^2 + \dots \end{aligned} \tag{25}$$

By (15) and (21) it follows that

$$F(\sigma_0) = (M^*M)^{-1} + (-\sigma_0)\widetilde{M}^{-1}S\widetilde{M}^{-1}S\widetilde{M}^{-1} + \sigma_0^2\widetilde{M}^{-1}S\widetilde{M}^{-1}(S\widetilde{M}^{-1})^2 + \dots \tag{26}$$

It is a well known result of the stability literature [8, Appendix D], [16, p. 370], that $\widetilde{M}^{-1}$ has a nonnegative Green's function kernel. However, even more can be said ([10], [11]). Consider first order differential operators given by

$$\begin{aligned} (D_j\varphi)(z) &= \frac{d}{dz} r_j(z)\,\varphi(z), \\ (D_j^*\varphi)(z) &= r_j(z)\,\frac{d\varphi(z)}{dz}, \qquad j = 1, 2, \dots n, \end{aligned}$$

with strictly positive weights $r_j(z)$, $j = 1, \dots n+1$, possessing $2n$ continuous derivatives in $[0, 1]$. The formal differential operators A_n and A_n^* which can be written as

$$(A_n \varphi)(z) = D_n \dots D_2 D_1 \varphi \ , \tag{27}$$

$$(A_n^* \varphi)(z) = D_1^* D_2^* \dots D_n^* \varphi \ , \tag{28}$$

form factors of the even order differential operator

$$E_{2n} = (-1)^n A_n^* r_{n+1} A_n . \tag{29}$$

The appropriate boundary conditions for E_{2n} might be: at the endpoint $z = 0$,

$$\begin{gathered}
(D_2^* D_3^* \dots D_n^* r_{n+1} D_n \dots D_2 D_1 \varphi)(0) + (-1)^n a_1 \varphi(0) = 0, \\
(D_3^* D_4^* \dots D_n^* r_{n+1} D_n \dots D_2 D_1 \varphi)(0) + (-1)^{n+1} a_2 (D_1 \varphi)(0) = 0, \\
\vdots \\
(r_{n+1} D_n \dots D_2 D_1 \varphi)(0) + (-1)^{2n-1} a_n (D_{n-1} \dots D_2 D_1 \varphi)(0) = 0
\end{gathered} \tag{30}$$

and at the endpoint $z = 1$,

$$\begin{gathered}
(D_2^* D_3^* \dots D_n^* r_{n+1} D_n \dots D_2 D_1 \varphi)(1) + (-1)^{n+1} b_1 \varphi(1) = 0, \\
(D_3^* D_4^* \dots D_n^* r_{n+1} D_n \dots D_2 D_1 \varphi)(1) + (-1)^{n+2} b_2 (D_1 \varphi)(1) = 0, \\
\vdots \\
(r_{n+1} D_n \dots D_2 D_1 \varphi)(1) + (-1)^{2n} b_n (D_{n-1} \dots D_2 D_1 \varphi)(1) = 0
\end{gathered} \tag{31}$$

where

$$0 \leqslant a_j \leqslant \infty , \qquad 0 \leqslant b_j \leqslant \infty , \qquad j = 1, 2, \dots , n$$

and for each j not both a_j and b_j are zero. (This restriction is to ensure that 0 is not an eigenvalue of E_{2n}.) The value $a_1 = \infty$ is interpreted as the boundary condition $\varphi = 0$; similarly $a_2 = \infty$ as the boundary condition $D_1 \varphi = 0$, etc. Then the Green's function kernel of E_{2n}, γ_{2n}, is totally positive (TP). This means that $\gamma_{2n} > 0$ on the interior of the square on which it is defined, and certain determinants of intermediate values are nonnegative. It is only the positivity characterization which is needed for the rest of this discussion. In particular, the formal differential operator m, which defines M^* and all of its restrictions, has the factorization

$$m\varphi = -e^{-kz} \frac{d}{dz} \left[e^{2kz} \frac{d}{dz} e^{-kz} \varphi \right] := E_2 \varphi \ . \tag{32}$$

Thus it is known that the Green's function kernel for $\widetilde{M}^{-1}$ is TP. Likewise since $(M^* M)\varphi = E_2 E_2 \varphi := E_4 \varphi$, on $\operatorname{dom}(M^* M)$, $(M^* M)^{-1}$ has a TP Green's function kernel. It also follows that $F(\sigma_0) = (M^*M + \sigma_0 M)^{-1}$ has a TP Green's function kernel for all real $\sigma_0 > -k^2$, since the corresponding formal differential operator is $m^2 + \sigma_0 m = m(m + \sigma_0)$, and $(m + \sigma_0)$ has the factorization

$$-(m + \sigma_0)\,\varphi = e^{-\mu z} \frac{d}{dz}\left[e^{2\mu z} \frac{d}{dz} e^{-\mu z} \varphi\right], \tag{33}$$

where $\mu = \sqrt{\sigma_0 + k^2}$. Together, (33) and (32) give that $F(\sigma_0)$ has a nonnegative kernel, when σ_0 is real, and $\sigma_0 > -k^2$. Hence, it is anticipated that the expansion (26) contains the desired result.

Consider then the term of order $-\sigma_0$ in (26). This is an integral operator defined on any function $w \in \mathfrak{H}$. That is, set

$$T = \widetilde{M}^{-1} S \widetilde{M}^{-1} S \widetilde{M}^{-1}. \tag{34}$$

It will be shown that T has a nonnegative kernel. Let the kernel of T be called h. We will write h as the sum of two kernels of form (29), that is, $h = h_1 + h_2$, where as functions of z

$$h_i(0) = h_i'(0) = h_i(1) = h_i'(1) = 0, \qquad i = 1, 2. \tag{35}$$

Owing to the fact that h (and thereby T) represents the inverse of a sixth order operator E_6, (29), two other boundary conditions must be imposed in order to describe it. From the nature of T, these conditions are

$$\int_0^1 e^{kz}(-D^2 + k^2)^2 h\,dz = 0 \quad \text{and} \quad \int_0^1 e^{-kz}(-D^2 + k^2)^2 h\,dz = 0.$$

After the integrals are performed and the conditions (35) are imposed the other two conditions on h become

$$\begin{aligned} e^{k} h'''(1) - ke^{k} h''(1) - h'''(0) + kh''(0) &= 0\,, \\ e^{-k} h'''(1) + ke^{-k} h''(1) - h'''(0) - kh''(0) &= 0\,. \end{aligned} \tag{36}$$

However, these conditions are not of the form (30) and (31). In order to employ the TP theory, search for such a decomposition, namely suppose that ($n = 3$, $r_1 = e^{-kz}$, $r_2 = e^{2kz}$, $r_3 = e^{-2kz}$, $r_4 = e^{2kz}$, then) suitable separated boundary conditions are

$$(r_4 D_3 D_2 D_1 h_i)(0) - a_i (D_2 D_1 h_i)(0) = 0 \quad \text{and}$$

$$(r_4 D_3 D_2 D_1 h_i)(1) + b_i (D_2 D_1 h_i)(1) = 0, \; i = 1, 2.$$

Then with (35), these reduce to

$$h'''_i(0) - (a_i + k)\, h''_i(0) = 0\,, \quad \text{and} \\ h'''_i(1) + (b_i - k)\, h''_i(1) = 0\,, \quad i = 1, 2\,. \tag{37}$$

The choice of constants $a_i \geqslant 0$ and $b_i \geqslant 0$ must be made so that given $h = h_1 + h_2$, (37) are compatible with (36). Such a decomposition will not be unique, but it can be done since (37) and (36) represent six homogeneous equations in the eight unknown boundary values. This system is compatible for nontrivial solutions as long as $a_1 \neq a_2$ and $b_1 \neq b_2$. This means that T in (34) has a nonnegative kernel, so it may be concluded T is a positive operator.

In a similar manner, the term of order σ_0^2 in (26) is representable as

$$\widetilde{M}^{-1} S \widetilde{M}^{-1} S \widetilde{M}^{-1} S \widetilde{M}^{-1}. \tag{38}$$

This is obtainable from a differential operator E_8, that is (29), with $n = 4$. For the kernel of this operator perform a decomposition into two parts so that to each (35) will apply. But now, not only do (36) apply as before, but a higher order counterpart as well. Through expressions such as (37), it is possible find suitable separated boundary conditions at higher orders as well. The net decomposition is that the kernel for (38) is written as the sum of four nonnegative kernels. This renders (38) as a positive operator. By this procedure, each successive term in (26) is expressible as a sum of 2^{n-2} positive operators for $n = 2, 3, 4, \ldots$ respectively.

What has been shown is that the expansion (20) for $F(\sigma_0)$ converges for $|\sigma_0|\,\|B\|_0 < 1$ and for σ_0 real, it gives that the operator $F(\sigma_0)$ is positive for $-k^2 < \sigma_0 < \|B\|^{-1}$. Perform another expansion about a real point $\sigma_1 > 0$, such that $\sigma_0 < \sigma_1 < \|B\|^{-1}$. By summing the series for $F(\sigma_1)$ find that from (19),

$$F(\sigma_1) = F(\sigma_0)[I + (\sigma_0 - \sigma_1)\, MF(\sigma_0) + (\sigma_0 - \sigma_1)^2 (MF(\sigma_0))^2 + \ldots]. \tag{39}$$

It has already been established that $F(\sigma_0)$ has positive coefficients in powers of $(-\sigma_0)$, when $-k^2 < \sigma_0 < \|B\|^{-1}$, as in (20). By similar reasoning, it is observed that $F(\sigma_1)$ has the same properties as $F(\sigma_0)$ since, for example, the second coefficient $F(\sigma_0)\, MF(\sigma_0)$ in (39) behaves like a typical factor in (25) when expanded. By Remark 3, $-\|\Gamma(0)\|^{-1} = -\|\widetilde{M}^{-1}\|^{-1} < -k^2$, and by (24), since $-\|B\|^{-1} \leqslant -\|\widetilde{M}^{-1}\|^{-1} < -k^2$, the region of positivity of $F(\sigma_1)$ lies inside the region of analyticity given by (20). However, the only singularity on the edge of the disc of convergence for (20), as is shown in the expression for $F(\sigma_1)$ given by (18), would occur where $\sigma_1 = -\|B\|^{-1}$, since $F(\sigma)$ has no singularities off the real line. Thus it can be concluded using analytic continuation that $F(\sigma)$

may be written as

$$F(\sigma) = F(\sigma_1)[I + (\sigma_1 - \sigma)\, MF(\sigma_1) + (\sigma_1 - \sigma)^2 (MF(\sigma_1))^2 + \dots], \tag{40}$$

where $|\sigma - \sigma_1| < \sigma_1 + \|B\|^{-1}$. This disc goes outside the original disc $|\sigma| < \|B\|^{-1}$. The positivity is thus preserved for σ_0 real and $-k^2 < \sigma_0 < 2\sigma_1 + \|B\|^{-1}$. By a sequence of such discs, with singularities of (18) only at the (negative) eigenvalues of (16), any point in the half-plane $\mathrm{Re}(\sigma) > -k^2$ can be covered. The positivity of the limiting operator thus holds for all real $\sigma_0 > -k^2$.

Computing the derivative expansions of $F(\sigma)$ in (19), each term is of the form (26), so that $\left(-\frac{d}{d\sigma}\right)^n F(\sigma_0)$ is positive for all n and $F(\sigma)$ has a power series expansion with nonnegative coefficients. ■

3.3. *Proof of PES.*

With the aid of the last lemma, it is possible now to complete the abstract analysis of the earlier formulation (11) and obtain the desired result.

The Principle of Exchange of Stabilities holds for (8)-(9), *at all Prandtl numbers, when the integrated internal heat sources $N(z)$ and variable gravity ratio $H(z)$ have the same sign throughout the layer.*

PROOF. The system (8)-(9) may be written as the single equation suggested by (11),

$$u = K(\sigma)\, u \tag{41}$$

where

$$K(\sigma) = k^2 R^2 F(\sigma)\, H(z)\, \Gamma(Pr\sigma)\, N(z).$$

The resolvent is examined as defined in (12). It has been demonstrated that the original system (8)-(9), and the transformed system (41), have spectra that agree except on the set Σ_k, when $0 < Pr \leqslant 1$ or on the set $\Sigma_{k/\sqrt{Pr}}$ when $Pr > 1$, which in either case is a subset of the negative real half-line. We have shown in Lemma 4 that $F(\sigma)$ is a positive operator, and that $\Gamma(Pr\sigma)$ and $F(\sigma)$ have power series expansions for real $\sigma_0 > -k^2/Pr$ and $\sigma_0 > -k^2$, respectively.

To verify condition (**P2**), again note that it is assumed that $H(z)\, N(z) \geqslant 0$, while k^2 and R^2 are clearly positive. Therefore, by the product rule for differentiation, one concludes that $K(\sigma)$ in (40) satisfies condition (**P2**).

It has been demonstrated that all of the terms in $K(\sigma)$ determine positive operators. Moreover, for σ real and sufficiently large, by Remark 3 and (18), the norms of the operators $\Gamma(\sigma)$ and $F(\sigma)$ become arbitrarily small. Hence, $\|K(\sigma)\|$ will be less than 1. Then $[I-K(\sigma)]^{-1}$ has a convergent Neumann series and hence is positive. This is the content of condition (**P1**). ■

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